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MULTIDIMENSIONAL RANDOM MOTION WITH UNIFORMLY DISTRIBUTED CHANGES OF DIRECTION AND ERLANG STEPS

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We study transport processes in $\mathbb{R}^n$, $n \geq 1$, that have nonexponentially distributed sojourn times or non-Markovian step durations. We use the idea that the probabilistic properties of a random vector are completely determined by those of its projection to a fixed line, and, using this idea, we avoid many difficulties appearing in the analysis of these problems in higher dimensions. As a particular case, we find the probability density function in three dimensions for 2-Erlang-distributed sojourn times.

1. Introduction

One-dimensional non-Markovian generalizations of the random telegraph process were obtained in [1, 2] with velocities alternating at Erlang-distributed sojourn times. The uniformly distributed direction of motion, or isotropic motion, was studied by Pinsky [3] for transport processes on a Riemannian manifold and by Orsingher and De Gregorio in higher dimensions [4]. However, most papers on multidimensional random motion are devoted to the analysis of models in which motions are driven by a homogeneous Poisson process (see [3–6] and references therein). The recent work of Le Caër [7] departs from this trend because he studies a random motion with uniformly distributed orientation and Pearson–Dirichlet-distributed steps in a multidimensional random-walk setting. In the present work, we consider random motions with uniformly distributed directions in the multidimensional space $\mathbb{R}^n$, $n \geq 1$, with Erlang-distributed step lengths. Our analysis is based on random evolutions in semi-Markov media.

Let us consider the renewal process $\xi(t) = \max\{m \geq 0\colon \tau_m \leq t\}$, $t \geq 0$, where $\tau_m = \sum_{k=1}^{m} \theta_k$, $\tau_0 = 0$, and $\theta_k \geq 0$, $k = 1, 2, \ldots,$ are i.i.d. random variables with a distribution function $G(t)$ such that there exists the probability density function

$$g(t) = \frac{d}{dt} G(t).$$

We assume that a particle starts from the origin of coordinates $(0, 0, \ldots, 0)$ in the space $\mathbb{R}^n$ at time $t = 0$ and continues its motion at constant absolute velocity v in the direction $\boldsymbol{\eta}_0^{(n)}$, where $\boldsymbol{\eta}_0^{(n)} = (x_1, x_2, \ldots, x_n)$ is a random n-dimensional vector uniformly distributed over the unit sphere $\Omega_1^{n-1} = \{(x_1, x_2, \ldots, x_n)\colon x_1^2 + x_2^2 + \ldots + x_n^2 = 1\}$.

At time τ_1, the particle changes its direction to $\boldsymbol{\eta}_1^{(n)}$, where $\boldsymbol{\eta}_0^{(n)}$ and $\boldsymbol{\eta}_1^{(n)}$ are i.i.d. random vectors on Ω_1^{n-1}. Then, at time τ_2, the particle changes its direction to $\boldsymbol{\eta}_2^{(n)}$, where $\boldsymbol{\eta}_2^{(n)}$ is also uniformly distributed on Ω_1^{n-1} and independent of $\boldsymbol{\eta}_0^{(n)}$ and $\boldsymbol{\eta}_1^{(n)}$, and so on.

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Denote by $\mathbf{x}^{(n)}(t)$, $t \geq 0$, the location of the particle at time t. We have

$$\mathbf{x}^{(n)}(t) = v \sum_{i=0}^{\xi(t)} \boldsymbol{\eta}_i^{(n)} \theta_i + v\, \boldsymbol{\eta}_{\xi(t)}^{(n)} \left(t - \tau_{\xi(t)}\right). \tag{1}$$

Basically, Eq. (1) determines the random evolution in the semi-Markov medium $\xi(t)$.

Lemma 1. *The probability distribution of the random vector* $\mathbf{x}^{(n)}(t)$ *is determined by the probability distribution of its projection*

$$x^{(n)}(t) = v \sum_{i=0}^{\xi(t)} \eta_i^{(n)} \theta_i + v\, \eta_{\xi(t)}^{(n)} (t - \tau_{\xi(t)})$$

to a fixed line, where $\eta_i^{(n)}$ *is the projection of* $\boldsymbol{\eta}_i^{(n)}$ *to the line.*

Proof. Consider the cumulative distribution function $F_{x^{(n)}(t)}(y) = P(x^{(n)}(t) \leq y)$. Then the characteristic function $\varphi_{\mathbf{x}^{(n)}(t)}(\boldsymbol{\alpha})$ of $\mathbf{x}^{(n)}(t)$ is given by

$$\varphi_{\mathbf{x}^{(n)}(t)} = \mathbf{E} \exp\left\{i \left(\boldsymbol{\alpha}, \mathbf{x}^{(n)}(t)\right)\right\} = \mathbf{E} \exp\left\{i \, \|\boldsymbol{\alpha}\| \left(\mathbf{e}, \mathbf{x}^{(n)}(t)\right)\right\}$$

$$= \mathbf{E} \exp\left\{i \, \|\boldsymbol{\alpha}\| x^{'(n)}(t)\right\} = \int_0^\infty \exp\{i \, \|\boldsymbol{\alpha}\| y\} \, dF_{x^{(n)}(t)}(y),$$

where $\|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2}$ and $x^{(n)}(t)$ is the projection of $\mathbf{x}^{(n)}(t)$ onto the unit vector $\mathbf{e}$, which has a cumulative distribution function $F_{x^{(n)}(t)}(y)$.

Lemma 1 is proved.

It is well known that if $f(x_1, x_2, \ldots, x_n) \in L_1(\mathbb{R}^n)$ depends only on $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$, i.e., $f(x_1, x_2, \ldots, x_n) = g(r)$, then the function

$$\varphi(s_1, s_2, \ldots, s_n) = \int_{\mathbb{R}^n} f(\mathbf{x}) \exp\left\{-i \sum_{k=1}^n s_k x_k\right\} d\mathbf{x}$$

depends only on $s = \|\mathbf{s}\|$. Such functions are called radial functions, and, for these functions, the Fourier transformation in several variables turns into a "Bessel transformation" in one variable as follows:

$$\varphi(\mathbf{s}) = \frac{(2\pi)^{n/2}}{s^{(n-2)/2}} \int_0^\infty g(r)\, r^{n/2} J_{(n-2)/2}(sr) dr,$$

where $J_p(x)$ denotes the pth-order Bessel function of the first kind [10].

Since $\varphi_{\mathbf{x}}(\boldsymbol{\alpha})$ depends only on $\alpha = \|\boldsymbol{\alpha}\|$, which means that $\varphi_{\mathbf{x}}(\boldsymbol{\alpha}) = \varphi(\alpha)$, we conclude that the probability density function $f_{\mathbf{x}(t)}(\mathbf{y})$ that corresponds to the distribution

$$F_{\mathbf{x}(t)}(\mathbf{y}) = \mathrm{P}\left(v \sum_{i=0}^{\xi(t)+1} \boldsymbol{\eta}_i^{(n)} \theta_i + v\, \boldsymbol{\eta}_{\xi(t)}^{(n)} \left(t - \tau_{\xi(t)}\right) \leq \mathbf{y} \right)$$

depends only on $r = \|\mathbf{y}\|$, i.e., $f_{\mathbf{x}(t)}(\mathbf{y}) = h(r)$, and we have

$$\varphi_{\mathbf{x}(t)}(\boldsymbol{\alpha}) = \frac{(2\pi)^{n/2}}{\alpha^{(n-2)/2}} \int_0^\infty h(r)\, r^{n/2} J_{(n-2)/2}(\alpha r) dr.$$

It also follows that if $h(r)$ is continuous on $[0, +\infty)$,

$$\int_0^\infty r^{n-1} h(r) dr < \infty,$$

and

$$\int_0^\infty \alpha^{n-1} \varphi(\alpha) d\alpha < \infty,$$

then [10]

$$f_{\mathbf{x}(t)}(\mathbf{y}) = h(r) = \frac{1}{(2\pi)^{n/2}\, r^{(n-2)/2}} \int_0^\infty \varphi(\alpha)\, \alpha^{n/2} J_{(n-2)/2}(\alpha r) d\alpha.$$

We now define

$$\hat{x}^{(n)}(t) = v \sum_{i=0}^{\xi(t)} \eta_i^{(n)} \theta_i \qquad \text{and} \qquad \Delta(t) = v \eta_{\xi(t)}^{(n)} \left(t - \tau_{\xi(t)}\right)$$

and denote the cumulative distribution function of $\hat{x}^{(n)}(t)$ (respectively, $\Delta(t)$) by $F_{\hat{x}^{(n)}(t)}(y)$ (respectively, $F_{\Delta(t)}(y)$). It is easy to verify that $\hat{x}^{(n)}(t)$ and $\Delta(t)$ are independent. Hence,

$$F_{x^{(n)}(t)}(y) = F_{\hat{x}^{(n)}(t)}(y) * F_{\Delta(t)}(y).$$

Therefore, by using Lemma 1, we can study the cumulative distribution function of $\mathbf{x}^{(n)}(t)$, but we need to know the cumulative distribution functions of $\hat{x}^{(n)}(t)$ and $\Delta(t)$.

Lemma 2. *Let $F_n(t)$ be the cumulative distribution function of $\eta_i^{(n)}\theta_i$. Then it has the following form:*

$$F_n(t) = \begin{cases} \dfrac{1}{2} + \dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \displaystyle\int_0^1 G\left(\frac{t}{x}\right)(1-x^2)^{(n-3)/2}dx & \text{if } t \ge 0, \\ \dfrac{1}{2} - \dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \displaystyle\int_0^1 G\left(-\frac{t}{x}\right)(1-x^2)^{(n-3)/2}dx & \text{if } t < 0. \end{cases} \tag{2}$$

Proof. Let $f_{\eta_i}(x)$ denote the probability density function of the projection $\eta_i^{(n)}$ of the vector $\boldsymbol{\eta}_i^{(n)}$ onto a fixed line. It was shown in [8] that $f_{\eta_i}(x)$ is of the following form:

$$f_{\eta_i^{(n)}}(x) = \begin{cases} \dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}(1-x^2)^{(n-3)/2} & \text{if } x \in [-1,1], \\ 0 & \text{if } x \notin [-1,1]. \end{cases} \tag{3}$$

Since $\eta_i^{(n)}$ and θ_i are independent, it is easy to verify that the cumulative distribution function of $\eta_i^{(n)}\theta_i$ has the form (2).

Lemma 2 is proved.

The process $\gamma(t) = t - \tau_{\xi(t)}$ is a Markov process, and it has the following generator operator A [9]:

$$A\varphi(s) = \varphi'(s) + \frac{g(s)}{1-G(s)}\left(\varphi(0) - \varphi(s)\right), \quad s \ge 0,$$

where $\varphi \in \mathcal{C}^1(\mathbb{R})$.

Lemma 3. *The cumulative distribution function $F_{\Delta(t)}(s) = \mathrm{P}\left(v\,\eta_{\xi(t)}^{(n)}\,(t - \tau_{\xi(t)}) \le s\right)$ is given by*

$$F_{\Delta(t)}(s) = \begin{cases} \dfrac{1}{2} + \dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \displaystyle\int_0^1 F_{\gamma(t)}\left(\frac{s}{vx}\right)(1-x^2)^{(n-3)/2}dx & \text{if } s \ge 0, \\ \dfrac{1}{2} - \dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \displaystyle\int_0^1 F_{\gamma(t)}\left(-\frac{s}{vx}\right)(1-x^2)^{(n-3)/2}dx & \text{if } s < 0. \end{cases}$$

Proof. The cumulative distribution function $F_{\gamma(t)}(u) = \mathrm{P}(\gamma(t) \le u)$ satisfies the following Markov renewal equation [9]:

$$F_{\gamma(t)}(u) = V(t,u) + \int_0^t g(s) F_{\gamma(t-s)}(u) ds, \tag{4}$$

where

$$V(t,u) = \mathrm{P}(\gamma(t) \le u, \tau_1 > t) = (1 - G(t))\, I_{\{t \le u\}}.$$

We define the function

$$R(t) = \sum_{k=0}^{\infty} g^{*(k)}(t),$$

where the symbol $*(n)$ denotes the k-fold convolution of $g(t)$ with itself. Then Eq. (4) can be rewritten in the form

$$F_{\gamma(t)}(u) = (V * R)(t,u) = \int_0^t V(t-s,u) dR(s).$$

Since $v\eta_{\xi(t)}^{(n)}$ and $\gamma(t)$ are independent, this concludes the proof.

2. Evolution in Odd-Dimensional Spaces

Now assume that $n = 2l + 3$, $l = 0, 1, 2, \ldots,$ and θ_k has an $(n-1)$-Erlang distribution, i.e.,

$$g(t) = \frac{\lambda^{n-1}}{\Gamma(n-1)} t^{n-2} e^{-\lambda t}.$$

It follows from Lemma 2 that, for $t \ge 0$, the probability density function $f_n(t)$ of the random variable $\eta_i^{(n)} \theta_i$ has the form

$$f_n(t) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)\Gamma(n-1)} \lambda \int_0^1 \frac{(\lambda t)^{2l+1}}{x^{2l+2}} e^{-\lambda t/x} (1-x^2)^l dx,$$

or, equivalently,

$$f_n(t) = \frac{\lambda \Gamma\left(l + \frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma(l+1)\Gamma(2l+2)} \sum_{k=0}^{l} \binom{l}{k} (-1)^k (\lambda t)^{2k} \int_{\lambda t}^{\infty} s^{2(l-k)} e^{-s} ds.$$

Furthermore, the following equivalent expression can be found after some algebraic simplifications:

$$f_n(t) = \frac{\lambda e^{-\lambda t}}{l!\, 2^{2l+1}} \sum_{k=0}^{l} (-1)^k \frac{(2(l-k)!)}{k!(l-k)!} \sum_{m=0}^{2(l-k)} \frac{(\lambda t)^{2k+m}}{m!}. \tag{5}$$

In the case where $t < 0$, we have $f_n(t) = f_n(-t)$.

3. Evolution in Three Dimensions

Let us consider the particular case where $n = 3$. Taking Lemma 2 into account, we conclude that $\eta_i^{(3)}$ is uniformly distributed on $[-1, 1]$.

Let random variables θ_k, $k = 0, 1, 2, \ldots,$ be 2-Erlang distributed, i.e., $g(t) = \lambda^2 t e^{-\lambda t}$, $\lambda > 0$, $t \geq 0$.

For this particular case, the Laplace transform $\widehat{R}(s)$ of $R(t)$ is of the form

$$\widehat{R}(s) = \int_0^\infty R(t)e^{-st}dt = \sum_{k=0}^\infty \int_0^\infty g^{*(k)}(t)e^{-st}dt = \sum_{k=0}^\infty \left(\frac{\lambda}{\lambda + s}\right)^{2k} = \frac{(\lambda + s)^2}{s^2 + 2\lambda s},$$

and the Laplace transform $\widehat{V}(s, u)$ of $V(t, u)$ can be written as follows:

$$\widehat{V}(s, u) = \frac{2\lambda + s - (\lambda us + \lambda^2 us + s + 2)e^{-(\lambda+s)u}}{(\lambda + s)^2}.$$

Therefore, the Laplace transform $\widehat{F}_\gamma(s, u)$ of $F_{\gamma(t)}(u)$ is given by

$$\widehat{F}_\gamma(s, u) = \widehat{R}(s)\widehat{V}(s, u) = \frac{2\lambda + s - (\lambda us + \lambda^2 us + s + 2)e^{-(\lambda+s)u}}{s(s + 2\lambda)}. \tag{6}$$

Applying the inverse Laplace transformation to $\widehat{F}_\gamma(s, u)$, we obtain the following relation for $t > u > 0$:

$$F_{\gamma(t)}(u) = e^{-2\lambda t} - (2 + \lambda u)e^{-\lambda t} \sinh(\lambda(t - u)) + 2e^{-\lambda t} \sinh(\lambda t) - (\lambda u + 1)e^{-\lambda(2t-u)}.$$

Thus, we have the limit result

$$\lim_{t \to +\infty} F_{\gamma(t)}(u) = 1 - e^{-\lambda u} - \frac{\lambda u}{2} e^{-\lambda u}.$$

Taking Lemma 3 into account, we can obtain the corresponding expression for $F_{\Delta(t)}(s)$.

It follows from Eq. (5) that $\eta_i \theta_i$ has the Laplace distribution with probability density function

$$f_3(t) = \frac{1}{2} \lambda e^{-\lambda |t|}.$$

Therefore, the Fourier transform of $\mathrm{P}\left(v \sum_{i=0}^k \eta_i \theta_i \leq y\right)$ is given by

$$\int_{-\infty}^\infty e^{-i\lambda y} d\mathrm{P}\left(v \sum_{i=0}^k \eta_i \theta_i \leq y\right) = \left(\frac{\lambda^2}{\lambda^2 + v^2\alpha^2}\right)^k.$$

On the other hand, since

$$F_{\hat{x}^{(3)}(t)}(y) = \mathrm{P}\left(v \sum_{i=0}^{\xi(t)} \eta_i^{(3)} \theta_i \leq y\right) = \sum_{k=0}^\infty \mathrm{P}\left(v \sum_{i=0}^k \eta_i^{(3)} \theta_i \leq y\right) \mathrm{P}(\xi(t) = k),$$

the characteristic function of $\hat{\mathbf{x}}^{(3)}(t)$, namely

$$\varphi_{\hat{\mathbf{x}}^{(3)}(t)}(\boldsymbol{\alpha}) = \mathbf{E}\left[e^{-i\alpha\hat{x}^{(3)}(t)}\right] = \int_{-\infty}^{\infty} e^{-i\alpha y}\, dF_{\hat{x}^{(3)}(t)}(y),$$

can be calculated as follows

$$\varphi_{\hat{\mathbf{x}}^{(3)}(t)}(\boldsymbol{\alpha}) = \sum_{k=0}^{\infty}\left(\frac{\lambda^2}{\lambda^2 + v^2\alpha^2}\right)^k \mathrm{P}(\xi(t) = k) = e^{-\lambda t}\sum_{k=0}^{\infty}\left(\frac{\lambda^2}{\lambda^2 + v^2\alpha^2}\right)^k \left(\frac{(\lambda t)^{2k}}{2k!} + \frac{(\lambda t)^{2k+1}}{(2k+1)!}\right).$$

We define

$$\Phi = \frac{\lambda^2}{\sqrt{\lambda^2 + v^2\alpha^2}}.$$

Then

$$\varphi_{\hat{\mathbf{x}}^{(3)}(t)}(\boldsymbol{\alpha}) = e^{-\lambda t}\left[\cosh \Phi t + \frac{\lambda^2 + v^2\alpha^2}{\lambda^2} \sinh \Phi t\right].$$

Therefore, by using the inverse Fourier transformation, we can obtain $F_{\hat{\mathbf{x}}^{(3)}(t)}(\mathbf{y})$.

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