Evolution process as an alternative to diffusion process and Black-Scholes Formula

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Abstract—In this paper, we study the one-dimensional transport process in the case of disbalance. In the hydrodynamic limit, this process approximates the diffusion process on a line. By using this property, we propose to apply transport processes instead of diffusion processes in some economical models, particularly in Black-Scholes formula. This application manages to avoid some drawbacks of diffusion processes.

Key words and phrases: Random evolutions; Black-Scholes formula; diffusion process; Brownian motion; transport process

1. INTRODUCTION

Wiener process is the well-known mathematical model used to study random movement of a particle, called the Brownian motion. It is also prominent in the mathematical theory of finance, in particular the Black-Scholes option pricing model. However, in some recent papers, for instance [3], considered alternative models of the Brownian motion based on the telegrapher process. This model manages to avoid such drawbacks of the Wiener process as a fractal trajectory of the process and others. Since the diffusion process used in the Black-Scholes formula, we propose to use Markovian evolutions in the case of disbalance instead of the diffusion process. Unlike diffusion process trajectories, Markovian evolutions have trajectories which are differentiable almost everywhere. Because of this, we think that the Markovian evolution model is more accurate in both the particle physics and economical theory applications.

2. TRANSPORT PROCESS AS AN ALTERNATIVE TO A DIFFUSION MODEL

Let \(\{\xi(t), t \geq 0\}\) be a Markov process on the phase space \(\{0, 1\}\) with the infinitesimal matrix

\[
Q = \lambda \begin{pmatrix}
-1 & 1 \\
1 & -1
\end{pmatrix},
\]
Now, consider the random velocity model or transport process

\[ x(t) = x_0 + \int_0^t v(s) \, ds, \]

where

\[ v(t) := \begin{cases} 
  v_0, & \text{if } \xi(t) = 0; \\
  v_1, & \text{if } \xi(t) = 1.
\end{cases} \]

The infinitesimal operator \( A \) of the bivariate process \( \{ \zeta(t) = (x(t), \xi(t)), t \geq 0 \} \) is of the following form

\[ A\varphi(x, i) = \begin{cases} 
  v_0 \frac{\partial}{\partial x} \varphi(x, 0) + \lambda \varphi(x, 1) - \lambda \varphi(x, 0), & \text{if } i = 0; \\
  v_1 \frac{\partial}{\partial x} \varphi(x, 1) + \lambda \varphi(x, 0) - \lambda \varphi(x, 1), & \text{if } i = 1.
\end{cases} \] (1)

where \( \varphi \in D(A) = \text{domain of the operator } A, \) and \( x \in \mathbb{R}. \)

We can interpret this operator in the following equivalent manner: Assume \( Z = \mathbb{R} \times \{0, 1\} \) and

\[ T_t \varphi(x, i) = \int_Z \varphi(z) P\{ \zeta(t) \in dz \mid \zeta(0) = (x, i) \}, \quad i \in \{0, 1\}. \]

Then,

\[ A\varphi(x, i) = \lim_{\Delta t \to 0^+} \frac{T_{\Delta t} \varphi(x, i) - \varphi(x, i)}{\Delta t}. \]

Let us define \( u_i(x, t) = T_t \varphi(x, i) \). Thus,

\[ \frac{\partial}{\partial t} u_i(x, t) = A T_t \varphi(x, i) = A u_i(x, t). \]

Now, let us consider the scaled evolution equation

\[ x_\varepsilon(t) = x_0 + \frac{1}{\varepsilon} \int_0^t \frac{s}{\varepsilon^2} \, ds, \]

with corresponding velocities \( \frac{v_i}{\varepsilon}, \ i = \{0, 1\}. \) Since the Markov process \( \varepsilon \left( \frac{t}{\varepsilon^2} \right) \) has scaled time, then its infinitesimal operator is of the form \( \frac{1}{\varepsilon^2} Q. \)

Hence, we obtain the following system of Kolmogorov backward differential equations

\[ \begin{align*}
  \frac{\partial}{\partial t} u_0^\varepsilon(x, t) &= \frac{v_0}{\varepsilon} \frac{\partial}{\partial x} u_0^\varepsilon(x, t) + \frac{\lambda}{\varepsilon^2} u_1^\varepsilon(x, t) - \frac{\lambda}{\varepsilon^2} u_0^\varepsilon(x, t), \\
  \frac{\partial}{\partial t} u_1^\varepsilon(x, t) &= \frac{v_1}{\varepsilon} \frac{\partial}{\partial x} u_1^\varepsilon(x, t) + \frac{\lambda}{\varepsilon^2} u_0^\varepsilon(x, t) - \frac{\lambda}{\varepsilon^2} u_1^\varepsilon(x, t).
\end{align*} \]

(2)

Equations (2) can be written in matrix form as follows

\[ \frac{\partial}{\partial t} \mathbf{u}^\varepsilon(x, t) = \frac{1}{\varepsilon} V \nabla \mathbf{u}^\varepsilon(x, t) + \frac{1}{\varepsilon^2} Q \mathbf{u}^\varepsilon(x, t) \]

(3)
where
\[ u^\varepsilon(x, t) = \begin{pmatrix} u_0^\varepsilon(x, t) \\ u_1^\varepsilon(x, t) \end{pmatrix}, \quad V = \begin{pmatrix} v_0 & 0 \\ 0 & v_1 \end{pmatrix}, \quad \nabla = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial x} \end{pmatrix}. \]

Let us define
\[ R_0 := \int_0^\infty (\Pi - e^{Qt}) \, dt, \]
where \( e^{Qt} = \{ p_{ij}(t); i, j \in \{0, 1\} \} \) is the set of time-dependent transition probabilities, and
\[ \Pi = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \]
is the projector operator on the null-space of \( Q \), i.e., \( \Pi Q = Q \Pi = 0. \)

Furthermore, it can be shown that \( R_0 = \Pi - (Q + \Pi)^{-1} \) [1]. So, \( R_0 \) is the potential operator of \( \xi(t) \).

The balance condition states that \( \frac{v_0 + v_1}{2} = 0 \), i.e., \( v_0 = -v_1 \) or equivalently \( v_0 = v \) and \( v_1 = -v \). This balance condition can also be expressed as \( \Pi V \Pi = 0 \) or \( \Pi V \nabla \Pi = 0 \), and it is assumed in most of the works for the diffusion approximation [1], [2].

In this paper we will consider the following disbalance condition: \( v_0 = v + \Delta_1 \) and \( v_1 = -v - \Delta_2 \), where \( \Delta_k = \varepsilon a_k, k = 1, 2 \). Then, it is easily verified that the infinitesimal operator of the process \( \zeta^\varepsilon(t) = \left( x_\varepsilon(t), \xi \left( \frac{t}{\varepsilon^2} \right) \right) \) is of the following form
\[ A_\varepsilon = \frac{1}{\varepsilon} V \nabla + A \nabla + \frac{1}{\varepsilon^2} Q, \]
where \( A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \).

Denote as \( u^\varepsilon_i(x, t) = T^\varepsilon \varphi(x, i) = \int_{\mathbb{Z}} \varphi(z) P(\zeta^\varepsilon(t) \in dz \mid \zeta^\varepsilon(0) = (x, i)), i = \{0, 1\}. \)

Then, similarly as above, we can write the matrix equation
\[ \frac{\partial}{\partial t} u^\varepsilon(x, t) = \frac{1}{\varepsilon} V \nabla u^\varepsilon(x, t) + A \nabla u^\varepsilon(x, t) + \frac{1}{\varepsilon^2} Q u^\varepsilon(x, t). \quad (4) \]

By using the technique of multiple scales, as it is described by A. Vasiljeva and V. Butuzov [4], we will find a solution of Eq. (4) in the following form
\[ u^\varepsilon(x, t) = u^{(0)}(x, t) + \sum_{n=1}^{\infty} \varepsilon^n \left( u^{(n)}(x, t) + r^{(n)}(x, t/\varepsilon^2) \right), \quad (5) \]
where \( u^{(n)}(x, t), n = 0, 1, 2, \ldots \) are the regular terms of the expansion whereas \( r^{(n)}(x, t/\varepsilon^2), \)
\( n = 1, 2, \ldots \) are the singular ones [1].
Then, by substituting Eq. (5) into Eq. (4) we obtain

\[ \begin{align*}
Q u^{(0)}(x, t) &= 0, \\
Q u^{(1)}(x, t) + V \nabla u^{(0)}(x, t) &= 0, \\
Q u^{(2)}(x, t) + V \nabla u^{(1)}(x, t) + A \nabla u^{(0)}(x, t) - \frac{\partial}{\partial t} u^{(0)}(x, t) &= 0, \\
& \vdots \\
Q u^{(k+2)}(x, t) + V \nabla u^{(k+1)}(x, t) + A \nabla u^{(k)}(x, t) - \frac{\partial}{\partial t} u^{(k)}(x, t) &= 0.
\end{align*} \]  

(6)

for \( k \geq 0. \)

Hence, \( u^{(0)}(x, t) \in \mathbb{N}_Q, \) where \( \mathbb{N}_Q \) is the null-space of \( Q, \) i.e., \( \Pi u^{(0)}(x, t) = u^{(0)}(x, t). \)

It follows from Eqs. (6) that

\[ u^{(1)}(x, t) = R_0 V \nabla u^{(0)}(x, t) + c_1(t), \]

(7)

where \( c_1(t) = \mathbb{N}_Q. \)

Similarly,

\[ Q u^{(2)}(x, t) = \frac{\partial}{\partial t} u^{(0)}(x, t) - V \nabla u^{(1)}(x, t) \]

\[ = \frac{\partial}{\partial t} u^{(0)}(x, t) - V \nabla R_0 V \nabla u^{(0)}(x, t) - V \nabla c_1(t) - A \nabla u^{(0)}(x, t). \]

Multiplying Eq. (8) by the operator \( \Pi \) we obtain

\[ \Pi Q u^{(2)}(x, t) = 0 = \frac{\partial}{\partial t} u^{(0)}(x, t) - \Pi V \nabla R_0 V \nabla u^{(0)}(x, t) - \Pi A \nabla u^{(0)}(x, t), \]

(8)

where we have used the fact that \( \Pi V \nabla c_1(t) = \Pi V \nabla \Pi c_1(t) = 0. \)

Therefore, the main term of the expansion (5) satisfies the diffusion equation (8).

We can obtain similar equations for the rest of the terms but we are just interested on the main term \( u^{(0)}(x, t). \)

Now, let write the matrix equation (4) in the following form

\[ \begin{pmatrix}
\frac{\partial}{\partial t} - v \frac{\partial}{\varepsilon \partial x} - a_1 \frac{\partial}{\partial x} + \lambda \\
\frac{\lambda}{\varepsilon^2} - v \frac{\partial}{\varepsilon \partial x} + a_2 \frac{\partial}{\partial x} + \lambda \\
\end{pmatrix} u^\varepsilon(x, t) = \Psi u^\varepsilon(x, t) = 0. \]

(9)

It is easy to verify that the function \( f_\varepsilon(x, t) = u_0^\varepsilon(x, t) + u_1^\varepsilon(x, t) \) is the solution of the following equation

\[ \det \Psi f_\varepsilon(x, t) = 0. \]

(10)

Writing Eq. (10) in more detail, we have

\[ \varepsilon^2 \left( \frac{\partial^2}{\partial t^2} + (a_2 - a_1) \frac{\partial^2}{\partial x \partial t} - \frac{v(a_2 + a_1)}{\varepsilon} \frac{\partial^2}{\partial x^2} - a_1 a_2 \frac{\partial^2}{\partial x^2} \right) f_\varepsilon(x, t) = 0. \]

(11)
Let us define the notation \( u^{(0)}(x,t) = (u_0(x,t), u_1(x,t)) \) and \( f_0(x,t) = u_0(x,t) + u_1(x,t) \). Since \( u^\varepsilon(x,t) \to u^{(0)}(x,t) \) as \( \varepsilon \to 0 \) [2], then we have \( \lim_{\varepsilon \to 0} f_\varepsilon(x,t) = f_0(x,t) \).

Hence, it follows from Eq. (11) that
\[
\left( \frac{\partial}{\partial t} - \frac{v^2}{2\lambda} \frac{\partial^2}{\partial x^2} + \frac{a_2 - a_1}{2} \frac{\partial}{\partial x} \right) f_0(x,t) = 0. \tag{12}
\]

So, if \( a_1 \neq a_2 \) then \( f_0(x,t) \) satisfies the diffusion equation (12) with drift coefficient \( \frac{a_1 - a_2}{2} \) and diffusion coefficient \( \frac{v^2}{2\lambda} \).

3. APPLICATION TO ECONOMIC MODEL OF MARKET

The well-known Black-Scholes formula gives the price \( X_t \) of a stock at time \( t \) such that
\[
X_t = X_0 \exp(\mu t + \sigma W_t), \tag{13}
\]
where \( \mu \) is the drift, \( \sigma \) is the volatility of the stock, and \( \{W_t\} \) is a Wiener process.

By substituting \( \mu = \frac{a_1 - a_2}{2} \) and \( \sigma = \frac{v}{\sqrt{2\lambda}} \) in Eq. (12) we obtain the diffusion equation for the process \( \{\mu t + \sigma W_t\} \). So, the Black-Scholes formula is obtained by considering an exponential Brownian motion for the share price \( X_t \).

The Wiener process provides a mathematical consistent model for Brownian motion. However, it has the following drawbacks to capture the physics of many applications [3]

1. Modulus of the velocity is infinite almost always at any instant of time
2. Zero length of free path
3. The path function of a particle is nowhere differentiable almost surely, and its Hausdorff dimension is equal to 1.5, i.e., the path function is fractal.

However, the actual movement of a physical particle as well as the actual evolution of share prices \( X_t \) are barely justified as fractal quantities. Taking into account these considerations we propose to use the process given by Eq. (11) instead of the diffusion process given in Eq. (12) for most applications. This recommendation is based on the fact that this process does not suffer of the drawback mentioned above. For instance, the process in Eq. (11) has almost always a finite modulus of velocity \( V_\varepsilon = \frac{v}{\varepsilon} \), nonzero free paths that depends on \( \Lambda_\varepsilon = \frac{\lambda}{\varepsilon^2} \), and its trajectories are almost surely differentiable.

Some difficulties may arise for calculating the coefficients \( V_\varepsilon \) and \( \Lambda_\varepsilon \). For such a purpose, we suggest to consider the time between two consequent renewal epochs (or equivalent two consequent impacts of a particle or two consequent changes of price) as an exponential distributed random variable with parameter \( \Lambda_\varepsilon \). Then, \( \Lambda_\varepsilon \) can be estimated from experimental data as well as the velocity \( V_\varepsilon \).

Hence, instead of the Black-Scholes formula we propose the following formula for the price \( X_t \) of a stock at time \( t \)
\[
X_t = X_0 \exp\left( \frac{1}{\varepsilon} \int_0^t v\left( \frac{s}{\varepsilon^2} \right) ds \right) \tag{14}
\]
with the corresponding coefficients \( \Lambda_\varepsilon \) and \( V_\varepsilon \).

We should mention that it is not clear how to calculate the parameter \( \sigma \) in application of the Black-Scholes formula to the price of a stock [6], and, in our opinion, such a problem does not exist in the calculation of \( \Lambda_\varepsilon \) for the process \( \xi \left( \frac{t}{\varepsilon^2} \right) \).
4. CONCLUSIONS

These are the ...

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