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Asymptotic expansion for the distribution of a Markovian random motion

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Abstract. In this paper, we study an asymptotic expansion for the distribution of a random motion of a particle driven by a Markov process in diffusion approximation. We show that the singularly perturbed equation of a Markovian random motion can be reduced to the regularly perturbed equation for the distribution of the random motion.

Key words. Markov stochastic evolution; asymptotic expansion; perturbed equation.

AMS classification. 60J25, 60J65.

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1. Introduction

The first CLT for additive functionals of a Markov chain with noncountable phase space was proved by Doeblin [2]. Additional functionals of Markov and semi-Markov processes with finite phase space have intensively been studied by V. S. Korolyuk, A. F. Turbin, M. Pinsky, V. M. Shurenkov and others [8], [9], [11].

In 1951, S. Goldstein introduced the telegrapher's stochastic process in his seminal paper [3], which is a random motion driven by a homogeneous Poisson process. This basic telegrapher process has been extended by M. Kac in [4]. Goldstein–Kac's telegraph process on the line and its weak convergence to the one-dimensional Brownian motion are well known.

This paper deals with the n-dimensional random motion which is an additional functional of some Markov process. This kind of model is well known and popular in the physical literature for the description of long polymer molecules. For example, one of the forms of the Airing model in [5] is similar to the model in this paper.

Let us consider the random motion of a particle in $\mathbb{R}^n$ driven by a Markov process $\xi(t)$, whose sojourn times at states are exponentially distributed with rate $\lambda > 0$ and transition probabilities $p_{ij} = \frac{1}{2n-1}\delta_{ij}$, $i, j \in E = \{1, 2, \ldots, 2n\}$, where E is the phase space of $\xi(t)$.

Let $\vec{b}_1, \ldots, \vec{b}_n$ be a Cartesian basis of $\mathbb{R}^n$. Put $\vec{e}_1 = \vec{b}_1$, $\vec{e}_2 = -\vec{b}_1$, $\vec{e}_3 = \vec{b}_2$, $\vec{e}_4 = -\vec{b}_2, \ldots, \vec{e}_{2n-1} = \vec{b}_n$, $\vec{e}_{2n} = -\vec{b}_n$ and $\vec{v}_i = v\vec{e}_i$, $i = 1, 2, \ldots, 2n$, where $v > 0$ is the constant speed of the particle.

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We assume that the particle moves in n-dimensional space in the following manner: If at some instant t the particle has velocity $\vec{v}_i$, then at a renewal moment of the Markov process the particle takes a new velocity $\vec{v}_j$, $j \neq i$, with probability $p_{ij} = \frac{1}{2n-1}$. The particle continues its motion with velocity $\vec{v}_j$ until the next renewal moment of the Markov process, and so on.

Let us denote by $\vec{r}(t) = (x_1(t), x_2(t), \dots, x_n(t))$, $t \geq 0$, the particle position at time t. Consider the function

$$\vec{C}(i) = (C_1(i), C_2(i), \dots, C_n(i)) = \vec{v}_i, \quad i \in E.$$

Then the position of the particle at time t can be expressed as

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{C}(\xi(t))dt.$$

2. Equation for the probability density of the particle position

Let us consider the bivariate stochastic process $\varsigma(t) = (\vec{r}(t), \xi(t))$ with the phase space $\mathbb{R}^n \times E$. It is well known that this process is Markovian and the generating operator of $\varsigma(t)$ is of the following form [7], [8]:

$$A\varphi(\vec{r}, i) = \vec{C}(i)\varphi'(\vec{r}, i) + \lambda[P\varphi(\vec{r}, i) - \varphi(\vec{r}, i)], \tag{2.1}$$

where

$$\vec{C}(i)\varphi'(\vec{r}, i) = C_1(i)\frac{\partial}{\partial x_1}\varphi(\vec{r}, i) + C_2(i)\frac{\partial}{\partial x_2}\varphi(\vec{r}, i) + \cdots + C_n(i)\frac{\partial}{\partial x_n}\varphi(\vec{r}, i)$$

and

$$P\varphi(\vec{r}, i) = \frac{\lambda}{2n-1}\sum_{j \in E \setminus i}\varphi(\vec{r}, j).$$

Now, let us consider the density function

$$f_i(t, x_1, \dots, x_n)dx_1 \dots dx_n = \mathbf{P}\{x_1 \leq \ x_1(t) \leq x_1 + dx_1, \dots, x_n \leq \ x_n(t) \leq x_n + dx_n\}.$$

It is easily verified that

$$f(t, x_1, \dots, x_n) = \sum_{i=1}^n f_i(t, x_1, \dots, x_n)$$

is the probability density of the particle position in $\mathbb{R}^n$ at time t.

Lemma 2.1. *The function f satisfies the following differential equation*

$$\prod_{i=1}^{2n}\left\{\frac{\partial}{\partial t}+(-1)^i v\frac{\partial}{\partial x_i}+\frac{2n\lambda}{2n-1}\right\}f + \frac{2n\lambda}{2n-1}\sum_{k=1}^{2n}\prod_{\substack{i=1\\ i\neq k}}^{2n}\left\{\frac{\partial}{\partial t}+(-1)^i v\frac{\partial}{\partial x_i}+\frac{2n\lambda}{2n-1}\right\}f=0.$$

Proof. For $i\in E$, the function f_i satisfies the first Kolmogorov equation, namely

$$\frac{\partial f_i(t,x_1,\dots,x_n)}{\partial t}=Af_i(t,x_1,\dots,x_n),\quad i\in E, \tag{2.2}$$

with initial conditions $f_i(0,x_1,\dots,x_n)=f_i^{(0)}$.

Equation (2.2) can be written in more detail as follows

$$\frac{\partial f_i(t,x_1,\dots,x_n)}{\partial t}+(-1)^i v\frac{\partial}{\partial x_i}f_i(t,x_1,\dots,x_n)+\lambda f_i(t,x_1,\dots,x_n) - \frac{\lambda}{2n-1}\sum_{j\in E\setminus i}f_j(t,x_1,\dots,x_n)=0,\quad i\in E. \tag{2.3}$$

Now, put

$$\vec{f}(t,x_1,\dots,x_n)=\{f_i(t,x_1,\dots,x_n), i\in E\}.$$

The set of equations (2.3) can be written in the following form

$$L_{2n}\vec{f}=0,$$

where

$$L_{2n}=\{l_{ij}\}_{ij\in E},\quad l_{ii}=\frac{\partial}{\partial t}+(-1)^i v\frac{\partial}{\partial x_i}+\lambda,\quad l_{ij}=\frac{-\lambda}{2n-1},\quad i\neq j.$$

The function f satisfies the equation [6]

$$\det(L_{2n})f=0, \tag{2.4}$$

with the initial condition $f(0,x_1,\dots,x_n)=\sum_{k=1}^n f_i^{(0)}$.

The determinant of the matrix L_{2n} is well known and it has the form

$$\det(L_{2n})=\prod_{i=1}^{2n}\left\{\frac{\partial}{\partial t}+(-1)^i v\frac{\partial}{\partial x_i}+\frac{2n\lambda}{2n-1}\right\} + \frac{2n\lambda}{2n-1}\sum_{k=1}^{2n}\prod_{\substack{i=1\\ i\neq k}}^{2n}\left\{\frac{\partial}{\partial t}+(-1)^i v\frac{\partial}{\partial x_i}+\frac{2n\lambda}{2n-1}\right\}.$$

□

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Since $v\frac{\partial}{\partial x_i}$ and $-v\frac{\partial}{\partial x_i}$ appear in L_{2n} symmetrically, it is easy to see that all monomials of the polynomial $\det(L_{2n})$ contain v^k only with even powers $k \geq 0$.

3. Reduction of singularly perturbed evolution equation to regularly perturbed equation

Let us put $v = \varepsilon^{-1}$ and $\lambda = \varepsilon^{-2}$, where $\varepsilon > 0$ is a small parameter. It is well known [7], [10], [12] that the solution of equation (2.3) in the hydrodynamical limit (as $\varepsilon \to 0$) weakly converges to the corresponding functional of a Wiener process.

By using a technique developed in [13], we can find the asymptotic expansion of the solution of equation (2.3), which consists of regular and singular terms. This technique involves tedious calculations [8], [10].

Proposition 3.1. *The equation* $\det(L_{2n})f = 0$ *is regularly perturbed, that is, multiplying it by* ε^{4n-2}*, we get*

$$\frac{\partial}{\partial t}f = \frac{2n-1}{2n^2}\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)f + D_\varepsilon f,$$

where $D_\varepsilon = \varepsilon^2 D_1 + \varepsilon^4 D_2 + \cdots$*, and* D_i*,* $i = 1, 2, \ldots$*, are the respective differential operators.*

Proof. To avoid cumbersome expressions, we consider the case when $n = 3$. Let us put $x = x_1$, $y = x_2$, $z = x_3$. In this case, equation (2.3) has the following form

$$\frac{\partial f_i(t,x,y,z)}{\partial t} + (-1)^i v\frac{\partial}{\partial x}f_i(t,x,y,z) + \lambda f_i(t,x,y,z) - \frac{\lambda}{5}\sum_{\substack{j\in E\\ j\neq i}} f_j(t,x,y,z) = 0, \quad i = 1,\ldots,6. \tag{3.1}$$

Putting $v = \varepsilon^{-1}$ and $\lambda = \varepsilon^{-2}$, we obtain the following singularly perturbed system of equations

$$\frac{\partial f_i(t,x,y,z)}{\partial t} + (-1)^i \varepsilon^{-1}\frac{\partial}{\partial x}f_i(t,x,y,z) + \varepsilon^{-2} f_i(t,x,y,z) - \varepsilon^{-2}\frac{1}{5}\sum_{\substack{j\in E\\ j\neq i}} f_j(t,x,y,z) = 0, \quad i = 1,\ldots,6. \tag{3.2}$$

Let us consider the equation $\det(L_6)f = 0$, where

$$f(t,x,y,z) = \sum_{i=1}^{6} f_i(t,x,y,z).$$

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It is easy to see that the elements of the matrix $L_6 = (l_{ij})_{i,j\in E}$ are as follows: $l_{ii} = \frac{\partial}{\partial t} + (-1)^i v\frac{\partial}{\partial t} + \lambda$ and $l_{ij} = \frac{\lambda}{5}$ for $i \neq j$, $i, j \in \{1, 2, \ldots, 6\}$.

Hence, the equation $\det(L_6)f = 0$ has the following form

$$\begin{aligned}\det(L_6)f(t,x,y,z) = \Big\{ & \frac{7776}{3125}\varepsilon^{-10}\frac{\partial}{\partial t} - \frac{432}{625}\varepsilon^{-10}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) \\ & - \frac{432}{125}\varepsilon^{-8}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{\partial}{\partial t} - \frac{144}{25}\varepsilon^{-6}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{\partial^2}{\partial t^2} \\ & + \frac{1296}{125}\varepsilon^{-2}\frac{\partial^2}{\partial t^2} + \frac{24}{25}\varepsilon^{-8}\left(\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^2}{\partial y^2\partial z^2}+\frac{\partial^2}{\partial x^2\partial z^2}\right) + \frac{72}{5}\varepsilon^{-4}\frac{\partial^4}{\partial t^4} \\ & - 4\varepsilon^{-4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{\partial^3}{\partial t^3} + 2\varepsilon^{-6}\left(\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^2}{\partial y^2\partial z^2}+\frac{\partial^2}{\partial x^2\partial z^2}\right)\frac{\partial}{\partial t} \\ & + 6\varepsilon^{-2}\frac{\partial^5}{\partial t^5} + \varepsilon^{-8}\left(\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^2}{\partial y^2\partial z^2}+\frac{\partial^2}{\partial x^2\partial z^2}\right)\frac{\partial^2}{\partial t^2} - \varepsilon^{-6}\frac{\partial^6}{\partial x^2\partial y^2\partial z^2} \\ & - \varepsilon^{-2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{\partial^4}{\partial t^4} + \frac{\partial^6}{\partial t^6} + \frac{432}{25}\varepsilon^{-6}\frac{\partial^3}{\partial t^3}\Big\} f(t,x,y,z) = 0, \end{aligned} \tag{3.3}$$

with the initial condition $f(0,x,y,z) = f_0$.

Multiplying equation (3.3) by ε^{10}, we obtain

$$\begin{aligned}\Big\{ & \frac{\partial}{\partial t} - \frac{5}{18}\Delta + \varepsilon^2\frac{25}{6}\left[\frac{\partial^2}{\partial t^2} - \frac{1}{3}\Delta\frac{\partial}{\partial t} + \frac{5}{54}\Delta^{(2)}\right] \\ & + \varepsilon^4\frac{125}{18}\left[\frac{\partial^3}{\partial t^3} - \frac{1}{3}\Delta\frac{\partial^2}{\partial t^2} + \frac{5}{216}\Delta^{(2)}\frac{\partial}{\partial t}\right] + \varepsilon^6\frac{625}{216}\left[\frac{\partial^4}{\partial t^4} - \frac{5}{18}\Delta\frac{\partial^3}{\partial t^3} + \frac{5}{72}\Delta^{(2)}\frac{\partial^2}{\partial t^2}\right] \\ & + \varepsilon^8\frac{3125}{1296}\left[\frac{\partial^5}{\partial t^5} - \frac{1}{6}\Delta^{(2)}\frac{\partial^4}{\partial t^4}\right] + \varepsilon^{10}\frac{3125}{7776}\frac{\partial^6}{\partial t^6}\Big\} f = 0, \end{aligned} \tag{3.4}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}, \quad \Delta^{(2)} = \frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^2\partial z^2}+\frac{\partial^4}{\partial x^2\partial z^2}.$$

□

Lemma 3.2. *The solution of equation* (3.4) *with the initial condition*

$$f(0,x,y,z) = u_0(0,x,y,z) + \varepsilon^2 u_1(0,x,y,z) + \varepsilon^4 u_2(0,x,y,z) + \cdots$$

has the asymptotic expansion

$$f(t,x,y,z) = u_0(t,x,y,z) + \varepsilon^2 u_1(t,x,y,z) + \varepsilon^4 u_2(t,x,y,z) + \cdots, \tag{3.5}$$

where the principal term $u_0(t,x,y,z)$ represents the solution of the equation

$$\frac{\partial}{\partial t}u_0(t,x,y,z) = \frac{5}{18}\Delta u_0(t,x,y,z).$$

Proof. To find the asymptotic expansion of the solution of (3.4), we use the method proposed in [13] and developed in [8]. In conformity with this method, the solution of (3.4) can be expanded into the series (3.5), where $\varepsilon > 0$ is small.

Substituting (3.5) into (3.4), we get the following equations for computing u_i, $i \geq 0$:

$$\begin{aligned}
\frac{\partial}{\partial t}u_0(t,x,y,z) &= \frac{5}{18}\Delta u_0(t,x,y,z),\\
\frac{\partial}{\partial t}u_1(t,x,y,z) &= \frac{5}{18}\Delta u_1(t,x,y,z)\\
&\quad + \varepsilon^2\frac{25}{6}\left[\frac{\partial^2}{\partial t^2} - \frac{1}{3}\Delta\frac{\partial}{\partial t} + \frac{5}{54}\Delta^{(2)}\right]u_0(t,x,y,z),\\
\frac{\partial}{\partial t}u_2(t,x,y,z) &= \frac{5}{18}\Delta u_2(t,x,y,z)\\
&\quad + \varepsilon^2\frac{25}{6}\left[\frac{\partial^2}{\partial t^2} - \frac{1}{3}\Delta\frac{\partial}{\partial t} + \frac{5}{54}\Delta^{(2)}\right]u_1(t,x,y,z)\\
&\quad + \frac{125}{18}\left[\frac{\partial^3}{\partial t^3} - \frac{1}{3}\Delta\frac{\partial^2}{\partial t^2} + \frac{5}{216}\Delta^{(2)}\frac{\partial}{\partial t}\right]u_0(t,x,y,z),\\
&\vdots\\
\frac{\partial}{\partial t}u_{m+5}(t,x,y,z) &= \frac{5}{18}\Delta u_{m+5}(t,x,y,z)\\
&\quad + \frac{25}{6}\left[\frac{\partial^2}{\partial t^2} - \frac{1}{3}\Delta\frac{\partial}{\partial t} + \frac{5}{54}\Delta^{(2)}\right]u_{m+4}(t,x,y,z)\\
&\quad + \frac{125}{18}\left[\frac{\partial^3}{\partial t^3} - \frac{1}{3}\Delta\frac{\partial^2}{\partial t^2} + \frac{5}{216}\Delta^{(2)}\frac{\partial}{\partial t}\right]u_{m+3}(t,x,y,z)\\
&\quad + \frac{625}{216}\left[\frac{\partial^4}{\partial t^4} - \frac{5}{18}\Delta\frac{\partial^3}{\partial t^3} + \frac{5}{72}\Delta^{(2)}\frac{\partial^2}{\partial t^2}\right]u_{m+2}(t,x,y,z)\\
&\quad + \frac{3125}{1296}\left[\frac{\partial^5}{\partial t^5} - \frac{1}{6}\Delta^{(2)}\frac{\partial^4}{\partial t^4}\right]u_{m+1}(t,x,y,z)\\
&\quad + \frac{3125}{7776}\frac{\partial^6}{\partial t^6}u_m(t,x,y,z) = 0,
\end{aligned}$$

for $m \geq 0$. □

Let us consider the function

$$\widetilde{f}_k^{(\varepsilon)}(t,x,y,z) = u_0(t,x,y,z) + \varepsilon^2 u_1(t,x,y,z) + \cdots + \varepsilon^{2k}u_k(t,x,y,z).$$

In [10], for the solution of a singularly perturbed equation of type (2.3), the remainder of the asymptotic expansion in the circuit of diffusion approximation was studied.

Taking into account that

$$f(t,x,y,z)=\sum_{i\in E} f_i(t,x,y,z),$$

it follows from the estimate of the remainder in [1], [8] that

$$\| f(t,x,y,z)-\widetilde{f}^{(\varepsilon)}(t,x,y,z)\| = O(\varepsilon^{2k}).$$

4. Conclusions

Singularly perturbed equations of the type (2.3) in the hydrodynamical limit (where $\frac{\lambda}{v^2}=O(1)$, $\lambda\downarrow 0$) have become the subject of a great deal of research [4], [8], [9], [10], [12] and others. By using the technique of professor A. F. Turbin [6], we reduce the singularly perturbed system of equations (2.3) to equation (2.4), which turns out to be regularly perturbed in the hydrodynamical limit to the diffusion process.

Therefore, in such cases we may simplify cumbersome calculations of terms of the asymptotic expansion for the solution of singulary perturbed equations for functionals of Markovian random motion.

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