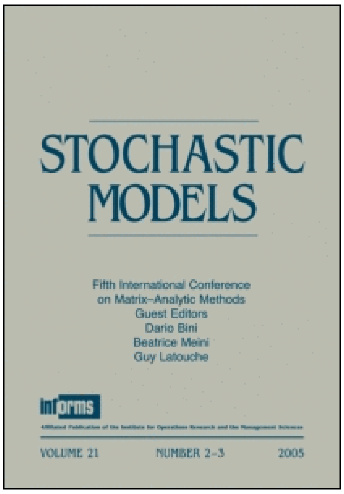

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Stationary Effectiveness of an Information Server with a Single Buffer and Bursty Demands of Two Different Customers

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STATIONARY EFFECTIVENESS OF AN INFORMATION SERVER WITH A SINGLE BUFFER AND BURSTY DEMANDS OF TWO DIFFERENT CUSTOMERS

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□ *In this article we study the stationary efficiency of a system consisting of a finite capacity buffer connected to two different customers with bursty on-off demands. We assume that the buffer is filled up at a constant rate. The dynamic of the overall system is modeled using a semi-Markov evolution environment and we derive design formulae involving the main parameters. It has been shown that it is possible to use the phase merging algorithm (PMA) to reduce the semi-Markov process to an approximated Markov process. We apply the PMA to the analysis of two different semi-Markov cases.*

Keywords Data servers; Information buffer; Markov operators; Phase-merging algorithm; Random evolutions; Semi-Markov processes.

Mathematics Subject Classification 60K20, 68M10.

1. INTRODUCTION

Random evolutions are the mathematical model of the evolutionary systems under the influence of random factors. In a general form, these models are described based on stochastic operator integral equations in a separable Banach space[7,8]. In this case the model of the evolutionary system is reduced to a first-order differential equation that determines the random evolution of the system. One of the parameters of this differential equation is a semi-Markov stochastic process which stands as the random media that influences the evolution of the system.

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As it is known, the theory of random evolutions was born after the application of probabilistic methods to the solution of some partial differential equations such as the heat and telegraph equations[4,5], after a generalization of the work of Kac regarding the motion on the real line. Then, the term was introduced by Reuben Hersh and Richard Griego being suggested by Peter Lax. The theory was mostly developed by authors such as Papanicolau, Hersh, Pinsky, Kertz, Watkins, and other important researchers. There are articles covering broad aspects of evolutions such as limit theorems[6,11], and diffusion processes and random motions[2,3]. The semi-Markov case has been considered by authors such as Swishchuk, Turbin, and Korolyuk. Of course this list is far from complete.

On the other hand, the problem of the availability of information for the supply of different customers normally involves a buffer. It is desirable to optimize the amount of stored information according to the expected customer needs and to the amount of incoming information (product) from the supply line.

In this article, we consider the case of an information server with a single buffer being filled at a constant rate while two different customers are connected to it. These customers demand a product in random alternating manner. We will assume that the alternating demands can be modeled by a semi-Markov stochastic process. We will reduce the semi-Markov process to a Markov process by lumping states according to the phase merging algorithm (PMA)[8,9,15]. As examples, we consider the m-Erlang and hyper-exponential probability distributions for the sojourn times.

The system functionality is as follows:

The customers switch from the active or "ON" state to the inactive or "OFF" state, and we consider that the switching process of the customers can be modeled as a semi-Markov process.

When active, one customer demands information at a rate f_0, whereas the other customer demands information at a rate f_1. When both customers are active, information is required at a rate $f_1 + f_0$. In each of these cases, if the buffer is empty ($v = 0$), an unproductive situation is considered. When no customer is active, then no product is required. The filling aggregate provides the buffer with a product at a constant rate F. This aggregate is active as long as the volume of information is below the maximum capacity of the buffer (V) (see Figure 1).

Let us denote $I(T)$ the amount of information delivered to customers S_1 and S_2, in a time interval $[0, T]$. Thus, we can define $K = \lim_{T\to\infty} \frac{I(T)}{T}$ as the steady state parameter for the system effectiveness (see Chapter 4 in Ref.[1]). Our main purpose in this work is to determine K as a function of the system parameters: the expected inactive sojourn time of each of the customers (consider $1/\lambda_0$ and $1/\lambda_1$), the expected active sojourn time ($1/\mu_0$ and $1/\mu_1$), the information demand (f_0 and f_1), and the incoming stream (F).

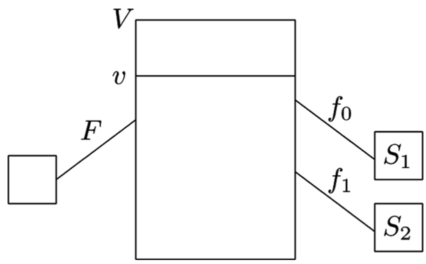

FIGURE 1 A system of two independent random state switching customers and one buffer filled up at a constant rate.

The dynamics of this linear system can be captured by a first-order differential equation having a random component, or the so-called random evolution process[7]. In Section 2, we elaborate our semi-Markov mathematical modeling. In Sections 3 and 4, we deal with the special Markov case. We obtain the stationary compound probability distribution for the buffer content level and the mathematical expression of the efficiency parameter in terms of the system values. Then, in Section 5, with the help of some plots we analyze some numerical results for different Markov and semi-Markov cases. In particular, we include the m-Erlang, exponential, and hyperexponential distributions for the active periods.

2. SEMI-MARKOV MATHEMATICAL MODEL

Consider the semi-Markov process $\{\chi(t)\}$, which is the superposition of two independent alternating semi-Markov processes with the phase space $\mathbb{Z} = \left\{(h, x^i) : h \in \mathbb{H}, x^i \in \mathbb{R}_+^{(2)}\right\}$, where $\mathbb{H} = \{h : h = (h_1, h_2), h_i = 0, 1;$ $i = 1, 2\}$, and $\mathbb{R}_+^{(2)} = \{\vec{x} : \vec{x} = (x, 0), x \geq 0\} \cup \left\{\vec{x} : \vec{x} = (0, x), x \geq 0\right\}$. We have defined h_i as

$$h_i = \begin{cases} 1, & \text{if } S_i \text{ is active;} \\ 0, & \text{if } S_i \text{ is not active,} \end{cases}$$

where S_i stands for subsystem i. The component x of the vector $(x, 0)$, respectively, $(0, x)$, is the residual life from the last state change of S_1 (respectively, S_2). The initial distribution of $\chi(t)$ is $P\{\chi(0) = (1, 1; 0, 0)\} = 1$.

Let us write this in more detail:

$(1, 1; 0, x)$	subsystem S_1 starts to be active and subsystem S_2 has been active for the time x,
$(1, 1; x, 0)$	subsystem S_2 starts to be active and subsystem S_1 has been active for the time x,
$(1, 0; 0, x)$	subsystem S_1 starts to be active and subsystem S_2 has been inactive for the time x,

$(1, 0; x, 0)$	subsystem S_2 starts to be inactive and subsystem S_1 has been active for the time x,
$(0, 1; 0, x)$	subsystem S_1 starts to be inactive and subsystem S_2 has been active for the time x,
$(0, 1; x, 0)$	subsystem S_2 starts to be active and subsystem S_1 has been inactive for the time x,
$(0, 0; 0, x)$	subsystem S_1 starts to be inactive and subsystem S_2 has been inactive for the time x,
$(0, 0; x, 0)$	subsystem S_2 starts to be inactive and subsystem S_1 has been inactive for the time x.

The embedded Markov chain of this semi-Markov process has the following transition probabilities[15]

$$\begin{aligned}
P[(h_1, h_2; 0, x), \{(\bar{h}_1, h_2; 0, u), u \le y\}] &= \frac{1}{\overline{F}^{(2)}_{h_2}(x)} \int_0^{y-x} \overline{F}^{(2)}_{h_2}(x+u)\mathrm{d}F^{(1)}_{h_1}(u),\\
P[(h_1, h_2; 0, x), \{(h_1, \bar{h}_2; u, 0), u \le y\}] &= \frac{1}{\overline{F}^{(2)}_{h_2}(x)} \int_x^{y+x} \overline{F}^{(1)}_{h_1}(u-x)\mathrm{d}F^{(2)}_{h_2}(u),\\
P[(h_1, h_2; x, 0), \{(\bar{h}_1, h_2; 0, u), u \le y\}] &= \frac{1}{\overline{F}^{(1)}_{h_1}(x)} \int_x^{y+x} \overline{F}^{(2)}_{h_2}(u-x)\mathrm{d}F^{(1)}_{h_1}(u),\\
P[(h_1, h_2; x, 0), \{(h_1, \bar{h}_2; u, 0), u \le y\}] &= \frac{1}{\overline{F}^{(1)}_{h_1}(x)} \int_0^{y-x} \overline{F}^{(1)}_{h_1}(u+x)\mathrm{d}F^{(2)}_{h_2}(u),
\end{aligned} \tag{1}$$

where $\bar{h}_i = 1 - h_i$, $\overline{F}(x) = 1 - F(x)$, and $F(x)$ is the cumulative distribution function.

The sojourn times corresponding to the stochastic process $\chi(t)$ with phase space $\mathbb{Z}$, have the following expected values

$$\begin{aligned}
m(h_1, h_2; x, 0) &= \frac{1}{\overline{F}^{(1)}_{h_1}(x)} \int_0^{\infty} \overline{F}^{(1)}_{h_1}(x+y)\overline{F}^{(2)}_{h_2}(y)\mathrm{d}y,\\
m(h_1, h_2; 0, x) &= \frac{1}{\overline{F}^{(2)}_{h_2}(x)} \int_0^{\infty} \overline{F}^{(1)}_{h_1}(y)\overline{F}^{(2)}_{h_2}(x+y)\mathrm{d}y.
\end{aligned}$$

Let $v(t)$ be the amount of information in the buffer at time t. It was shown in Rodriguez-Said et al.[14] that we can use the PMA to reduce the random evolution $v(t)$ in the semi-Markov medium $\chi(t)$ to the Markov evolution $\bar{v}(t)$ in the Markov medium $\bar{\chi}(t)$. Then we have the following transition probabilities of the embedded Markov chain:

$$P\{(h_1,h_2)(\bar{h}_1,h_2)\}$$
$$= \frac{\int_0^\infty\int_0^\infty \overline{F}^{(2)}_{h_2}(x+u)\mathrm{d}F^{(1)}_{\bar{h}_1}(u)\mathrm{d}x + \int_0^\infty\int_0^\infty \overline{F}^{(2)}_{h_2}(u)\mathrm{d}uF^{(1)}_{\bar{h}_1}(x+u)\mathrm{d}x}{\int_0^\infty \overline{F}^{(1)}_{h_1}(x)\mathrm{d}x + \int_0^\infty \overline{F}^{(1)}_{h_2}(x)\mathrm{d}x}, \tag{2}$$

$$P\{(h_1,h_2)(h_1,\bar{h}_2)\}$$
$$= \frac{\int_0^\infty\int_0^\infty \overline{F}^{(1)}_{h_1}(x+u)\mathrm{d}F^{(2)}_{\bar{h}_2}(u)\mathrm{d}x + \int_0^\infty\int_0^\infty \overline{F}^{(1)}_{h_1}(u)\mathrm{d}uF^{(2)}_{\bar{h}_2}(x+u)\mathrm{d}x}{\int_0^\infty \overline{F}^{(1)}_{h_1}(x)\mathrm{d}x + \int_0^\infty \overline{F}^{(1)}_{h_2}(x)\mathrm{d}x}, \tag{3}$$

where $\bar{h}_i = 1 - h_i$, $\overline{F}(x) = 1 - F(x)$, and $F(x)$ is the cumulative distribution function.

The mean sojourn times of the process $\bar{\chi}(t)$ in states from $\mathbb{X} = \{00, 01, 10, 11\}$ are given by

$$m(h_1,h_2) = \int_0^\infty \rho(h_1,h_2;x,0)m(h_1,h_2;x,0)\mathrm{d}x + \int_0^\infty \rho(h_1,h_2;0,x)m(h_1,h_2;0,x)\mathrm{d}x, \tag{4}$$

$$= c_{s0}\Big(\int_0^\infty\int_0^\infty \overline{F}^{(1)}_{h_1}(y)\overline{F}^{(2)}_{h_2}(x+y)\mathrm{d}y\,\mathrm{d}x + \int_0^\infty\int_0^\infty \overline{F}^{(1)}_{h_1}(x+y)\overline{F}^{(2)}_{h_2}(y)\mathrm{d}y\,\mathrm{d}x\Big), \tag{5}$$

where

$$c_{s0}^{-1}(h_1,h_2) = \int_0^\infty \Big(\overline{F}^{(1)}_{h_1}(x) + \overline{F}^{(2)}_{h_2}(x)\Big)\mathrm{d}x.$$

Then, by using the PMA, the random evolution $v(t)$ in the semi-Markov medium $\chi(t)$ can be reduced to the Markov evolution $\bar{v}(t)$ in the Markov medium $\bar{\chi}(t)$.

In order to simplify notation, we make the following correspondence: $00 \Leftrightarrow 0$, $01 \Leftrightarrow 1$, $10 \Leftrightarrow 2$, and $11 \Leftrightarrow 3$. Consider $\Theta = \{0,1,2,3\} = \mathbb{X}$.

Let us define the function $f(w)$, where $w \in \mathbb{W} = \Theta \times [0, V]$ as follows:

$$f(w) := \begin{cases} f_0, & \text{if } w = \{1, v\},\ 0 < v \le V; \\ f_1, & \text{if } w = \{2, v\},\ 0 < v \le V; \\ f_0 + f_1, & \text{if } w = \{3, v\},\ 0 < v \le V; \\ 0 & \text{in other cases.} \end{cases} \tag{6}$$

This is the productivity of the system.

Let us assume the joint stochastic process with a two-dimensional phase space $\xi(t) = (\bar{\chi}(t), \bar{v}(t))$. Then, we can state the following equality:

$$K = \lim_{T\to\infty} \frac{V(T)}{T} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(\xi(t))\mathrm{d}t. \tag{7}$$

It follows from ergodic theory[13] that if the process $\xi(t)$ has a stationary distribution $\rho(\cdot)$, then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\xi(t))\mathrm{d}t = \int_{\mathbb{W}} f(w)\mathrm{d}\rho(w). \tag{8}$$

Hence, by using equation (7) we obtain

$$K = \int_{\mathbb{W}} f(w)\mathrm{d}\rho(w) = \int_{\mathbb{W}} f(w)\rho(\mathrm{d}w). \tag{9}$$

In summary, by using the PMA, the random evolution $v(t)$ in the semi-Markov medium $\chi(t)$ can be reduced to the Markov evolution $\bar{v}(t)$ in the Markov medium $\bar{\chi}(t)$. So, as an example, we consider an evolution in a Markov medium.

3. MARKOV MATHEMATICAL MODEL

Let us introduce the following stochastic process $\{\bar{\chi}(t)\}$ where

$$\bar{\chi}(t) = \begin{cases} 0, & \text{if no customer is active;} \\ 1, & \text{if customer } S_1 \text{ is active;} \\ 2, & \text{if customer } S_2 \text{ is active;} \\ 3, & \text{if customers } S_1 \text{ and } S_2 \text{ are active.} \end{cases}$$

The stochastic process $\bar{\chi}(t)$ is a Markov process on the phase space (or states) $\Theta = \{0, 1, 2, 3\}$. Hence, the generating operator (or matrix) of $\bar{\chi}(t)$ can be written as[13]:

$$Q = q[P - I] = \begin{bmatrix} -(\lambda_0 + \lambda_1) & \lambda_0 & \lambda_1 & 0 \\ \mu_0 & -(\mu_0 + \lambda_1) & 0 & \lambda_1 \\ \mu_1 & 0 & -(\mu_1 + \lambda_0) & \lambda_0 \\ 0 & \mu_1 & \mu_0 & -(\mu_0 + \mu_1) \end{bmatrix},$$

where $q = [q_i\delta_{ij}; i, j \in \{0, 1, 2, 3\}]$ is a diagonal matrix of sojourn times intensities of different states and $q_0 = \lambda_1 + \lambda_0$, $q_1 = \lambda_1 + \mu_0$, $q_2 = \lambda_0 + \mu_1$, and

$q_3 = \mu_1 + \mu_0$. Here, as usual, the Kronecker's delta is defined as

$$\delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

We should notice that $q_\theta = \left(m(h_1, h_2)\right)^{-1}$, with the equivalence $(h_1 h_2) = \{00, 01, 10, 11\} \Leftrightarrow \{0, 1, 2, 3\} = \Theta \ni \theta$.

The elements of the P matrix are the transition probabilities of the Markov chain embedded in the Markov process $\bar{\chi}(t)$, i.e.,

$$P = \begin{bmatrix} 0 & \frac{\lambda_0}{\lambda_1+\lambda_0} & \frac{\lambda_1}{\lambda_1+\lambda_0} & 0 \\ \frac{\mu_0}{\lambda_1+\mu_0} & 0 & 0 & \frac{\lambda_1}{\lambda_1+\mu_0} \\ \frac{\mu_1}{\mu_1+\lambda_0} & 0 & 0 & \frac{\lambda_0}{\mu_1+\lambda_0} \\ 0 & \frac{\mu_1}{\mu_0+\mu_1} & \frac{\mu_0}{\mu_0+\mu_1} & \mu_0 \end{bmatrix}.$$

Consider a function $\overline{C}(w)$ on the space $\mathbb{W} = \{0, 1, 2, 3\} \times [0, V]$ defined as

$$\overline{C}(w) = \begin{cases} F & w = \{0, v\},\ 0 < v < V; \\ F - f_0 & w = \{1, v\},\ 0 < v < V; \\ F - f_1 & w = \{2, v\},\ 0 < v < V; \\ F - (f_0 + f_1) & w = \{3, v\},\ 0 < v < V; \\ 0 & \text{in other cases.} \end{cases} \tag{10}$$

Denote by $\bar{v}(t)$ the amount of information in the buffer at time t. It is easily verified that $\bar{v}(t)$ satisfies the following equation:

$$\frac{\mathrm{d}\bar{v}(t)}{\mathrm{d}t} = \overline{C}(\bar{\chi}(t), \bar{v}(t)), \tag{11}$$

with the initial condition $\bar{v}(0) = \bar{v}_0 \in [0, V]$. Equation (11) determines the random evolution of the system in the Markov medium $\bar{\chi}(t)$[12].

Assume now the joint stochastic process with a two-dimensional phase space $\xi = \left(\bar{\chi}(t), \bar{v}(t)\right)$. Then, the parameter K can be calculated from the stationary distribution ρ of the process $\xi(t)$, as it is shown in equations (7)–(9).

4. STATIONARY DISTRIBUTION

The sojourn time distribution functions, say $F_\theta(t)$, have the following form for the different states: $F_0(t) = 1 - e^{-(\lambda_1+\lambda_0)t}$, $F_1(t) = 1 - e^{-(\lambda_1+\mu_0)t}$,

$F_2(t) = 1 - e^{-(\mu_1+\lambda_0)t}$, and $F_3(t) = 1 - e^{-(\mu_1+\mu_0)t}$. Now, denote as $f_\theta(t) = \frac{dF_\theta(t)}{dt}$ and $r_\theta = \frac{f_\theta(t)}{1-F_\theta(t)}$ for all $\theta \in \Theta$, i.e., $r_0 = \lambda_1 + \lambda_0$, $r_1 = \lambda_1 + \mu_0$, $r_2 = \mu_1 + \lambda_0$, and $r_3 = \mu_1 + \mu_0$. Then, the two component process $\xi(t) = \big(\bar{\chi}(t), \bar{v}(t)\big)$ is a Markov process with the generator[7,12]:

$$A\varphi(\theta, \bar{v}) = \overline{C}(\theta, \bar{v})\frac{\partial}{\partial \bar{v}}\varphi(\theta, \bar{v}) + r_\theta[P\varphi(\theta, \bar{v}) - \varphi(\theta, \bar{v})],$$

where $P\varphi(\theta, \bar{v}) = \sum_{y\in\Theta} p_{\theta y}\varphi(y, \bar{v})$, or equivalently,

$$A\varphi(\theta, \bar{v}) = \overline{C}(\theta, \bar{v})\frac{\partial}{\partial \bar{v}}\varphi(\theta, \bar{v}) + Q\varphi(\theta, \bar{v}),$$

where $Q = r[P - I]$.

Denote as ρ the stationary distribution of the process $\xi(t)$. Then, for every function $\varphi(\cdot)$ belonging to the domain of the operator A, we have

$$\int_{\mathbb{W}} A\varphi(z)\rho(dz) = 0. \tag{12}$$

The analysis of the process $\xi(t)$ properties leads up to the conclusion that, for the case $\max(f_0, f_1) < F < f_1 + f_0$, the stationary distribution ρ has atoms at points $(3, 0)$, $(0, V)$, $(1, V)$, and $(2, V)$. We denote them as $\rho[3, 0]$, $\rho[0, V]$, $\rho[1, V]$, and $\rho[2, V]$. We denote the continuous part of ρ as $\rho(\theta, v)$.

Let us write equation (12) in more detail for the case, $\max(f_0, f_1) < F < f_1 + f_0$ as follows:

$$\begin{aligned}
&\int_{\mathbb{W}} A\varphi(z)\rho(dz) \\
&\quad = \int_{0+}^{V-} \Bigg\{\bigg[F\frac{\partial}{\partial v}\varphi(0, v) - (\lambda_0 + \lambda_1)\varphi(0, v) + \lambda_0\varphi(1, v) + \lambda_1\varphi(2, v)\bigg]\rho(0, v) \\
&\qquad + \bigg[(F - f_0)\frac{\partial}{\partial v}\varphi(1, v) + \mu_0\varphi(0, v) - (\mu_0 + \lambda_1)\varphi(1, v) + \lambda_1\varphi(3, v)\bigg]\rho(1, v) \\
&\qquad + \bigg[(F - f_1)\frac{\partial}{\partial v}\varphi(2, v) + \mu_1\varphi(0, v) - (\mu_1 + \lambda_0)\varphi(2, v) + \lambda_0\varphi(3, v)\bigg]\rho(2, v) \\
&\qquad + \bigg[(F - f_0 - f_1)\frac{\partial}{\partial v}\varphi(3, v) + \mu_1\varphi(1, v) + \mu_0\varphi(2, v) - (\mu_0 + \mu_1)\varphi(3, v)\bigg] \\
&\qquad\qquad \times \rho(3, v)\Bigg\}\, dv \\
&\qquad + \big[-(\lambda_0 + \lambda_1)\varphi(0, V) + \lambda_0\varphi(1, V) + \lambda_1\varphi(2, V)\big]\rho[0, V] \\
&\qquad + \big[\mu_0\varphi(0, V) - (\mu_0 + \lambda_1)\varphi(1, V) + \lambda_1\varphi(3, V)\big]\rho[1, V]
\end{aligned}$$

$$+\left[\mu_1\varphi(0,V)-(\mu_1+\lambda_0)\varphi(2,V)+\lambda_0\varphi(3,V)\right]\rho[2,V]$$
$$+\left[\mu_1\varphi(1,0)+\mu_0\varphi(2,0)-(\mu_0+\mu_1)\varphi(3,0)\right]\rho[3,0]=0 \tag{13}$$

Let A^* be the conjugate or adjoint operator of A. Then, by changing the order of integration in equation (13), we can obtain the following expressions for the continuous part of $A^*\rho$:

$$\begin{cases} -F\dfrac{\partial}{\partial v}\rho(0,v)-(\lambda_0+\lambda_1)\rho(0,v)+\mu_0\rho(1,v)+\mu_1\rho(2,v)=0 \\ -(F-f_0)\dfrac{\partial}{\partial v}\rho(1,v)+\lambda_0\rho(0,v)-(\mu_0+\lambda_1)\rho(1,v)+\mu_1\rho(3,v)=0 \\ -(F-f_1)\dfrac{\partial}{\partial v}\rho(2,v)+\lambda_1\rho(0,v)-(\mu_1+\lambda_0)\rho(2,v)+\mu_0\rho(3,v)=0 \\ -(F-f_0-f_1)\dfrac{\partial}{\partial v}\rho(3,v)+\lambda_1\rho(1,v)+\lambda_0\rho(2,v)-(\mu_0+\mu_1)\rho(3,v)=0 \end{cases}. \tag{14}$$

The expressions for the atoms for the case $\max(f_1, f_0) < F < f_1 + f_0$ are given by

$$\begin{cases} -F\rho(0,0+)=0 \\ -(F-f_0)\rho(1,0+)+\mu_1\rho[3,0]=0 \\ -(F-f_1)\rho(2,0+)+\mu_0\rho[3,0]=0 \\ -(F-f_0-f_1)\rho(3,0+)-(\mu_0+\mu_1)\rho[3,0]=0 \end{cases} \tag{15}$$

and

$$\begin{cases} F\rho(0,V-)-(\lambda_0+\lambda_1)\rho[0,V]+\mu_0\rho[1,V]+\mu_1\rho[2,V]=0 \\ (F-f_0)\rho(1,V-)+\lambda_0\rho[0,V]-(\mu_0+\lambda_1)\rho[1,V]=0 \\ (F-f_1)\rho(2,V-)+\lambda_1\rho[0,V]-(\mu_1+\lambda_0)\rho[2,V]=0 \\ (F-f_0-f_1)\rho(3,V-)+\lambda_1\rho[1,V]+\lambda_0\rho[2,V]=0 \end{cases}. \tag{16}$$

In these equations we have defined the notation

$$\rho(\theta,0+):=\lim_{v\downarrow 0}\rho(\theta,v)$$

and

$$\rho(\theta,V-):=\lim_{v\uparrow V}\rho(\theta,v).$$

It follows from equation (14) that

$$F\rho(0,v) + (F - f_0)\rho(1,v) + (F - f_1)\rho(2,v) \\ + (F - f_0 - f_1)\rho(3,v) = c = \text{constant}. \tag{17}$$

The constant c can be proved to be equal to 0 from equations (15) and (16).

Using equations (14)–(17) we can solve for the complete expressions of the continuous part of the stationary distribution $\rho(\theta, v)$ and atoms $\rho[3,0]$, $\rho[0,V]$, $\rho[1,V]$, and $\rho[2,V]$ obtaining the following:

$$\begin{aligned}
\rho(0,v) &= c_1 e^{\delta_1 v} + c_2 e^{\delta_2 v} + c_3 e^{\delta_3 v},\\
\rho(1,v) &= k_{11} c_1 e^{\delta_1 v} + k_{12} c_2 e^{\delta_2 v} + k_{13} c_3 e^{\delta_3 v},\\
\rho(2,v) &= k_{21} c_1 e^{\delta_1 v} + k_{22} c_2 e^{\delta_2 v} + k_{23} c_3 e^{\delta_3 v},\\
\rho(3,v) &= k_{31} c_1 e^{\delta_1 v} + k_{32} c_2 e^{\delta_2 v} + k_{33} c_3 e^{\delta_3 v},\\
\rho[0,V] &= K_{00}\rho(0,V-) + K_{01}\rho(1,V-) + K_{02}\rho(2,V-),\\
\rho[1,V] &= K_{10}\rho(0,V-) + K_{11}\rho(1,V-) + K_{12}\rho(2,V-),\\
\rho[2,V] &= K_{20}\rho(0,V-) + K_{21}\rho(1,V-) + K_{22}\rho(2,V-),\\
\rho[3,0] &= K_{30}\rho(3,0+).
\end{aligned}$$

The constants $k_{ij},\ i,j = 1,2,3$ and $K_{ij},\ i,j = 0,1,2$ are known as well as c_n, $n = 1,2,3$ and K_{30}. See the Appendix.

5. NUMERICAL RESULTS AND STATIONARY EFFICIENCY

With the expression of c_1, it is now possible to evaluate the complete expression of the stationary distribution. For example, on the case $f_0 = 3/2$, $f_1 = 1$, $\lambda_0 = 3/10$, $\lambda_1 = 2/10$, $\mu_0 = 1/10$, $\mu_1 = 1/15$, $V = 100$, $F = 7/4$ ($\max(f_0, f_1) < F < f_0 + f_1$) we obtain from equation (27):

$$\rho(0,v) = \frac{\sqrt{2}}{5}\frac{-2 + e^{16(-4+\sqrt{2})v/105} + e^{-16(4+\sqrt{2})v/105}}{-676\sqrt{2} + (2\sqrt{2}-1)e^{-(320\sqrt{2}+1280)/21} + (2\sqrt{2}+1)e^{(320\sqrt{2}-1280)/21}}.$$

We can define the following stationary distribution:

$$\rho(v) = \begin{cases} \rho(0,v) + \rho(1,v) + \rho(2,v), & 0 < v < V \\ \rho[3,0] & v = 0 \\ \rho[0,V] + \rho[1,V] + \rho[2,V] & v = V \end{cases}.$$

We can plot both this analytical result and simulation results to illustrate some common cases (see Figure 2). On these plots $F = 7/4$, $F = 2$ are considered, and

$$F = \frac{f_0\lambda_0}{\lambda_0 + \mu_0} + \frac{f_1\lambda_1}{\lambda_1 + \mu_1} = \frac{15}{8}, \tag{18}$$

i.e., when F is equal to the expected average demand of the two customers.

In Figure 2 we can see three (solid) lines that correspond to the stationary probability density of the level of the buffer for three different cases. The case $F = 7/4$ is an example where the incoming stream is less than the expected average demand of the customers. One can see from the figure that this case is the one with the higher probability of finding the buffer empty or nearly empty. The case $F = 2$ is an example where the incoming stream is greater than the expected average demand of the customers. One can see from the figure that this case is the one with the higher probability of finding the buffer full or nearly full. The case $F = 15/8$ is an example where the incoming information stream is equal to the expected average demand of the customers. This case has some balance tendency and is the one that uses more uniformly the whole dynamical range of the buffer. We will see that only within the two latter cases is it possible to reach the maximum buffer efficiency which is equal to the expected average demand of the customers.

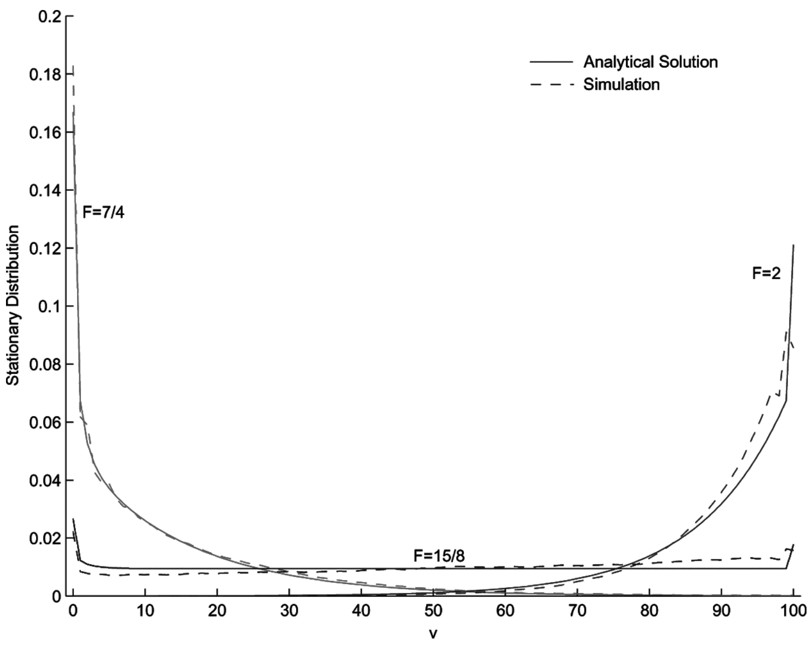

FIGURE 2 Stationary distribution of the buffer for the case $\max(f_1, f_2) < F < f_1 + f_2$ for three different values of F.

The simulation model used here is based on discrete-event simulation. We generate the sojourn time random variable for every state and customer. The inverse distribution approach is used to achieve this goal from a generated uniform distribution. At the beginning of the simulation and when a customer changes state, the sojourn time is generated according to the distribution of the actual state to establish the epoch of the next transition. As the simulation evolves, the index that represents the level of the buffer is refreshed according to the change that takes place regarding the state of the customers (see equation (10)). The position of that index within 100 intervals inside $[0, V]$ is kept to obtain the relative frequency of the localization of the index, i.e., the level of the buffer. The simulation runs for 1×10^8 time units and we typically have more than 1×10^7 state changes for each customer, although it shows convergence at 1×10^7 time units.

Some sort of perturbation close to $v = 100$ can be noticed in the curves from the simulation in Figure 2. That perturbation comes from the system functionality for the case $\max(f_0, f_1) < F < f_0 + f_1$, i.e., if any of the two customers are active, the level of the buffer can be increased. However, the filling rate is expected to stop when the level of the buffer reaches its maximum. The result is that, if any of the two customers are active, the level of the buffer can be increased to its maximum; then the filling rate is turned off and the level starts to decrease. At any moment that the level is sensed not to be at its maximum again, then the filling rate is restored. On that scenario, for the time that this single customer is active, the level in the buffer swings between its maximum and some close point below. This is the reason why some small peaks can be observed in the computer simulation at some point close to V.

For the sake of the analytical solution, the atoms $\rho[1, V]$ and $\rho[2, V]$ were considered as a more steady approximation of the real system behavior. It is worth saying that this behavior is not observed for other choices of F. For example, if we choose $F < \min(f_1, f_0)$, we obtain the exact analytical solution.

Now we can recall the function $f(w)$ and show some plots regarding the efficiency parameter K from equation (9). Let us write this expression in more detail for this system with $\max(f_0, f_1) < F < f_0 + f_1$:

$$\begin{aligned}
K &= \int_0^V \{f_0\rho(1, v) + f_1\rho(2, v) + (f_0 + f_1)\rho(3, v)\}\mathrm{d}v + f_0\rho[1, V] + f_1\rho[2, V] \\
&= c_1\left(\frac{e^{\delta_1 V} - 1}{\delta_1}\right)(f_0 + f_1K_{11} + f_2K_{21} + f_3K_{31}) \\
&\quad + c_2\left(\frac{e^{\delta_2 V} - 1}{\delta_2}\right)(f_0 + f_1K_{11} + f_2K_{21} + f_3K_{31}) \\
&\quad + c_3\left(\frac{e^{\delta_3 V} - 1}{\delta_3}\right)(f_0 + f_1K_{11} + f_2K_{21} + f_3K_{31}).
\end{aligned}$$

$K(V)$ is shown in Figure 3, i.e., K as a function of the maximum capacity of the buffer, for the same three cases of F from Figure 2.

Once again, the case $F = 15/8$ stands as an example where the incoming information stream is equal to the expected average demand of the customers. It is possible to see that this case is capable in reaching the maximum system efficiency (equal to 15/8) as the buffer grows in capacity V. The case of $F = 7/4$, where the incoming stream is less that the expected average demand of the customers, never reaches the maximum efficiency no matter what the size of the buffer is. The case of $F = 2$, where the incoming stream is greater that the expected demand of the customers, quickly reaches the maximum efficiency.

It can be proved that in every case

$$\lim_{V\to\infty}[K(V)]_{F\geq\frac{f_1\lambda_1}{\lambda_1+\mu_1}+\frac{f_0\lambda_0}{\lambda_0+\mu_0}} = \frac{f_1\lambda_1}{\lambda_1+\mu_1} + \frac{f_0\lambda_0}{\lambda_0+\mu_0}. \tag{19}$$

In addition, it can be proven that in every case

$$\lim_{V\to\infty}[K(V)]_{F<\frac{f_1\lambda_1}{\lambda_1+\mu_1}+\frac{f_0\lambda_0}{\lambda_0+\mu_0}} < F < \frac{f_1\lambda_1}{\lambda_1+\mu_1} + \frac{f_0\lambda_0}{\lambda_0+\mu_0}. \tag{20}$$

That is, if the buffer is big enough, no F larger than the average system demand is required to meet the system maximum efficiency. On the other

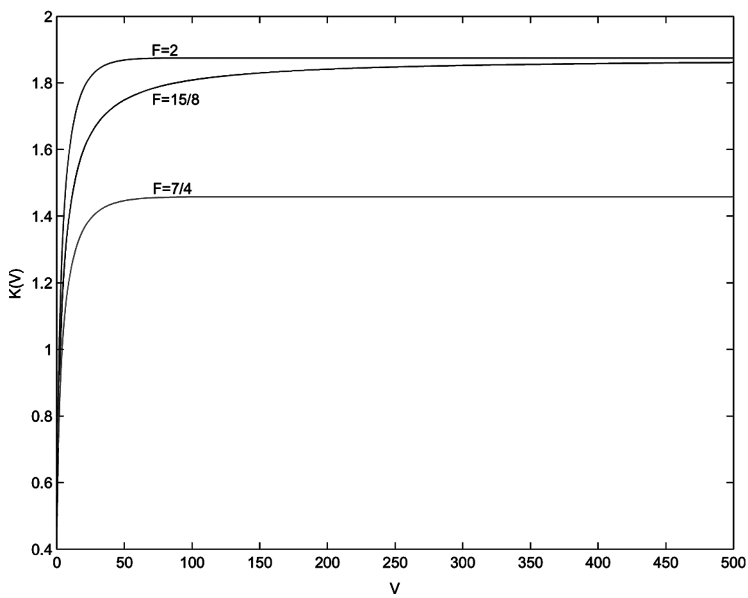

FIGURE 3 Efficiency parameter as a function of the buffer capacity for the case $\max(f_0, f_1) < F < f_1 + f_2$ and three different values of F.

hand, if the incoming stream F is smaller than the expected long-term average system demand, the system efficiency $K(V)$ is even smaller than the incoming stream.

These results are the same even for other choices of F besides $\max(f_0, f_1) < F < f_0 + f_1$. For example, if we choose $F < \min(f_0, f_1)$, we obtain the same results as those in (19) and (20).

According to Section 2, the phase merging algorithm can be used to obtain a random evolution $\bar{v}(t)$ in an approximated Markov environment from an evolution $v(t)$ in a semi-Markov environment. Then, some plots regarding the semi-Markov case can be displayed. As a first example, we can consider a m-Erlang scenario, where both active and inactive sojourn times are considered with such distribution. This is for the actual sojourn time distributions we have:

$$F_0^1(u) = \int_0^u \frac{\lambda_0 e^{-\lambda_0 x}(\lambda_0 x)^{m_0^{(1)}-1}}{(m_0^{(1)}-1)!}\,\mathrm{d}x,$$

$$F_1^1(u) = \int_0^u \frac{\mu_0 e^{-\mu_0 x}(\mu_0 x)^{m_1^{(1)}-1}}{(m_1^{(1)}-1)!}\,\mathrm{d}x,$$

$$F_0^2(u) = \int_0^u \frac{\lambda_1 e^{-\lambda_1 x}(\lambda_1 x)^{m_0^{(2)}-1}}{(m_0^{(2)}-1)!}\,\mathrm{d}x,$$

$$F_1^2(u) = \int_0^u \frac{\mu_1 e^{-\mu_1 x}(\mu_1 x)^{m_1^{(2)}-1}}{(m_1^{(2)}-1)!}\,\mathrm{d}x.$$

Here, $m_j^{(i)}$ $(i \in \{1,2\}, j \in \{0,1\})$ stands for the number of exponentials that form the m-Erlang distribution of the i subsystem in the j state. As we know, if we make $m_j^{(i)} = 1$, we get an exponential distribution.

We use equations (2) and (3) to calculate the transition probabilities of the embedded Markov chain. We obtain

$$P = \begin{bmatrix} 0 & \frac{m_0^{(2)}\lambda_0}{m_0^{(2)}\lambda_0 + m_0^{(1)}\lambda_1} & \frac{m_0^{(1)}\lambda_1}{m_0^{(2)}\lambda_0 + m_0^{(1)}\lambda_1} & 0 \\ \frac{m_0^{(2)}\mu_0}{m_1^{(1)}\lambda_1 + m_0^{(2)}\mu_0} & 0 & 0 & \frac{m_1^{(1)}\lambda_1}{m_1^{(1)}\lambda_1 + m_0^{(2)}\mu_0} \\ \frac{m_0^{(1)}\mu_1}{m_1^{(2)}\lambda_0 + m_0^{(1)}\mu_1} & 0 & 0 & \frac{m_1^{(2)}\lambda_0}{m_1^{(2)}\lambda_0 + m_0^{(1)}\mu_1} \\ 0 & \frac{m_1^{(1)}\mu_1}{m_1^{(2)}\mu_0 + m_1^{(1)}\mu_1} & \frac{m_1^{(2)}\mu_0}{m_1^{(2)}\mu_0 + m_1^{(1)}\mu_1} & 0 \end{bmatrix}.$$

After that, we use equation (5) to calculate the mean sojourn times and, consequently, the sojourn time intensities. Let us remember that $q_\theta = (m(h_1, h_2))^{-1}$ with the equivalence $(h_1, h_2) = \{00, 01, 10, 11\} \Leftrightarrow \{0, 1, 2, 3\} =$

$\Theta \ni \theta$. Also let us remember that $q = [q_i\delta_{ij}; i, j \in \{0, 1, 2, 3\}]$ is a diagonal matrix. Then we obtain

$$q = \begin{bmatrix} \frac{\lambda_0}{m_0^{(1)}} + \frac{\lambda_1}{m_0^{(2)}} & 0 & 0 & 0 \\ 0 & \frac{\lambda_1}{m_0^{(2)}} + \frac{\mu_0}{m_1^{(1)}} & 0 & 0 \\ 0 & 0 & \frac{\lambda_0}{m_0^{(1)}} + \frac{\mu_1}{m_1^{(2)}} & 0 \\ 0 & 0 & 0 & \frac{\mu_0}{m_1^{(1)}} + \frac{\mu_1}{m_1^{(2)}} \end{bmatrix}.$$

Now we can calculate the generating operator $Q = q[P - I]$ and solve for the continuous part and atoms of a stationary distribution just as we did in Section 3 for an evolution in a Markov media.

We can consider the inactive sojourn time with an exponential distribution $\left(m_0^{(i)} = 1, i = 1, 2\right)$ and plot several m-Erlang cases for the active sojourn time. This example is illustrated in Figures 4 and 5.

We considered as before

$$F = \frac{f_0\lambda_0}{\lambda_0 + \mu_0} + \frac{f_1\lambda_1}{\lambda_1 + \mu_1}, \tag{21}$$

$$\max(f_0, f_1) < F < f_0 + f_1\ . \tag{22}$$

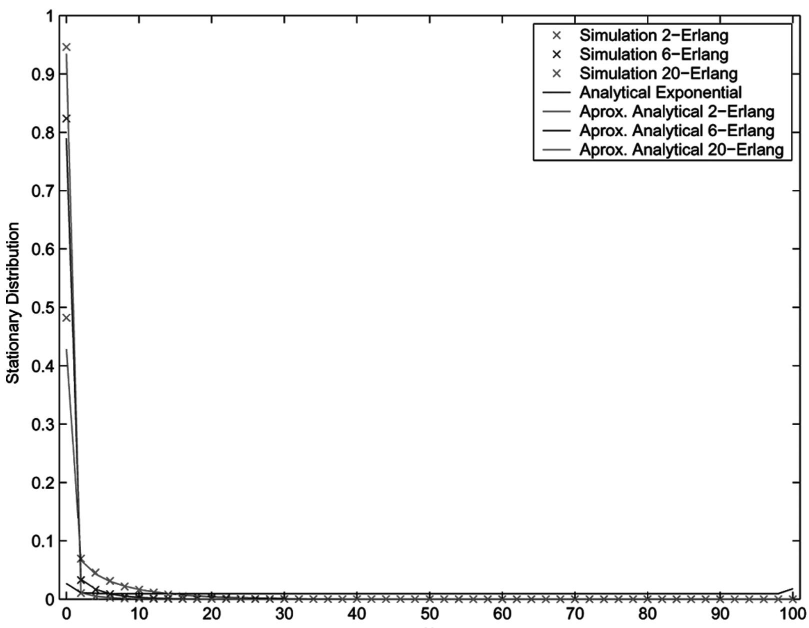

FIGURE 4 Stationary distribution of the buffer with m-Erlang distributed active sojourn time and exponentially distributed inactive sojourn time.

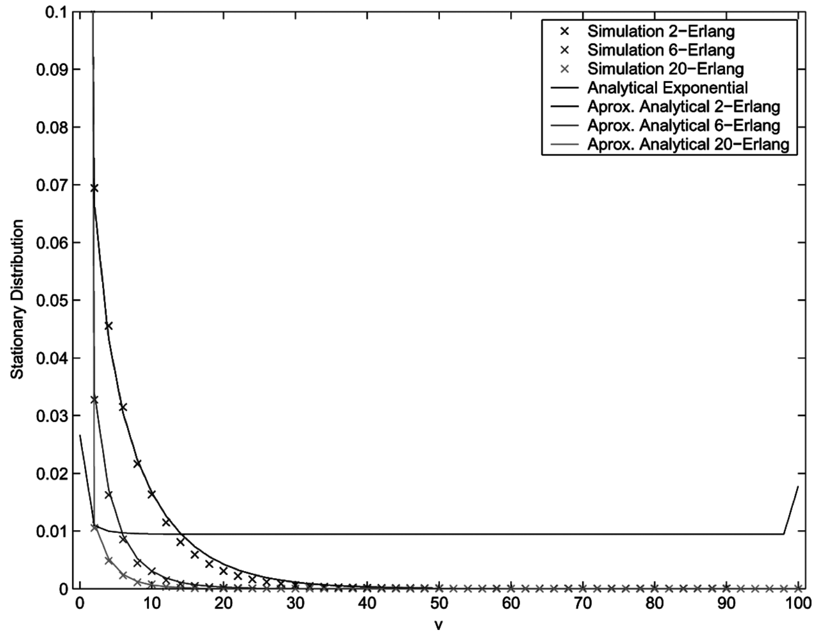

FIGURE 5 A closer view of the stationary distribution of the buffer with m-Erlang distributed active sojourn time and exponentially distributed inactive sojourn time.

We also used $f_0 = 3/2$, $f_1 = 1$, $\lambda_0 = 3/10$, $\lambda_1 = 2/10$, $\mu_0 = 1/10$, $\mu_1 = 1/15$, and $V = 100$.

In Figures 4 and 5 we can see a good match between the analytical solutions and simulations for every m-Erlang case. In Figure 4 one can see the complete stationary distributions including the atoms of the distribution at $v = 0$. Figure 5 is a close view of the low part of these distributions.

In this case we see that the behavior of the curves is modified as the order of the m-Erlang distributions is increased. For active sojourn times with higher order m-Erlang distribution, the expected values of these sojourn times for subsystems S_1 and S_2 are longer. Therefore we can see in Figures 4 and 5 that this causes the stationary distribution to be biased to the empty side of the buffer.

Finally, we can introduce the hyperexponential semi-Markov case as another example. For this case, the distribution of the active and inactive sojourn times were taken as

$$
\begin{aligned}
F_0^1(u) &= 1 - p\exp(-\lambda_0 u) - (1-p)\exp(-\lambda_{0b} u),\\
F_1^1(u) &= 1 - p\exp(-\mu_0 u) - (1-p)\exp(-\mu_{0b} u),\\
F_0^2(u) &= 1 - p\exp(-\lambda_1 u) - (1-p)\exp(-\lambda_{1b} u),\\
F_1^2(u) &= 1 - p\exp(-\mu_1 u) - (1-p)\exp(-\mu_{1b} u).
\end{aligned}
$$

The following choices were taken as an instance: $\lambda_{0b} = n\lambda_0$, $\lambda_{1b} = n\lambda_1$, $\mu_{0b} = n\mu_0$, and $\mu_{1b} = n\mu_1$, $n > 0$. We use equations (2) and (3) to calculate the transition probabilities of the embedded Markov chain. We obtain

$$P = \begin{bmatrix} 0 & \frac{\lambda_0}{\lambda_1+\lambda_0} & \frac{\lambda_1}{\lambda_1+\lambda_0} & 0 \\ \frac{\mu_0}{\lambda_1+\mu_0} & 0 & 0 & \frac{\lambda_1}{\lambda_1+\mu_0} \\ \frac{\mu_1}{\mu_1+\lambda_0} & 0 & 0 & \frac{\lambda_0}{\mu_1+\lambda_0} \\ 0 & \frac{\mu_1}{\mu_0+\mu_1} & \frac{\mu_0}{\mu_0+\mu_1} & \mu_0 \end{bmatrix}.$$

This is the same transition probability matrix of the Markov case that does not depend on the choice of n.

After that we use equation (5) to calculate the mean sojourn times and, consequently, the sojourn time intensities. The result is

$$q = \begin{bmatrix} \frac{n(\lambda_0+\lambda_1)}{np+1-p} & 0 & 0 & 0 \\ 0 & \frac{n(\lambda_1+\mu_0)}{np+1-p} & 0 & 0 \\ 0 & 0 & \frac{n(\lambda_0+\mu_1)}{np+1-p} & 0 \\ 0 & 0 & 0 & \frac{n(\mu_0+\mu_1)}{np+1-p} \end{bmatrix}.$$

Now we can calculate the generating operator $Q = q[P - I]$ and solve for the continuous part and atoms of the stationary distribution just as we did in Section 3 for the evolution in a Markov media.

We can make some plots for this semi-Markov example. In Figure 6, the behavior of the approximation can be appreciated along with some plots from simulations for this semi-Markov case. In Figure 6, we choose $n = 2$ as an instance. Besides, we took again $f_0 = 3/2$, $f_1 = 1$, $\lambda_0 = 3/10$, $\lambda_1 = 2/10$, $\mu_0 = 1/10$, $\mu_1 = 1/15$, and $V = 100$, as well as F from conditions (21) and (22).

It can be noticed that we can use the stationary probability density obtained before for the Markov case to obtain the approximated stationary density for these semi-Markov cases. Meaning that for the m-Erlang case we may only use the substitutions

$$\lambda_0 \to \frac{\lambda_0}{m_0^{(1)}}, \quad \lambda_1 \to \frac{\lambda_1}{m_0^{(2)}}, \quad \mu_0 \to \frac{\mu_0}{m_1^{(1)}}, \quad \text{and} \quad \mu_1 \to \frac{\mu_1}{m_1^{(2)}}$$

directly in equations (27)–(30) as well as in the expressions for the atoms to obtain the approximated stationary density for this semi-Markov case. For

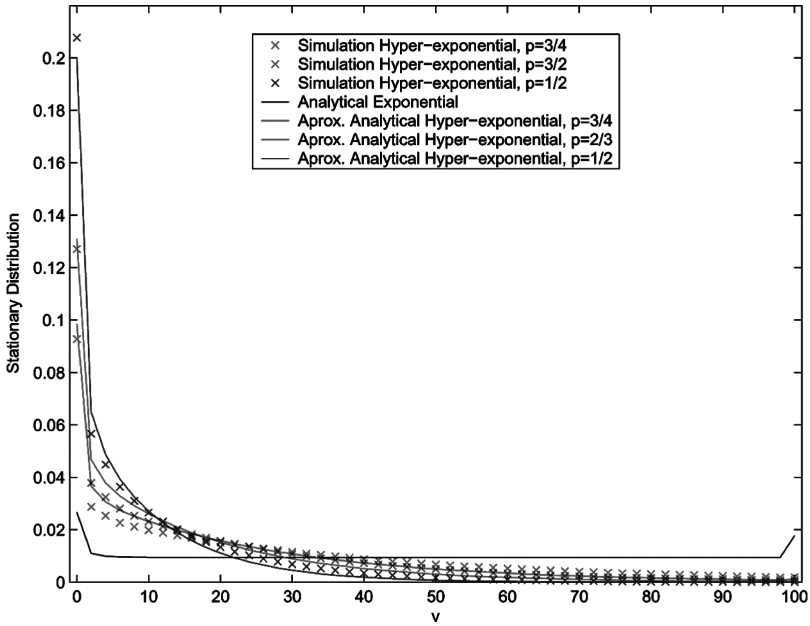

FIGURE 6 Stationary distribution of the hyperexponential semi-Markov case for the active and inactive sojourn time.

the hyperexponential case, we should use the substitutions

$$\lambda_0 \to \frac{\lambda_0}{np+1-p}, \quad \lambda_1 \to \frac{\lambda_1}{np+1-p}, \quad \mu_0 \to \frac{\mu_0}{np+1-p} \quad \text{and}$$
$$\mu_1 \to \frac{\mu_1}{np+1-p}.$$

Also, we can use the same substitutions in the expression (18) to obtain the condition that leads to the best usage of the buffer in terms of the stationary efficiency of the system.

6. CONCLUSIONS

It is possible to use the PMA to reduce a semi-Markov process to an approximated Markov process. Once this is done, it is possible to find some closed-form expression for the stationary distribution of the system. It has been seen that the approximation that the algorithm gives may be good enough for some applications. We showed plots of some analytical results and computer simulations regarding the Markov and semi-Markov cases.

Also, it has been seen that the approximation can also be considered to obtain expressions for the stationary efficiency of the system for some semi-Markov cases. Besides the driving function $C(w)$ given in equation (10) typical in the random evolution formulation[10], we introduce an additional function $f(w)$ in equation (6) to capture the functionality of our scheme. Two cases were studied regarding the incoming stream F in terms of the stationary efficiency K and we showed some typical cases for this parameter. Some graphics of the performance parameter K were added and we analyzed an optimization condition which is related to the long-term average demand of the customers.

Our buffer is a subsystem aimed at increasing the availability of information. It is worth mentioning that all the results shown here also match those of a system with an overflowed buffer. This is a system where the incoming information stream is not turned off when the buffer reaches its maximum capacity and then the incoming information overflows the buffer.

APPENDIX

Here, we find the stationary distribution of the system and atoms using equations (14)–(17).

Using equation (14) we obtain

$$\frac{\partial^3}{\partial v^3}\rho(0,v) = \frac{A_0}{D_0}\frac{\partial^2}{\partial v^2}\rho(0,v) + \frac{B_0}{D_0}\frac{\partial}{\partial v}\rho(0,v) + \frac{G_0}{D_0}\rho(0,v), \tag{23}$$

$$\rho(1,v) = \frac{A_1}{D_1}\frac{\partial^2}{\partial v^2}\rho(0,v) + \frac{B_1}{D_1}\frac{\partial}{\partial v}\rho(0,v) + \frac{G_1}{D_1}\rho(0,v), \tag{24}$$

$$\rho(2,v) = \frac{A_2}{D_2}\frac{\partial}{\partial v}\rho(0,v) + \frac{B_2}{D_2}\rho(0,v) + \frac{G_2}{D_2}\rho(1,v), \tag{25}$$

$$\rho(3,v) = \frac{A_3}{D_3}\rho(0,v) + \frac{B_3}{D_3}\rho(1,v) + \frac{G_3}{D_3}\rho(2,v), \tag{26}$$

where

$$\begin{aligned}
A_0 = F^3\lambda_1 + (F-f_0)\big(F^2\{2(\lambda_0+\mu_0+\mu_1)+\lambda_1\} - F\{f_0(2\lambda_0+\lambda_1+\mu_1) \\
+ f_1[2(\lambda_1+\mu_0+\mu_1)+3\lambda_0]\} + \lambda_0 f_1^2 + \lambda_0 f_0 f_1\big) \\
- (F-f_1)\big(F[\lambda_1 f_0 + f_1(2\lambda_1+\mu_0)] - \lambda_1 f_0 f_1\big) - \lambda_1 f_1 f_0^2,
\end{aligned}$$

$$\begin{aligned}
B_0 = -F\big(\lambda_0\mu_0 f_0 + f_1[\lambda_1(\lambda_0+\mu_1)+\mu_0\mu_1]\big) + (F-f_0)\big(F\{\mu_1[2(\lambda_1+\mu_0) \\
+ 3\lambda_0+\mu_1] + 3\lambda_0\lambda_1\} - \lambda_0 f_0(\lambda_0+\lambda_1+\mu_1) - f_1(2\lambda_0(\lambda_1+\mu_1)+\lambda_1\mu_1)\big) \\
+ (F-f_1)\big(F\big[\lambda_1^2 + (\lambda_0+\mu_0)^2 + \mu_0(3\lambda_1+\mu_1)\big] - f_0[\lambda_0(\lambda_0+\mu_0) \\
+ \lambda_1(\lambda_1+2\mu_0)] - \lambda_1 f_1(\lambda_0+\lambda_1+\mu_0)\big),
\end{aligned}$$

$$G_0 = (\lambda_0 + \lambda_1 + \mu_0 + \mu_1)(\lambda_1 + \mu_1)(\lambda_0 + \mu_0)\left(F - \frac{\lambda_1 f_1}{\mu_1 + \lambda_1} - \frac{\lambda_0 f_0}{\lambda_0 + \mu_0}\right),$$

$$D_0 = -F(F - f_1)(F - f_0)(F - f_1 - f_0),$$

and

$$A_1 = F(F - f_1)(F - f_0)(F - f_0 - f_1),$$

$$\begin{aligned} B_1 = (F - f_0)\big\{&F^2(\lambda_1 + \mu_1) - F[f_0(2\lambda_0 + \lambda_1 + \mu_1) + f_1(\lambda_1 + \mu_1)] + f_0 f_1(\lambda_0 + \lambda_1)\big\} \\ &+ (F - f_1)\{F^2(2\lambda_0 + 2\mu_0) - F[f_0(2\lambda_0 + \mu_0) \\ &\qquad + f_1(\lambda_0 + \lambda_1 + \mu_0)] + f_0 f_1(\lambda_0 + \lambda_1)\}, \end{aligned}$$

$$\begin{aligned} G_1 = (F - f_0)&\big\{F\big[\lambda_0^2 + \mu_0(\mu_1 + \lambda_1)\big] - \lambda_0 f_0(\lambda_0 + \lambda_1 + \mu_1) - f_1(\lambda_0^2 + \lambda_1\mu_0)\big\} \\ &+ (F - f_1)\big\{F\big[\lambda_0(\mu_1 + \lambda_1) + \mu_0(\lambda_0 + \lambda_1 + \mu_1)\big] - \lambda_0 f_0(\lambda_1 + \mu_1) - \lambda_1\mu_0 f_1\big\}, \end{aligned}$$

$$\begin{aligned} D_1 = \mu_0\{&-(F - f_0)(\mu_1 F + \lambda_0 f_0) + (F - f_1) \\ &\times [F(\lambda_0 - \lambda_1 + \mu_0) - f_0(\lambda_0 - \lambda_1) + f_1\lambda_1]\}, \end{aligned}$$

as well as, $A_2 = F$, $B_2 = \lambda_0 + \lambda_1$, $G_2 = -\mu_0$, $D_2 = \mu_1$, and, $A_3 = -F$, $B_3 = -(F - f_0)$, $G_3 = -(F - f_1)$, and $D_3 = F - f_0 - f_1$.

Solving equation (23) we obtain

$$\rho(0, v) = c_1 e^{\delta_1 v} + c_2 e^{\delta_2 v} + c_3 e^{\delta_3 v}, \tag{27}$$

where δ_1, δ_2, and δ_3 are the roots of the polynomial

$$x^3 + \frac{A_0}{D_0}x^2 + \frac{B_0}{D_0}x + \frac{G_0}{D_0} = 0.$$

Using expression (27) into (24) we obtain

$$\rho(1, v) = k_{11} c_1 e^{\delta_1 v} + k_{12} c_2 e^{\delta_2 v} + k_{13} c_3 e^{\delta_3 v}, \tag{28}$$

where

$$k_{11} = \frac{G_1 + B_1\delta_1 + A_1\delta_1^2}{D_1}, \quad k_{12} = \frac{G_1 + B_1\delta_2 + A_1\delta_2^2}{D_1}, \quad \text{and}$$
$$k_{13} = \frac{G_1 + B_1\delta_3 + A_1\delta_3^2}{D_1}.$$

Also, using expression (28) into (25) we obtain

$$\rho(2, v) = k_{21} c_1 e^{\delta_1 v} + k_{22} c_2 e^{\delta_2 v} + k_{23} c_3 e^{\delta_3 v}, \tag{29}$$

where

$$k_{21} = \frac{B_2D_1 + G_2G_1 + (G_2B_1 + A_2D_1)\delta_1 + G_2A_1\delta_1^2}{D_1D_2},$$

$$k_{22} = \frac{B_2D_1 + G_2G_1 + (G_2B_1 + A_2D_1)\delta_2 + G_2A_1\delta_2^2}{D_1D_2},$$

$$k_{23} = \frac{B_2D_1 + G_2G_1 + (G_2B_1 + A_2D_1)\delta_3 + G_2A_1\delta_3^2}{D_1D_2}.$$

Using expression (29) into (26) we obtain

$$\rho(3, v) = k_{31}c_1e^{\delta_1 v} + k_{32}c_2e^{\delta_2 v} + k_{33}c_3e^{\delta_3 v}, \tag{30}$$

where

$$\begin{aligned} k_{31} &= \frac{A_3}{D_3} + B_3\frac{G_1 + B_1\delta_1 + A_1\delta_1^2}{D_1D_3} \\ &\quad + G_3\frac{B_2D_1 + G_2G_1 + (G_2B_1 + A_2D_1)\delta_1 + G_2A_1\delta_1^2}{D_1D_2D_3}, \\ k_{32} &= \frac{A_3}{D_3} + B_3\frac{G_1 + B_1\delta_2 + A_1\delta_2^2}{D_1D_3} \\ &\quad + G_3\frac{B_2D_1 + G_2G_1 + (A_2D_1 + G_2B_1)\delta_2 + G_2A_1\delta_2^2}{D_1D_2D_3}, \\ k_{33} &= \frac{A_3}{D_3} + B_3\frac{G_1 + B_1\delta_3 + A_1\delta_3^2}{D_1D_3} \\ &\quad + G_3\frac{G_2G_1 + B_2D_1 + (A_2D_1 + G_2B_1)\delta_3 + G_2A_1\delta_3^2}{D_1D_2D_3}. \end{aligned}$$

In equations (27)–(30) the constants k_{ij}, $i, j = 1, 2, 3$ are known. However c_1, c_2, and c_3 need to be found. On the case where $\max(f_1, f_0) < F < f_1 + f_0$, we obtain from equation (15) that $\rho(0, 0+) = 0$, and then we obtain

$$c_3 = -c_1 - c_2. \tag{31}$$

Using the second and third expressions from equations (15), we can eliminate $\rho[3, 0]$ and obtain $c_2 = c_{12}c_1$ and $c_3 = c_{13}c_1$ where

$$c_{12} = -\frac{F(\mu_0(-k_{11} + k_{13}) + \mu_1(k_{21} - k_{23})) + f_0\mu_0(k_{11} - k_{13}) + f_1\mu_1(-k_{21} + k_{23})}{F(\mu_0(-k_{12} + k_{13}) + \mu_1(k_{22} - k_{23})) + f_0\mu_0(k_{12} - k_{13}) + f_1\mu_1(-k_{22} + k_{23})},$$

$$c_{13} = \frac{F(\mu_0(-k_{11} + k_{12}) + \mu_1(k_{21} - k_{22})) + f_0\mu_0(k_{11} - k_{12}) + f_1\mu_1(-k_{21} + k_{22})}{F(\mu_0(-k_{12} + k_{13}) + \mu_1(k_{22} - k_{23})) + f_0\mu_0(k_{12} - k_{13}) + f_1\mu_1(-k_{22} + k_{23})}.$$

The constant c_1 can be calculated from the normalization condition

$$\int_{\mathbb{W}} \rho(z)\mathrm{d}z = 1. \tag{32}$$

c_1 is a factor of every term in the continuous part of the stationary distribution, $\rho(\cdot)$. Now, we need to find an expression for the atoms of the stationary distribution in terms of the solutions in the continuous part. Using the last expression of equation (15), we can see that

$$\rho[3,0] = \frac{-(F - f_0 - f_1)}{\mu_0 + \mu_1}\rho(3,0+), \tag{33}$$

for $\max(f_1, f_0) < F < f_1 + f_0$.

Let us say $\rho[3,0] = K_{33}\rho(3,0+)$, where the constant K_{31} is known. For the rest of the atoms we need to solve the system of equation (16). We can use the first, second, and third expressions of (16) to find

$$\rho[0,V] = \frac{\begin{array}{c}(\lambda_0+\mu_1)(\lambda_1+\mu_0)F\rho(0,V-) + \mu_0(\lambda_0+\mu_1)(F-f_0)\rho(1,V-)\\ +\,\mu_1(\lambda_1+\mu_0)(F-f_1)\rho(2,V-)\end{array}}{\lambda_0\lambda_1(\lambda_0+\lambda_1+\mu_0+\mu_1)},$$

$$\rho[1,V] = \frac{\begin{array}{c}(\lambda_0+\mu_1)F\rho(0,V-) + (\lambda_0+\lambda_1+\mu_1)(F-f_0)\rho(1,V-)\\ +\,\mu_1(F-f_1)\rho(2,V-)\end{array}}{\lambda_1(\lambda_0+\lambda_1+\mu_0+\mu_1)},$$

$$\rho[2,V] = \frac{\begin{array}{c}(\lambda_1+\mu_0)F\rho(0,V-) + \mu_0(F-f_0)\rho(1,V-)\\ +\,(\lambda_0+\lambda_1+\mu_0)(F-f_1)\rho(2,V-)\end{array}}{\lambda_0(\lambda_0+\lambda_1+\mu_0+\mu_1)}.$$

As a check, we can substitute these results into the last expression of equation (16). We obtain

$$F\rho(0,V) + (F-f_0)\rho(1,V) + (F-f_1)\rho(2,V) + (F-f_0-f_1)\rho(3,V) = 0,$$

which also comes after equation (17). So, these results seem to be correct.

Just as we did for equation (33), let us say,

$$\rho[0,V] = K_{00}\rho(0,V-) + K_{01}\rho(1,V-) + K_{02}\rho(2,V-),$$
$$\rho[1,V] = K_{10}\rho(0,V-) + K_{11}\rho(1,V-) + K_{12}\rho(2,V-),$$
$$\rho[2,V] = K_{20}\rho(0,V-) + K_{21}\rho(1,V-) + K_{22}\rho(2,V-),$$

where the constants K_{ij}, $i,j = 0,1,2$ are known.

Now, we can use equation (32) to find c_1.

For the case $\max(f_0, f_1) < F < f_0 + f_1$, we have

$$\int_0^V \{\rho(0,v) + \rho(1,v) + \rho(2,v) + \rho(3,v)\}\mathrm{d}v + \rho[0,V] + \rho[1,V] + \rho[2,V] + \rho[3,0] = 1. \tag{34}$$

Writing equation (34) in more detail and solving for c_1, we have

$$\begin{aligned}
c_1^{-1} = {} & \frac{(1 + k_{11} + k_{21} + k_{31})(e^{\delta_1 V} - 1)}{\delta_1} + \frac{c_{12}(1 + k_{12} + k_{22} + k_{32})(e^{\delta_2 V} - 1)}{\delta_2} \\
& + \frac{c_{13}(1 + k_{13} + k_{23} + k_{33})(e^{\delta_3 V} - 1)}{\delta_3} + K_{33}(k_{31} + c_{12}k_{32} + c_{13}k_{33}) \\
& + [K_{00} + K_{10} + K_{20} + k_{11}(K_{01} + K_{11} + K_{21}) + k_{21}(K_{02} + K_{12} + K_{22})]e^{\delta_1 V} \\
& + c_{12}[K_{00} + K_{10} + K_{20} + k_{12}(K_{01} + K_{11} + K_{21}) + k_{22}(K_{02} + K_{12} + K_{22})]e^{\delta_2 V} \\
& + c_{13}[K_{00} + K_{10} + K_{20} + k_{13}(K_{01} + K_{11} + K_{21}) + k_{23}(K_{02} + K_{12} + K_{22})]e^{\delta_3 V}
\end{aligned}$$

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