Statistics and Probability Letters 83 (2013) 1638–1643

Contents lists available at SciVerse ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Random motion with gamma steps in higher dimensions

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ARTICLE INFO

Article history:
Received 14 September 2012
Received in revised form 15 March 2013
Accepted 18 March 2013
Available online 26 March 2013

MSC:
primary 60K35
secondary 60K99
60K15

Keywords:
Random evolutions
Semi-Markov processes
Gamma distributions
Renewal equation

ABSTRACT

We consider isotropic random motion where the direction alternations occur according to the renewal epochs of a Gamma distribution with shape parameter $(n-2)/2$, $n = 4, 5, 6, \dots,$ in higher dimensions. We formulate a general renewal-type equation for the characteristic function and we solve the renewal equation in an iterative manner.

1. Introduction

Many variations have been proposed to the basic Markov telegraph process. One-dimensional non-Markovian generalizations of the telegrapher's random process were obtained in Di Crescenzo (2001) and Pogorui and Rodriguez-Dagnino (2005) with velocities alternating at k-Erlang-distributed sojourn times, see also Pogorui and Rodriguez-Dagnino (2011) and Pogorui (2011). Isotropic random motions with motion driven by a homogeneous Poisson process in higher spaces or dimensions has been studied by Orsingher and De Gregorio in higher dimensions (Orsingher and De Gregorio, 2007), see also De Gregorio and Orsingher (2012) and references therein. The exponential pdf assumed in the Poisson process is not the best option for many important applications since it gives higher probability to very short intervals. The work of Le Caër (2010, 2011) departs from this trend since he is studying isotropic random motion with Pearson–Dirichlet distributed steps in higher dimensions. In this work, we consider isotropic random motions in higher dimensions with a special kind of Gamma distributed step with shape parameter β equal to 1, 3/2, 2, 5/2, and so on. The maximum or mode of this distribution occurs at $(\beta - 1)/\lambda$, where $\lambda > 0$ is the scale parameter. So, the applications are extended to the cases that have more likely longer step sizes.

Let $\{\xi(t), t \geq 0\}$ be an ordinary renewal counting process such that $\xi(t) = \max\{m \geq 0 : \tau_m \leq t\}$, where $\tau_m = \sum_{k=1}^{m} \theta_k$, $\tau_0 = 0$ and $\theta_k \geq 0$, $k = 1, 2, \dots,$ are nonnegative random variables denoting the interoccurrence times of change of direction of the particle. We assume that these random variables are independent and identically distributed (iid) with a distribution function $G_\theta(t)$ and that there exists the probability density function (pdf) $g_\theta(t) = \frac{d}{dt} G_\theta(t)$.

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http://dx.doi.org/10.1016/j.spl.2013.03.011

2. Basic modeling statements

We will study the random motion of a particle that starts from the coordinate origin $\mathbf{0} = (0, 0, \dots, 0)$ of the space $\mathbb{R}^n$, at time $t = 0$, and it continues its motion with a constant absolute velocity $v(t)$ along the direction $\boldsymbol{\eta}_0^{(n)}$, where $\boldsymbol{\eta}_0^{(n)} = (x_{01}, x_{02}, \dots, x_{0n}) = (x_1, x_2, \dots, x_n)$ is a random n-dimensional vector uniformly distributed on the unit sphere $\Omega_1^{n-1} = \{(x_1, x_2, \dots, x_n) : x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}$.

At instant τ_1 the particle changes its direction to $\boldsymbol{\eta}_1^{(n)} = (x_{11}, x_{12}, \dots, x_{1n})$, where $\boldsymbol{\eta}_0^{(n)}$ and $\boldsymbol{\eta}_1^{(n)}$ are iid random vectors on Ω_1^{n-1}. Then, at instant τ_2 the particle changes its direction to $\boldsymbol{\eta}_2^{(n)} = (x_{21}, x_{22}, \dots, x_{2n})$, where $\boldsymbol{\eta}_2^{(n)}$ is also uniformly distributed on Ω_1^{n-1} and independent of $\boldsymbol{\eta}_0^{(n)}$ and $\boldsymbol{\eta}_1^{(n)}$, and so on.

Denote by $\mathbf{X}^{(n)}(t)$, $t \geq 0$, the particle position at time t after the occurrence of a random number of direction alternations given by the counting process $\{\xi(t)\}$ in the interval $(0, t)$. We have that

$$\mathbf{X}^{(n)}(t) = \sum_{j=1}^{\xi(t)} \boldsymbol{\eta}_{j-1}^{(n)} \int_{\tau_{j-1}}^{\tau_j} v(y)\, dy + \boldsymbol{\eta}_{\xi(t)}^{(n)} \int_{\tau_{\xi(t)}}^{t} v(y)\, dy. \tag{1}$$

Basically, Eq. (1) determines the random evolution in the semi-Markov (or renewal) medium $\xi(t)$.

The probabilistic properties of the random vector $\mathbf{X}^{(n)}(t)$ are completely determined by those of its projection

$$X^{(n)}(t) = \sum_{j=1}^{\xi(t)} \eta_{j-1}^{(n)} \int_{\tau_{j-1}}^{\tau_j} v(y)\, dy + \eta_{\xi(t)}^{(n)} \int_{\tau_{\xi(t)}}^{t} v(y)\, dy$$

on a fixed line, where $\eta_j^{(n)}$ is the projection of $\boldsymbol{\eta}_j^{(n)}$ on the line.

Indeed, let us consider the cumulative distribution function (cdf) $F_X(y) = P\left(X^{(n)}(t) \leq y\right)$. Then, the characteristic function (Fourier transform) $H(t, \alpha)$ of $\mathbf{X}^{(n)}(t)$, where $\alpha = \|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}$, is given by

$$\begin{aligned} H(t, \alpha) &= \mathbf{E}\left[\exp\left\{i\left(\boldsymbol{\alpha}, \mathbf{X}^{(n)}(t)\right)\right\}\right] \\ &= \mathbf{E}\left[\exp\left\{i\,\|\boldsymbol{\alpha}\|\left(\mathbf{e}, \mathbf{X}^{(n)}(t)\right)\right\}\right] = \mathbf{E}\left[\exp\left\{i\,\|\boldsymbol{\alpha}\| X^{(n)}(t)\right\}\right] \\ &= \int_{-\infty}^{\infty} \exp\{i\,\|\boldsymbol{\alpha}\| y\}\, dF_X(y), \end{aligned}$$

where $X^{(n)}(t)$ is the projection of $\mathbf{X}^{(n)}(t)$ onto the unit vector $\mathbf{e}$.

Let us denote by $f_\eta(x)$ the pdf of the projection $\eta_j^{(n)}$ of the vector $\boldsymbol{\eta}_j^{(n)}$ onto a fixed line. It is shown in Pogorui (2010) that $f_\eta(x)$ is of the following form

$$f_{\eta^{(n)}}(x) = \begin{cases} \dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} (1 - x^2)^{(n-3)/2}, & \text{if } x \in [-1, 1], \\ 0, & \text{if } x \notin [-1, 1]. \end{cases} \tag{2}$$

Let us denote by

$$\varphi(t, \alpha) = \mathbf{E}\left[\exp\left\{i\eta_j^{(n)} \alpha \int_0^t v(y) dy\right\}\right] = \int_{-\infty}^{\infty} e^{ix\alpha \int_0^t v(y)dy} f_{\eta^{(n)}}(x) dx$$

the characteristic function of $\eta_j^{(n)}$. We should notice that the function $\varphi(\cdot)$ is also used in Pogorui and Rodriguez-Dagnino (2011, Eq. 2.5). It is well known that for $v(t) = v = \text{constant} > 0$ we have, Pogorui and Rodriguez-Dagnino (2011),

$$\varphi(t, \alpha) = 2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right) \frac{J_{\frac{n-2}{2}}(\alpha t v)}{(\alpha t v)^{\frac{n-2}{2}}}.$$

It is easily verified that in the case of nonconstant velocity, we have

$$\varphi(t, \alpha) = 2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right) \frac{J_{\frac{n-2}{2}}\left(\alpha \int_0^t v(y) dy\right)}{\left(\alpha \int_0^t v(y) dy\right)^{\frac{n-2}{2}}}.$$

3. Renewal equation for characteristic function

We have found the characteristic function $H(t,\alpha)$ of the random motion $\mathbf{X}^{(n)}(t)$ satisfies a Volterra integral equation as it is shown in the following theorem.

Theorem 1. *The characteristic function $H(t,\alpha)$, $t\geq 0$ is a solution of the following Volterra integral equation*

$$H(t,\alpha)=(1-G_\theta(t))\,\varphi(t,\alpha)+\int_0^t g_\theta(u)\varphi(u,\alpha)H(t-u,\alpha)\,du \tag{3}$$

or in the convolution form

$$H(t,\alpha)=(1-G_\theta(t))\,\varphi(t,\alpha)+(g_\theta(t)\times\varphi(t,\alpha))*H(t,\alpha).$$

The function $H(t,\alpha)$ is a unique solution of Eq. (3) in the class of continuous functions.

Proof.

$$\begin{aligned}
H(t,\alpha)&=\mathbf{E}\left[\exp\left\{i\left(\boldsymbol{\alpha},\mathbf{X}^{(n)}(t)\right)\right\}\right]\\
&=\mathbf{E}\left[\exp\left\{i\left(\boldsymbol{\alpha},\sum_{j=1}^{\xi(t)}\boldsymbol{\eta}_{j-1}^{(n)}\int_0^{\tau_1}v(y)\,dy+\boldsymbol{\eta}_{\xi(t)}^{(n)}\int_{\tau_{\xi(t)}}^{t}v(y)\,dy\right)\right\}\right]\\
&=\mathbf{E}\left[I_{\{\tau_1>t\}}\,e^{i\int_0^t v(y)dy\,(\boldsymbol{\alpha},\boldsymbol{\eta}_0^{(n)})}\right]+\int_0^t\mathbf{E}\left[I_{\{\tau_1\in du\}}\,e^{i\int_0^u v(r)dr\,(\boldsymbol{\alpha},\boldsymbol{\eta}_0^{(n)})}\right]H(t-u,\alpha)\\
&=(1-G_\theta(t))\,\mathbf{E}\left[e^{i\int_0^t v(y)dy\,(\boldsymbol{\alpha},\boldsymbol{\eta}_0^{(n)})}\right]+\int_0^t g_\theta(u)\mathbf{E}\left[e^{i\int_0^u v(r)dr\,(\boldsymbol{\alpha},\boldsymbol{\eta}_0^{(n)})}\right]H(t-u,\alpha)du.
\end{aligned}$$

Now, we should notice that

$$\varphi(t,\alpha)=\mathbf{E}\left[e^{i\int_0^t v(y)dy\,(\boldsymbol{\alpha},\boldsymbol{\eta}_0^{(n)})}\right]$$

and this completes the proof. □

Now, by passing to the Laplace transform $\widehat{H}(s,\alpha)=\mathcal{L}\left(H(t,\alpha)\right)=\int_0^\infty H(t,\alpha)e^{-st}dt$ in Eq. (3) we obtain

$$\widehat{H}(s,\alpha)=\frac{\int_0^\infty(1-G_\theta(t))\,\varphi(t,\alpha)e^{-st}dt}{1-\int_0^\infty g_\theta(t)\varphi(t,\alpha)e^{-st}dt}. \tag{4}$$

Eq. (4) can be useful to obtain solutions of Eq. (3).

We should remark that Theorem 1 has been proved in De Gregorio and Orsingher (2012) by considering an exponential pdf for $g_\theta(t)$, and in Pogorui and Rodriguez-Dagnino (2011) for Erlang $g_\theta(t)$. In this case it has been proved for any pdf $g_\theta(t)$, and it includes, of course, the special kind of Gamma pdf that we consider in this paper.

We will denote by $f_n(t,\mathbf{x})$ the probability density function (pdf) of the particle position at time t and a point $\mathbf{x}\in\mathbb{R}^n$. It is easily seen that $f_n(t,\mathbf{x})=\mathcal{F}^{-1}(H(t,\alpha))$. Our purpose, in the rest of the paper, is to study $f_n(t,\mathbf{x})$.

To gain some insights into this problem we will provide an example that might be an approximate model for a particle that starts with a huge velocity (probably after a collision) in a very dense medium, i.e., the velocity decays very fast with time. The time duration in that particular direction is given by a gamma pdf with scale parameter equals to 1 and shape parameter equals to 3/2.

Example 1. Let us consider three-dimensional motion, i.e., $n=3$. Suppose that $v(t)=\frac{1}{\sqrt{t}}$ and $g_\theta(t)=\frac{2}{\sqrt{\pi}}\sqrt{t}e^{-t}$, and we calculate $f_3(t,\mathbf{x})$.

In this case

$$\varphi(t,\alpha)=\sqrt{\frac{\pi}{2}}\frac{J_{\frac12}(\alpha\sqrt{t}\,)}{(\alpha\sqrt{t}\,)^{\frac12}}.$$

It is not hard to calculate the following Laplace transform

$$\int_0^\infty g_\theta(t)\varphi(t,\alpha)e^{-st}dt=\sqrt{2}\int_0^\infty\sqrt{t}\,\frac{J_{\frac12}(\alpha\sqrt{t}\,)}{(\alpha\sqrt{t}\,)^{\frac12}}\,e^{-(s+1)t}dt=\frac{e^{-\frac{\alpha^2}{s+1}}}{(s+1)^{\frac32}}.$$

Now, by applying the inverse Laplace transform $\mathcal{L}^{-1}$, we obtain for $i \geq 1$

$$\mathcal{L}^{-1}\left(\left(\int_0^\infty g_\theta(t)\varphi(t,\alpha)e^{-st}\,dt\right)^i\right) = \mathcal{L}^{-1}\left(\frac{e^{-\frac{\alpha^2 i}{s+1}}}{(s+1)^{\frac{3i}{2}}}\right)$$
$$= e^{-t}\left(\frac{2}{\alpha}\sqrt{\frac{t}{i}}\right)^{\frac{3i-2}{2}} J_{\frac{3i-2}{2}}\left(\sqrt{t\,i}\,\alpha\right).$$

We need to apply the inverse Fourier transform $\mathcal{F}^{-1}$ with respect to α, we will do it for $i = 1$. Thus,

$$\mathcal{F}^{-1}\left(e^{-t}\left(\frac{2}{\alpha}\sqrt{t}\right)^{\frac{1}{2}} J_{\frac{1}{2}}\left(\sqrt{t}\,\alpha\right)\right) = \frac{2}{\sqrt{\pi}}\sqrt{t}\,e^{-t}\frac{\delta(t-x^2)}{4\pi t},$$

where $x = \|\mathbf{x}\|$.

Let us calculate the following convolution

$$\left(\left(1-\frac{2}{\sqrt{\pi}}\int_0^t\sqrt{w}\,e^{-w}dw\right)\frac{\delta(t-\|\mathbf{x}\|^2)}{4\pi t}\right) * \left(\frac{2}{\sqrt{\pi}}\sqrt{t}\,e^{-t}\frac{\delta(t-\|\mathbf{x}\|^2)}{4\pi t}\right)$$
$$= \int_0^t\int_{\mathbb{R}^3}\left(\left(1-\frac{2}{\sqrt{\pi}}\int_0^{t-v}\sqrt{w}\,e^{-w}dw\right)\frac{\delta(t-v-\|\mathbf{x}-\mathbf{u}\|^2)}{4\pi t}\right)\times\left(\frac{2}{\sqrt{\pi}}\sqrt{v}\,e^{-v}\frac{\delta(v-\|\mathbf{u}\|^2)}{4\pi v}\right)d\mathbf{u}\,dv$$
$$= \int_{\mathbb{R}^3}\left(1-\frac{2}{\sqrt{\pi}}\int_0^{\|\mathbf{x}-\mathbf{u}\|^2}\sqrt{w}\,e^{-w}dw\right)\frac{\delta(t-\|\mathbf{u}\|^2-\|\mathbf{x}-\mathbf{u}\|^2)}{8t\pi^{5/2}\|\mathbf{u}\|}\,d\mathbf{u}$$
$$= \frac{1}{8t\pi^{5/2}}\int_{\|\mathbf{u}\|<\sqrt{t}}\left(1-\frac{2}{\sqrt{\pi}}\int_0^{\|\mathbf{x}-\mathbf{u}\|^2}\sqrt{w}\,e^{-w}dw\right)\frac{d\mathbf{u}}{\|\mathbf{u}\|}.$$

Therefore,

$$f_3(t,\mathbf{x}) = \mathcal{F}^{-1}(H(t,\alpha))(\mathbf{x}) = \left(1-\frac{2}{\sqrt{\pi}}\int_0^t\sqrt{w}\,e^{-w}dw\right)\frac{\delta(t-x^2)}{4\pi t}$$
$$+\frac{1}{8t\pi^{5/2}}\int_{\|\mathbf{u}\|<\sqrt{t}}\left(1-\frac{2}{\sqrt{\pi}}\int_0^{\|\mathbf{x}-\mathbf{u}\|^2}\sqrt{w}\,e^{-w}dw\right)\frac{d\mathbf{u}}{\|\mathbf{u}\|}$$
$$+\left(\left(1-\frac{2}{\sqrt{\pi}}\int_0^x\sqrt{t}\,e^{-t}dt\right)\frac{\delta(t-x^2)}{4\pi t}\right) * \sum_{i=2}^\infty\frac{e^{-t}(i\,t-x^2)^{(3i-5)/2}}{2^{3i/4}\Gamma\left(\frac{3i-3}{2}\right)i^{(3i-2)/2}\,t^{(3i-2)/8}}.$$

4. Constant velocity

Now, suppose that $v(t) = v = constant$, and let us introduce the following function

$$H_n(t,\alpha) = e^{-\lambda t}\sum_{j=1}^\infty j(2\lambda)^{j\beta}\frac{J_{\frac{jn}{2}}(vt\alpha)}{t(v\alpha)^{\frac{jn}{2}}} \tag{5}$$

where $\beta = \frac{n-2}{2}$.

Theorem 2. *Suppose that*

$$g_\theta(t) = \frac{\lambda^\beta t^{\beta-1}}{\Gamma(\beta)}\,e^{-\lambda t}I_{\{t\geq 0\}}\quad\text{for } n\geq 4 \text{ or } \beta = 1, 3/2, 2, 5/2, \ldots$$

i.e., θ_k *is gamma distributed. Then,*

$$H_n(t,\alpha) = g_\theta(t)\varphi(t,\alpha) + \int_0^t g_\theta(u)\varphi(u,\alpha)H_n(t-u,\alpha)\,du,$$

or in more detail we have

$$H_n(t,\alpha) = \lambda(n-2)\,e^{-\lambda t}(2\lambda t)^{\beta-1}\frac{J_\beta(\alpha t v)}{(\alpha t v)^\beta} + \lambda(n-2)\int_0^t e^{-\lambda u}(2\lambda u)^{\beta-1}\frac{J_\beta(\alpha u v)}{(\alpha u v)^\beta}H_n(t-u,\alpha)\,du. \tag{6}$$

Proof. Let us write the function $H_n(t,\alpha)$ in the following form

$$H_n(t,\alpha) = e^{-\lambda t}\sum_{j=1}^{\infty} a_{j\beta}\frac{J_{j\beta}(vt\alpha)}{t}.$$

It is not hard to verify that

$$\begin{aligned}&(n-2)e^{-\lambda t}\frac{1}{2v\alpha}\left(\frac{2\lambda}{\alpha v}\right)^{\beta} a_{j\beta}\int_0^t \frac{J_{j\beta}(vu\alpha)}{vu\alpha}\,\frac{J_{\beta}(v(t-u)\alpha)}{v(t-u)\alpha}\,du\\ &\quad = (n-2)e^{-\lambda t}\frac{1}{2v\alpha}\left(\frac{2\lambda}{\alpha v}\right)^{\beta} a_{j\beta}\left(\frac{1}{\beta}+\frac{1}{j\beta}\right) J_{(j+1)\beta}(vu\alpha),\end{aligned}$$

where we have applied Formula 6.533 (2) in Gradshteyn and Ryzhik (1992) that we repeat here for an easy reference

$$\int_0^t \frac{J_\mu(u)}{\mu}\,\frac{J_\nu(t-u)}{t-u}\,du = \left(\frac{1}{\mu}+\frac{1}{\nu}\right)\frac{J_{\mu+\nu}(t)}{t}, \quad \text{for } \mu>0, \nu>0. \tag{7}$$

Hence, we have

$$\frac{1}{v\alpha}\left(\frac{2\lambda}{\alpha v}\right)^{\beta}\frac{j+1}{j}\,a_{j\beta} = a_{(j+1)\beta} \quad \text{for } j = 1, 2, \ldots. \tag{8}$$

Eq. (8) implies that

$$a_{j\beta} = \frac{j(2\lambda)^{j\beta}}{(v\alpha)^{\frac{jn}{2}}}$$

and it concludes the proof. □

Now, let us assume that n is even, then by taking into account Eq. (3), we should solve the following equation

$$H(t,\alpha) = \sum_{i=0}^{\beta-1} e^{-\lambda t}\frac{(\lambda t)^i}{i!}\,\frac{2^{\beta}\Gamma(\beta+1)}{(vt\alpha)^{\beta}} J_\beta(vt\alpha) + \int_0^t g_\theta(u)\varphi(u,\alpha)H(t-u,\alpha)\,du. \tag{9}$$

Denote by $H^{(k)}(t,\alpha)$, $k = 0, 1, 2, \ldots, n/2-2$ a solution of the following equation

$$H^{(k)}(t,\alpha) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}\,\frac{2^{\beta}\Gamma(\beta+1)}{(vt\alpha)^{\beta}} J_\beta(vt\alpha) + \int_0^t g_\theta(u)\varphi(u,\alpha)H^{(k)}(t-u,\alpha)\,du. \tag{10}$$

It is easily seen that $H(t,\alpha) = \sum_{k=0}^{n/2-2} H^{(k)}(t,\alpha)$ is the solution of Eq. (9).

Lemma 1. *For each $k = 0, 1, 2, \ldots, n/2-2$ the following equations hold*

$$H^{(k)}(t,\alpha) = e^{-\lambda t}\frac{\lambda^k t^k}{k!}\varphi(t,\alpha) + \lambda\int_0^t e^{-\lambda u}\frac{\lambda^k u^k}{k!}\varphi(u,\alpha)H_n(t-u,\alpha)\,du.$$

Proof. Denote by $q_k(t) = e^{-\lambda t}\frac{\lambda^k t^k}{k!}$, the discrete version of $g_\theta(t)$, and by taking the Laplace transform in Eqs. (6) and (10) we can obtain

$$\widehat{H}_n(s,\alpha) = \frac{\frac{1}{\lambda}\int_0^\infty q_k(t)\varphi(t,\alpha)\,e^{-st}\,dt}{1-\int_0^\infty q_k(t)\varphi(t,\alpha)\,e^{-st}\,dt} = \frac{1}{\lambda}\sum_{j=1}^{\infty}\left(\int_0^\infty q_k(t)\varphi(t,\alpha)\,e^{-st}\,dt\right)^j, \tag{11}$$

and

$$\widehat{H}^{(k)}(s,\alpha) = \left(1+\sum_{j=1}^{\infty}\left(\int_0^\infty q_k(t)\varphi(t,\alpha)\,e^{-st}\,dt\right)^j\right)\int_0^\infty q_k(t)\varphi(t,\alpha)\,e^{-st}\,dt. \tag{12}$$

We conclude the proof after taking the inverse Laplace transform of Eqs. (11) and (12). □

Now, we need to calculate $\mathcal{F}^{-1}(H_n(t,\alpha))$, where $\mathcal{F}^{-1}(\cdot)$ is the n-dimensional inverse Fourier transform with respect to $\boldsymbol{\alpha}$.

We can reduce the n-dimensional inverse Fourier transform to the Hankel transform, Bochner and Chandrasekharan (1949). First, let us calculate the following inverse Fourier transform that is valid for $x = \| \mathbf{x} \| < vt$ and $j \geq 1$,

$$\mathcal{F}^{-1}\left(\frac{J_{jn/2}(vt\alpha)}{t(v\alpha)^{jn/2}}\right) = \frac{1}{(2\pi)^{n/2}}\int_0^\infty \frac{J_{jn/2}(vt\alpha)}{t(v\alpha)^{jn/2}}\,\alpha^{n-1}\frac{J_{(n-2)/2}(x\alpha)}{(x\alpha)^{(n-2)/2}}\,d\alpha. \tag{13}$$

Let use the following integral (for $a \geq b$)

$$\int_0^\infty J_{\nu+1}(az)J_\mu(bz)z^{\mu-\nu}dz = \frac{(a^2-b^2)^{\nu-\mu}\,b^\mu}{2^{\nu-\mu}a^{\nu+1}\Gamma(\nu-\mu+1)}, \quad \nu+1>\mu>0. \tag{14}$$

We should notice that the integral in Eq. (14) is a special case of the integral in p. 69 of McLachlan (1955), and that the corresponding integral 6.575 (1) in Gradshteyn and Ryzhik (1992) seems to have a mistake in the condition $\mu > \nu + 1$.

Thus, Eq. (13) can be solved by Eq. (14) and we have after substituting Eq. (5) that

$$\begin{aligned} H_n(t,x) &= \mathcal{F}^{-1}(H_n(t,\alpha)) = e^{-\lambda t}\sum_{j=1}^\infty j(2\lambda)^{j\beta}\mathcal{F}^{-1}\left(\frac{J_{jn/2}(vt\alpha)}{t(v\alpha)^{jn/2}}\right) \\ &= e^{-\lambda t}\sum_{j=1}^\infty j(2\lambda)^{j\beta}\frac{\left(\frac{v^2t^2-x^2}{2}\right)^{(n-1)j/2}}{(2\pi)^{n/2}t\,v^{jn/2}x^\beta\,\Gamma\left(\frac{(n-1)j}{2}+1\right)}. \end{aligned}$$

By applying Lemma 1 we can calculate $H^{(m)}(t,\alpha)$ for $m = 0, 1, \ldots, n/2-2$. Then, by taking the inverse Fourier transform we finally obtain

$$f_n(t,\mathbf{x}) = \sum_{m=0}^{n/2-2}\mathcal{F}^{-1}(H^{(m)}(t,\alpha)).$$

We should remark the correspondence $H_n(t,\alpha) = H^{(n/2-2)}(t,\alpha)$.

Acknowledgment

We thank Tecnológico de Monterrey through the Research Chair in Telecommunications, for the support provided in the development of this work.

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