Multidimensional Random Motion with Uniformly Distributed Changes of Direction and Erlang Steps

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Abstract

In this paper we study transport processes in $\mathbb{R}^n$, $n \geq 1$, having non-exponential distributed sojourn times or non-Markovian step durations. We use the idea that the probabilistic properties of a random vector are completely determined by those of its projection on a fixed line, and using this idea we avoid many of the difficulties appearing in the analysis of these problems in higher dimensions. As a particular case, we find the probability density function in three dimensions for 2-Erlang distributed sojourn times.

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1. Introduction

One-dimensional non-Markovian generalizations of the telegrapher's random process were obtained in [1, 2] with velocities alternating at Erlang-distributed sojourn times. Uniformly distributed direction of motion or isotropic motion has been studied by Pinsky [3] for transport processes on Riemannian manifold and by Orsingher and De Gregorio in higher dimensions [4]. However, most of the papers on multidimensional random motion are devoted to analysis of

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models in which motions are driven by a homogeneous Poisson process (see [3]-[6] and references therein). The recent work of Le Caer [7] departs from this trend since he is studying uniformly distributed orientation random motion with Pearson-Dirichlet distributed steps in a multidimensional random walk setting. In this work, we consider random motions with uniformly distributed directions on the multidimensional space $\mathbb{R}^n$, $n \geq 1$, with Erlang distributed step lengths. Our analysis is based on random evolutions on a semi-Markov media.

Let us consider the renewal process $\xi(t) = \max\{m \geq 0 : \tau_m \leq t\}$, $t \geq 0$, where $\tau_m = \sum_{k=1}^{m} \theta_k$, $\tau_0 = 0$ and $\theta_k \geq 0$, $k = 1, 2, ...$, are i.i.d. random variables with a distribution function $G(t)$ such that there exists the probability density function (pdf) $g(t) = \frac{d}{dt}G(t)$.

We assume that a particle starting from the coordinate origin $(0, 0, \dots, 0)$ of the space $\mathbb{R}^n$, at time $t = 0$, continues its motion with a constant absolute velocity v along the direction $\boldsymbol{\eta}_0^{(n)}$, where $\boldsymbol{\eta}_0^{(n)} = (x_1, x_2, \cdots, x_n)$ is a random n-dimensional vector uniformly distributed on the unit sphere $\Omega_1^{n-1} = \{(x_1, x_2, \dots, x_n) : x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}$.

At instant τ_1 the particle changes its direction to $\boldsymbol{\eta}_1^{(n)}$, where $\boldsymbol{\eta}_0^{(n)}$ and $\boldsymbol{\eta}_1^{(n)}$ are i.i.d. random vectors on Ω_1^{n-1}. Then, at instant τ_2 the particle changes its direction to $\boldsymbol{\eta}_2^{(n)}$, where $\boldsymbol{\eta}_2^{(n)}$ is also uniformly distributed on Ω_1^{n-1} and independent of $\boldsymbol{\eta}_0^{(n)}$ and $\boldsymbol{\eta}_1^{(n)}$, and so on.

Denote by $\mathbf{x}^{(n)}(t)$, $t \geq 0$, the particle position at time t. We have that

$$\mathbf{x}^{(n)}(t) = v \sum_{i=0}^{\xi(t)} \boldsymbol{\eta}_i^{(n)} \theta_i + v\, \boldsymbol{\eta}_{\xi(t)}^{(n)} (t - \tau_{\xi(t)}). \tag{1}$$

Basically, Eq. (1) determines the random evolution in the semi-Markov medium $\xi(t)$.

Lemma 1. *The probability distribution of the random vector* $\mathbf{x}^{(n)}(t)$ *is determined by the probability distribution of its projection* $x^{(n)}(t) = v \sum_{i=0}^{\xi(t)} \eta_i^{(n)} \theta_i + v\, \eta_{\xi(t)}^{(n)} (t - \tau_{\xi(t)})$ *on a fixed line, where* $\eta_i^{(n)}$ *is the projection of* $\boldsymbol{\eta}_i^{(n)}$ *on the line.*

Proof. Let us consider the cumulative distribution function (cdf) $F_{x^{(n)}(t)}(y) = P(x^{(n)}(t) \leq y)$. Then, the characteristic function $\varphi_{\mathbf{x}^{(n)}(t)}(\boldsymbol{\alpha})$ of $\mathbf{x}^{(n)}(t)$ is given by

$$\varphi_{\mathbf{x}^{(n)}(t)} = \mathbf{E}\exp\left\{i\left(\boldsymbol{\alpha},\mathbf{x}^{(n)}(t)\right)\right\} = \mathbf{E}\exp\left\{i\,\|\boldsymbol{\alpha}\|\left(\mathbf{e},\mathbf{x}^{(n)}(t)\right)\right\}$$

$$= \mathbf{E}\exp\left\{i\,\|\boldsymbol{\alpha}\|x^{'(n)}(t)\right\} = \int_0^\infty \exp\{i\,\|\boldsymbol{\alpha}\|y\}\,dF_{x^{(n)}(t)}(y),$$

where $\|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2+\alpha_2^2+\cdots+\alpha_n^2}$, $x^{(n)}(t)$ is the projection of $\mathbf{x}^{(n)}(t)$ onto the unit vector $\mathbf{e}$ and it has a cdf $F_{x^{(n)}(t)}(y)$. $\square$

It is well known that if $f(x_1,x_2,\ldots,x_n)\in L_1(\mathbb{R}^n)$ depends only on $\|\mathbf{x}\| = \sqrt{x_1^2+x_2^2+\cdots+x_n^2}$, i.e., $f(x_1,x_2,\ldots,x_n)=g(r)$, then the function

$$\varphi(s_1,s_2,\ldots,s_n) = \int_{\mathbb{R}^n} f(\mathbf{x})\exp\left\{-i\sum_{k=1}^n s_k x_k\right\}d\mathbf{x}$$

depends only on $s=\|\mathbf{s}\|$. Such functions are called radial functions and for these functions the Fourier transform in several variables goes over into the "Bessel transform" in one variable as follows:

$$\varphi(\mathbf{s}) = \frac{(2\pi)^{n/2}}{s^{(n-2)/2}}\int_0^\infty g(r)\,r^{n/2}J_{(n-2)/2}(sr)dr,$$

where $J_p(x)$ denotes the Bessel function, of the first kind, of order p [10].

Since $\varphi_{\mathbf{x}}(\boldsymbol{\alpha})$ depends only on $\alpha=\|\boldsymbol{\alpha}\|$, meaning that $\varphi_{\mathbf{x}}(\boldsymbol{\alpha})=\varphi(\alpha)$ then the pdf $f_{\mathbf{x}(t)}(\mathbf{y})$ corresponding to the distribution

$$F_{\mathbf{x}(t)}(\mathbf{y}) = P\left(v\sum_{i=0}^{\xi(t)+1}\boldsymbol{\eta}_i^{(n)}\theta_i + v\,\boldsymbol{\eta}_{\xi(t)}^{(n)}\left(t-\tau_{\xi(t)}\right)\le\mathbf{y}\right)$$

depends only on $r=\|\mathbf{y}\|$, that is, $f_{\mathbf{x}(t)}(\mathbf{y})=h(r)$ and we have

$$\varphi_{\mathbf{x}(t)}(\boldsymbol{\alpha}) = \frac{(2\pi)^{n/2}}{\alpha^{(n-2)/2}}\int_0^\infty h(r)\,r^{n/2}J_{(n-2)/2}(\alpha r)dr.$$

It also follows that if $h(r)$ is continuous on $[0,+\infty)$ and $\int_0^\infty r^{n-1}h(r)dr<\infty$, and if $\int_0^\infty \alpha^{n-1}\varphi(\alpha)d\alpha<\infty$, then [10]:

$$f_{\mathbf{x}(t)}(\mathbf{y}) = h(r) = \frac{1}{(2\pi)^{n/2}\,r^{(n-2)/2}}\int_0^\infty \varphi(\alpha)\,\alpha^{n/2}J_{(n-2)/2}(\alpha r)d\alpha.$$

Now, let us define $\hat{x}^{(n)}(t) = v\sum_{i=0}^{\xi(t)}\eta_i^{(n)}\theta_i$ and $\Delta(t) = v\,\eta_{\xi(t)}^{(n)}\left(t-\tau_{\xi(t)}\right)$, and we will denote as $F_{\hat{x}^{(n)}(t)}(y)$ (resp. $F_{\Delta(t)}(y)$) the cdf of $\hat{x}^{(n)}(t)$ (resp. $\Delta(t)$).

It is easy to verify that $\hat{x}^{(n)}(t)$ and $\Delta(t)$ are independent. Hence, we have $F_{x^{(n)}(t)}(y) = F_{\hat{x}^{(n)}(t)}(y) * F_{\Delta(t)}(y)$.

Therefore, by using Lemma 1 we can study the cdf of $\mathbf{x}^{(n)}(t)$ but we need to know the cdf of $\hat{x}^{(n)}(t)$ and $\Delta(t)$.

Lemma 2. *Let $F_n(t)$ be the cdf of $\eta_i^{(n)}\theta_i$ and it is of the following form*

$$F_n(t) = \begin{cases} \dfrac{1}{2} + \dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \displaystyle\int_0^1 G\left(\frac{t}{x}\right)(1-x^2)^{(n-3)/2}dx, & \text{if} \quad t \geq 0, \\ \dfrac{1}{2} - \dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \displaystyle\int_0^1 G\left(-\frac{t}{x}\right)(1-x^2)^{(n-3)/2}dx, & \text{if} \quad t < 0. \end{cases} \tag{2}$$

Proof. Let us denote by $f_{\eta_i}(x)$ the pdf of the projection $\eta_i^{(n)}$ of the vector $\boldsymbol{\eta}_i^{(n)}$ onto a fixed line. It is showed in [8] that $f_{\eta_i}(x)$ is of the following form

$$f_{\eta_i^{(n)}}(x) = \begin{cases} \dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}(1-x^2)^{(n-3)/2}, & \text{if} \quad x \in [-1,1], \\ 0, & \text{if} \quad x \notin [-1,1]. \end{cases} \tag{3}$$

Since $\eta_i^{(n)}$ and θ_i are independent it is easy to verify that the cdf of $\eta_i^{(n)}\theta_i$ is of the form (2). $\square$

The process $\gamma(t) = t - \tau_{\xi(t)}$ is a Markov process and it has the following generator operator A [9]

$$A\varphi(s) = \varphi'(s) + \frac{g(s)}{1-G(s)}\left(\varphi(0) - \varphi(s)\right), \qquad s \geq 0,$$

where $\varphi \in \mathcal{C}^1(\mathbb{R})$.

Lemma 3. *The cdf $F_{\Delta(t)}(s) = P\left(v\,\eta_{\xi(t)}^{(n)}\,(t-\tau_{\xi(t)}) \leq s\right)$ is given by*

$$F_{\Delta(t)}(s) = \begin{cases} \dfrac{1}{2} + \dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \displaystyle\int_0^1 F_{\gamma(t)}\left(\frac{s}{vx}\right)(1-x^2)^{(n-3)/2}dx, & \text{if} \quad s \geq 0, \\ \dfrac{1}{2} - \dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \displaystyle\int_0^1 F_{\gamma(t)}\left(-\frac{s}{vx}\right)(1-x^2)^{(n-3)/2}dx, & \text{if} \quad s < 0. \end{cases}$$

Proof. The cdf $F_{\gamma(t)}(u) = P(\gamma(t) \le u)$ satisfies the following Markov renewal equation [9]

$$F_{\gamma(t)}(u) = V(t,u) + \int_0^t g(s) F_{\gamma(t-s)}(u) ds, \tag{4}$$

where $V(t,u) = P(\gamma(t) \le u, \tau_1 > t) = (1 - G(t)) \, I_{\{t \le u\}}$.

Let us define the function $R(t) = \sum_{k=0}^{\infty} g^{*(k)}(t)$, where the symbol $*(n)$ denotes the k-fold convolution of $g(t)$ with itself. Then, Eq. (4) can be rewritten as

$$F_{\gamma(t)}(u) = (V * R)(t,u) = \int_0^t V(t-s,u) dR(s).$$

Since $v\, \eta^{(n)}_{\xi(t)}$ and $\gamma(t)$ are independent that concludes the proof. $\square$

2. Evolution in odd-dimensional spaces

Now, let us assume that $n = 2l+3$, $l = 0,1,2,\ldots$ and θ_k has a $(n-1)$-Erlang distribution, that is $g(t) = \dfrac{\lambda^{n-1}}{\Gamma(n-1)} t^{n-2} e^{-\lambda t}$. It follows from Lemma 2 that the pdf $f_n(t)$ of the random variable $\eta_i^{(n)} \theta_i$ has the form

$$f_n(t) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)\Gamma(n-1)} \lambda \int_0^1 \frac{(\lambda t)^{2l+1}}{x^{2l+2}} e^{-\lambda t/x} (1-x^2)^l dx$$

or equivalently,

$$f_n(t) = \frac{\lambda \Gamma\left(l+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma(l+1)\Gamma(2l+2)} \sum_{k=0}^{l} \binom{l}{k} (-1)^k (\lambda t)^{2k} \int_{\lambda t}^{\infty} s^{2(l-k)} e^{-s} ds,$$

for $t \ge 0$. Furthermore, the following equivalent expression can be found after some algebraic simplifications

$$f_n(t) = \frac{\lambda e^{-\lambda t}}{l!\, 2^{2l+1}} \sum_{k=0}^{l} (-1)^k \frac{(2(l-k)!)}{k!(l-k)!} \sum_{m=0}^{2(l-k)} \frac{(\lambda t)^{2k+m}}{m!}. \tag{5}$$

We have $f_n(t) = f_n(-t)$ for the case when $t < 0$.

3. Evolution in three dimensions

Let us consider the particular case when $n = 3$. Thus, by taking into account Lemma 2, we have that $\eta_i^{(3)}$ is uniformly distributed on $[-1,1]$.

Let random variables θ_k, $k = 0, 1, 2, \ldots$ be 2-Erlang distributed, i.e., $g(t) = \lambda^2 t e^{-\lambda t}$, $\lambda > 0$, $t \geq 0$.

For this particular case, the Laplace transform of $R(t)$, say $\widehat{R}(s)$, is of the form

$$\widehat{R}(s) = \int_0^\infty R(t)e^{-st}dt = \sum_{k=0}^\infty \int_0^\infty g^{*(k)}(t)e^{-st}dt = \sum_{k=0}^\infty \left(\frac{\lambda}{\lambda+s}\right)^{2k} = \frac{(\lambda+s)^2}{s^2+2\lambda s},$$

and the Laplace transform $\widehat{V}(s,u)$ of $V(t,u)$ can be written as

$$\widehat{V}(s,u) = \frac{2\lambda + s - (\lambda us + \lambda^2 us + s + 2)e^{-(\lambda+s)u}}{(\lambda+s)^2}.$$

Therefore, the Laplace transform $\widehat{F}_\gamma(s,u)$ of $F_{\gamma(t)}(u)$ is given by

$$\widehat{F}_\gamma(s,u) = \widehat{R}(s)\widehat{V}(s,u) = \frac{2\lambda + s - (\lambda us + \lambda^2 us + s + 2)e^{-(\lambda+s)u}}{s(s+2\lambda)}. \tag{6}$$

After applying the inverse Laplace transform to $\widehat{F}_\gamma(s,u)$, we obtain for $t > u > 0$

$$F_{\gamma(t)}(u) = e^{-2\lambda t} - (2+\lambda u)e^{-\lambda t}\sinh(\lambda(t-u)) + 2e^{-\lambda t}\sinh(\lambda t) - (\lambda u + 1)e^{-\lambda(2t-u)}.$$

Thus, we have the limit result

$$\lim_{t\to+\infty} F_{\gamma(t)}(u) = 1 - e^{-\lambda u} - \frac{\lambda u}{2}\, e^{-\lambda u}.$$

Taking into account Lemma 3 we can obtain the corresponding expression for $F_{\Delta(t)}(s)$.

It follows from Eq. (5) that $\eta_i\theta_i$ has the Laplace distribution with pdf $f_3(t) = \frac{1}{2}\,\lambda e^{-\lambda|t|}$.

Therefore, the Fourier transform of $P\left(v \sum_{i=0}^k \eta_i\theta_i \leq y\right)$ is given by

$$\int_{-\infty}^\infty e^{-i\lambda y} dP\left(v\sum_{i=0}^k \eta_i\theta_i \leq y\right) = \left(\frac{\lambda^2}{\lambda^2 + v^2\alpha^2}\right)^k.$$

On the other hand, since

$$F_{\hat{x}^{(3)}(t)}(y) = P\left(v\sum_{i=0}^{\xi(t)} \eta_i^{(3)}\theta_i \leq y\right) = \sum_{k=0}^\infty P\left(v\sum_{i=0}^k \eta_i^{(3)}\theta_i \leq y\right) P(\xi(t) = k)$$

then the characteristic function of $\hat{\mathbf{x}}^{(3)}(t)$, say,

$$\varphi_{\hat{\mathbf{x}}^{(3)}(t)}(\boldsymbol{\alpha}) = \mathbf{E}[e^{-i\alpha\hat{x}^{(3)}(t)}] = \int_{-\infty}^{\infty} e^{-i\alpha y} dF_{\hat{x}^{(3)}(t)}(y)$$

can be calculated as follows

$$\varphi_{\hat{\mathbf{x}}^{(3)}(t)}(\boldsymbol{\alpha}) = \sum_{k=0}^{\infty} \left(\frac{\lambda^2}{\lambda^2 + v^2\alpha^2} \right)^k P(\xi(t) = k)$$

$$= e^{-\lambda t} \sum_{k=0}^{\infty} \left(\frac{\lambda^2}{\lambda^2 + v^2\alpha^2} \right)^k \left(\frac{(\lambda t)^{2k}}{2k!} + \frac{(\lambda t)^{2k+1}}{(2k+1)!} \right)$$

Let us define $\Phi = \dfrac{\lambda^2}{\sqrt{\lambda^2 + v^2\alpha^2}}$, then

$$\varphi_{\hat{\mathbf{x}}^{(3)}(t)}(\boldsymbol{\alpha}) = e^{-\lambda t} \left[\cosh \Phi t + \frac{\lambda^2 + v^2\alpha^2}{\lambda^2} \sinh \Phi t \right].$$

Therefore, by using the inverse Fourier transform, we can obtain $F_{\hat{\mathbf{x}}^{(3)}(t)}(\mathbf{y})$.

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