ESTIMATION OF STATIONARY PRODUCTIVITY OF ONE-PHASE SYSTEM WITH A STORAGE

ANATOLY A. POGORUI

Abstract. In this paper we study a system consisting of an unreliable aggregate, a customer and an accumulator located between the aggregate and the customer. Operation of the system is modeled by random evolution in a semi-Markov medium. The stationary average productivity of the system is investigated as a function of the system parameters.

1. INTRODUCTION

The first investigations in the reliability theory of systems with storages were the papers of B.A. Sevastyanov [1], G.N. Cherkesov [2], J.A.Bazucott [3].

The technique of jump Markovian processes is used in these works in the reliability theory of systems with storage, including also [4], [5]. In some works, for example [6], the requirement of exponential distributions of faultless operation and elements restoration is removed and semi-Markov models of operation system with storages are studied.

A general demerit of the mentioned above methods is a condition instant addition of reserves in storages. It is rather severe assumption for the most of investigated system.

In papers [7], [8] the apparatus of Markovian evolutions is used for investigation of a two-phase system with two storages. This method removes the requirement of instant addition of reserves in storages. In [9] the system effectiveness is studied by using the phase merging algorithm [10].

In this paper operation of the system is modeled by random evolution in an alternating semi-Markov medium.

2. ONE-PHASE SYSTEM WITH A STORAGE

Consider the system, which consists of an aggregate A a bunker (accumulator, container) B and a customer C. The aggregate A is unreliable: the time of non-failure operation τ_1 of A is a random variable with a general distribution function $G_1(t)$, and the renewal time τ_0 of A is a random variable with a general distribution function $G_0(t)$.

Let us denote by V the volume of the accumulator B. The system operates in the following way:

- If the aggregate A operates and the container B is filled then the product goes from A to C with the rate a_0.

2000 *Mathematics Subject Classification.* Primary 60K15; Secondary 60K20.

Key words and phrases. Stationary average productivity, stationary distribution, semi-Markov process, ergodic theory.

- If the aggregate A operates and the container B is not filled up, then the product goes from A to C with the rate a_0. In addition, the product goes from A to B with the rate b.
- If the aggregate A has failed and the container B is not empty then the product goes from B to C with the rate a_1.
- If the aggregate A has failed and the container B is empty then the customer C does not receive the production.

Denote by $V(T)$ the product volume which is received by the customer during time $[0,T]$. The limit $K = lim_{T\to\infty} V_T/T$, if it exists, will be called the stationary average productivity of the system.

Our purpose is to calculate K as a function of the following system parameters: $a_0, a_1, b, V, G_0(t), G_1(t)$.

Let us consider the following random process $n(t)$:

$$n(t) = \begin{cases} 0, & \text{if aggregate } A \text{ fails;} \\ 1, & \text{if aggregate } A \text{ operates.} \end{cases}$$

The process $n(t)$ is semi-Markov with the phase space $E = \{0,1\}$. A sojourn time of $n(t)$ in state 0 (resp. 1) is τ_0 (resp. τ_1). Let $v(t)$ be the volume of the product in the container B at time t. We introduce the following functions on $W = 0,1 \times [0,V]$:

$$C(w) = \begin{cases} b, & w = (1,v), v < V; \\ -c_0, & w = (0,v), v > 0; \\ 0, & \text{other cases.} \end{cases}$$

$$f(w) = \begin{cases} a_0, & w = (1,v), v < V; \\ a_1, & w = (0,v), v > 0; \\ 0, & w = (0,0). \end{cases}$$

It is easy to see that $v(t)$ satisfies the following evolution equation

$$\begin{aligned} \frac{dv(t)}{dt} &= C(n(t), v(t)), \\ v(t) &= v_0 \in [0,V]. \end{aligned}$$

This equation determines the random evolution in the semi-Markov medium $n(t)$ [10].

Let us consider the following bivariate random process $\xi(t) = (n(t), v(t))$. Then we have $V(T) = \int_0^T f(\xi(t))dt$ and $\lim_{T\to\infty} \frac{V_T}{T} = \lim_{T\to\infty} 1/T \int_0^T f(\xi(t))dt = K$ (provided that K exists). If the random process $\xi(t)$ has the stationary distribution $\xi(\cdot)$ then according to the ergodic theory, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\xi(t))dt = \int_W f(w)\rho(dw). \tag{1}$$

Hence, to compute the stationary productivity K of the system, it suffices to determine the stationary distribution $\rho(\cdot)$. We assume that $G_i(t)$ is nondegenerate and there are exist $g_i(t) = dG_i(t)/dt$, $m_i = \int_0^\infty t g_i(t)dt$, $i = 0,1$.

Let us put $\widehat{g}_i(s) = \int_0^\infty e^{st} g_i(t)dt$, $r_i(t) = \frac{g_i(t)}{1-G_i(t)}$, $i = 0,1$. Suppose the following conditions are fulfilled:

C_1. *If $m_0 a_1 - m_1 b \neq 0$, then there exists $s_0 \neq 0$ such that*

$$\widehat{g}_0(a_1 s_0)\widehat{g}_1(-b s_0) = 1, \tag{2}$$

C_2.

$$I_0 = \int_0^\infty \exp\{a_1 s_0 t - \int_0^t r_0(l)dl\}dt < \infty, \tag{3}$$

$$I_1 = \int_0^\infty \exp\{-b s_0 t - \int_0^t r_1(l)dl\}dt < \infty. \tag{4}$$

In the case when $m_0 a_1 = m_1 b$ instead of C_2, we will assume that the following condition being satisfied

C_2'. *There exist*

$$m_0^{(2)} = \int_0^\infty t^2 g_0(t)dt, \quad m_1^{(2)} = \int_0^\infty t^2 g_1(t)dt.$$

Let us consider sufficient conditions under which C_1 is fulfilled.

Lemma 1. *If $m_0 a_1 \neq m_1 b$ and there exist $l_1 < l_2$, $p_1 < p_2$, $\sigma_1 > 0$, $\sigma_2 > 0$ such that $g_0(t) > \sigma_1$, $t \in [l_1, l_2]$, $g_1(t) > \sigma_2$, $t \in [p_1, p_2]$ and $bp_1 < a_1 l_2$, $a_1 l_1 < bp_2$ then there exists $s_0 \neq 0$, which satisfies condition C_1.*

Proof. Let us put $f(s) = \widehat{g}_0(a_1 s)\widehat{g}_0(-bs)$. We have $f'(0) = m_0 a_1 - m_1 b \neq 0$ and $f(0) = 1$. If $m_0 a_1 - m_1 b < 0$ then

$$f(s) \geq \int_{l_1}^{l_2} e^{a_1 st} g_0(t)dt \int_{p_1}^{p_2} e^{-bst} g_1(t)dt$$
$$\geq \frac{\sigma_1 \sigma_2}{a_1 b}(e^{a_1 l_2 s} - e^{a_1 l_1 s})(e^{-bp_1 s} - e^{-bp_2 s}) \to +\infty, \quad s \to +\infty.$$

The case $f'(0) = m_0 a_1 - m_1 b > 0$ can be proved in much the same way as previous one.

Remark. We should notice that condition C_2 is a consequence of condition C_1, provided that there exist $\delta_1, \delta_2 > 0$ and $T < +\infty$ such that $r_i(t) \geq \delta_i$ for $t > T$, $i = 1, 2$.

Theorem 1. *If $m_0 a_1 \neq m_1 b$ and there exists $s_0 \neq 0$, such that conditions C_1, C_2 are fulfilled then there exists the stationary distribution of $\xi(t)$ with the absolutely continuous part:*

$$\rho(0, v) = c s_0 I_0 e^{s_0 v}, v > 0;$$
$$\rho(1, v) = c\widehat{g}_0(a_1 s_0) s_0 I_1 e^{s_0 v}, v < V. \tag{5}$$

and the singular part (or atoms) $\rho[0, 0]$, $\rho[1, V]$ at points $(0, 0)$, $(1, V)$ respectively:

$$\rho[0, 0] = c(I_0 - m_0), \quad \rho[1, V] = c\widehat{g}_0(a_1 s)(m_1 - I_1)e^{s_0 V}, \tag{6}$$

where

$$c = [I_0 e^{s_0 V} - I_1 g_0(a_1 s_0) - m_0 + m_1 g_0(a_1 s_0)e^{s_0 V}]^{-1}.$$

If $m_0 a_1 = m_1 b$ and instead of C_2 conditions C_2' is fulfilled then there exists the stationary distribution of $\xi(t)$ with the absolutely continuous part:

$$\rho(i, v) = c m_i, i = 0, 1$$

and the singular part (or atoms) at points $(0,0)$, $(1,V)$*:*

$$\rho[0,0]=ca_1(m_0^2)/2, \rho[1,V]=cb(m_1^2)/2,$$

where $c^{-1}=V(m_0+m_1)+a_1(m_0^2)/2+b(m_1^2)/2$.

Proof. Let us consider the three-component process $\xi(t)=(\tau(t),n(t),v(t))$ in the phase space $Z=[0,\infty)\times 0,1\times[0,V]$, where $\tau(t)=t-sup\{u<t:x(u)\neq x(t)\}$.

It is well known that the process $\xi(t)$ is markovian and its infinitesimal operator is of the following form [10], [11]:

$$\begin{aligned} A\varphi(\tau,n,v) &= \frac{\partial}{\partial\tau}\varphi(\tau,n,v)+r_x(\tau)[P\varphi(0,n,v)-\varphi(\tau,n,v)] \\ &+ C(n,v)\frac{\partial}{\partial v}\varphi(\tau,n,v), \end{aligned}$$

where the function $\varphi(\tau,n,v)$ is continuously differentiable with respect to τ and v. In addition $\varphi(\tau,n,v)$ satisfies boundary conditions $\varphi'_\tau(\tau,1,0)=\varphi'_\tau(\tau,0,V)=0$, $(\tau,n,v)\in Z$. Here $P\varphi(0,1,v)=\varphi(0,0,v)$, $P\varphi(0,0,v)=\varphi(0,1,v)$.

In more details the operator A has the form:

$$\begin{aligned} A\varphi(\rho,1,v) &= \frac{\partial}{\partial\tau}\varphi(\tau,n,v)+r_1(\tau)[P\varphi(0,0,v)-\varphi(\tau,1,v)] \\ &+ b\frac{\partial}{\partial v}\varphi(\tau,1,v), \quad 0\le v\le V, \\ A\varphi(\rho,0,v) &= \frac{\partial}{\partial\tau}\varphi(\tau,0,v)+r_0(\tau)[P\varphi(0,1,v)-\varphi(\tau,0,v)] \\ &- a_1\frac{\partial}{\partial v}\varphi(\tau,0,v), \quad 0\le v\le V, \\ A\varphi(\rho,0,0) &= \frac{\partial}{\partial\tau}\varphi(\tau,0,0)+r_0(\tau)[P\varphi(0,1,0)-\varphi(\tau,0,0)] \\ A\varphi(\rho,1,V) &= \frac{\partial}{\partial\tau}\varphi(\tau,1,V)+r_1(\tau)[P\varphi(0,0,V)-\varphi(\tau,1,V)] \\ \frac{\partial}{\partial\tau}\varphi(\tau,1,0) &= \frac{\partial}{\partial\tau}\varphi(\tau,0,V)=0. \end{aligned}$$

If there exists the stationary distribution $\rho(\cdot)$ of $\xi(t)$ then for any function $\varphi(\cdot)$ from the domain of A we have

$$\int_Z A\varphi(z)\rho(dz)=0. \qquad (7)$$

The analysis of the properties of the process $\xi(t)$ leads up to the conclusion that the stationary distribution $\rho(\cdot)$ has singularities at points at points $(\tau,0,0)$, $(\tau,1,V)$, which is denoted by $\rho[\tau,0,0]$, $\rho[\tau,1,V]$.

Passing from the operator A to the conjugate operator A^* and taking into account Eqs.(7), we have

$$\begin{aligned} \frac{\partial}{\partial \tau}\rho(\tau,0,v)+r_0(\tau)\rho(\tau,0,v)-a_1\frac{\partial}{\partial v}\rho(\tau,0,v) &= 0, &(8)\\ \frac{\partial}{\partial \tau}\rho(\tau,1,v)+r_1(\tau)\rho(\tau,1,v)+b\frac{\partial}{\partial v}\rho(\tau,1,v) &= 0, &(9)\end{aligned}$$

$$\int_0^\infty r_1(\tau)\rho(\tau,1,v)d\tau=\rho(0,0,v), \quad (10)$$

$$\int_0^\infty r_0(\tau)\rho(\tau,0,v)d\tau=\rho(0,1,v), \quad (11)$$

$$\rho(\tau,0,v)=\rho(\tau,1,v)=0,$$

$$\frac{\partial}{\partial \tau}\rho[\tau,0,0]+r_0(\tau)\rho[\tau,0,0]-a_1\rho(\tau,0,0+)=0, \quad (12)$$

$$\frac{\partial}{\partial \tau}[\tau,1,V]+r_1(\tau)\rho[\tau,1,V]-b\rho(\tau,1,V-)=0, \quad (13)$$

$$\rho[\tau,1,V]=\rho[\tau,0,0]=\rho[0,1,V]=\rho[0,0,0]=0.$$

Taking into account the boundary conditions, we get

$$\begin{aligned} \int_0^\infty r_0(\tau)\rho[\tau,0,0]d\tau &= b\int_0^\infty \rho(\tau,1,0+)d\tau, &(14)\\ \int_0^\infty r_1(\tau)\rho[\tau,1,V]d\tau &= a_1\int_0^\infty \rho(\tau,0,V-)d\tau, &(15)\end{aligned}$$

By solving Eqs.(8),(9), we obtain

$$\begin{aligned} \rho(\tau,0,v) &= f_0(v+a_1\tau)e^{-\int_0^\tau r_0(t)dt}, &(16)\\ \rho(\tau,1,v) &= f_1(v-b\tau)e^{-\int_0^\tau r_1(t)dt}, &(17)\end{aligned}$$

$f_1\in C^1, i=0,1.$

Substituting Eqs.(16), (17) into Eqs.(10), (11), we have

$$\begin{aligned} \int_0^\infty f_0(v+a_1t)e^{-\int_0^t r_0(s)ds}r_0(t)dt &= f_1(v), &(18)\\ \int_0^\infty f_1(v-bt)e^{-\int_0^t r_1(s)ds}r_1(t)dt &= f_0(v). &(19)\end{aligned}$$

It follows from Eqs.(18) and (19) that

$$\int_0^\infty\int_0^\infty f_i(v-bt+a_1l)e^{-\int_0^t r_1(s)ds}e^{-\int_0^l r_0(s)ds}r_1(t)r_0(l)dtdl=f_i(v), \quad (20)$$

$i=0,1.$

Taking into account Eqs.(18)-(20), we will find the function $f_i(v)$ in the following form

$$f_i(v)=c_ie^sv, c_i>0, i=0,1. \quad (21)$$

Next, substituting (21) in Eq.(20) and taking into account that $e^{-\int_0^t r_i(s)ds}=1-G_i(t)$, we have

$$\widehat{g}_0(a_1s)\widehat{g}_1(-bs)=1.$$

It follows from condition C_1 and Eqs.(16)-(19) that

$$\begin{aligned}\rho(\tau,0,v) &= c_0 e^{s_0 v + a_1 s_0} e^{-\int_0^\tau r_0(t)dt}, \qquad (22)\\ \rho(\tau,1,v) &= c_0 \widehat{g}_0(a_1 s) e^{s_0 v - b s_0 \tau} e^{-\int_0^\infty r_1(t)dt}. \qquad (23)\end{aligned}$$

It easily verified that if $m_0 a_1 \neq m_1 b$, then only $s_0 \neq 0$ that satisfies condition C_1 in addition to that satisfies Eqs. (14), (15). In the case, when $m_0 a_1 = m_1 b$, Eqs. (14),(15) have solution $s_0 = 0$.

Substituting (22) and (23) into (12) and (13) respectively, we have

$$\begin{aligned}\rho[\tau,0,0] &= c_0 a_1 \int_0^\tau e^{a_1 s_0 t} dt e^{-\int_0^\tau r_0(s)ds} \qquad (24)\\ \rho[\tau,1,V] &= c_0 \widehat{g}_0(a_1 s_0) b e^{s_0 V} \int_0^\tau e^{-b s_0 t} dt e^{-\int_0^\tau r_1(s)ds} \qquad (25)\end{aligned}$$

The normalizing coefficient c_0 of Eqs.(22)-(25) we can obtain from the condition

$$\sum_{i=0}^{1} \int_0^V \int_0^\infty \rho(\tau,i,v) d\tau dv + \int_0^\infty \rho[\tau,0,0] d\tau + \int_0^\infty \rho[\tau,1,V] d\tau = 1.$$

To complete the proof, note that

$$\begin{aligned}\rho(i,v) &= \int_0^\infty \rho(\tau,i,v)d\tau, \quad i = 0,1,\\ \rho[0,0] &= \int_0^\infty \rho[\tau,0,0]d\tau, \quad \rho[1,V] = \int_0^\infty \rho[\tau,1,V]d\tau.\end{aligned}$$

It follows from the theorem and Eq.(1) that if $m_0 a_1 \neq m_1 b$ then the stationary average productivity of the system is of the following form

$$\begin{aligned}K = c[a_0 I_0(e^{s_0 V} - 1) + a_1 \widehat{g}_0(a_1 s_0) I_1 (e^{s_0 V} - 1) +\\ + a_0 \widehat{g}_0(a_1 s_0)(m_1 - I_1) e^{s_0 V}],\end{aligned}$$

where $c = [I_0 e^{s_0 V} - I_1 \widehat{g}_0(a_1 s_0) - m_0 + m_1 \widehat{g}_0(a_1 s_0) e^{s_0 V}]^{-1}$.

In the case when $m_0 a_1 = m_1 b$ which we will call the balance case, we have

$$K = c[(m_0 a_1 + m_1 b)V + a_0 b(m_1^(2)))/2],$$

where $c^{-1} = V(m_0 + m_1) + a_1(m_0^(2)))/2 + b(m_1^(2)))/2$.

3. EXAMPLE

Let $g_0(t) = q^2 t e^(- qt)$, $q > 0$, $g_1(t) = p^2 t e^(- pt)$, $p > 0$. In this case Eq.(2) has the form

$$\widehat{g}_0(a_1 s)\widehat{g}_1(-bs) = \left(\frac{q}{q - a_1 s}\right)^2 \left(\frac{p}{p + bs}\right)^2 = 1, \qquad (26)$$

with the conditions

$$a_1 s < q, bs > -p. \qquad (27)$$

By solving Eq.(26), we obtain the following solutions that satisfy the conditions (27): $s_0^{'} = 0$ and $s_0^{''} = \frac{qb - a_1 p}{a_1 b}$.

Since $r_0(t) = \frac{q^2 t}{1+qt} \to q > 0$ as $q \to \infty$ and $r_1(t) = \frac{p^2 t}{1+pt} \to p > 0$ as $p \to \infty$, it follows from the remark that there exist the integrals I_0, I_1. Hence, we can apply the theorem.

If we have the balance case that is $qb = a_1 p$ then

$$K = \frac{2(a_1/p + b/q)V + 3a_0 b/q^2}{2V(1/p + 1/q) + 3a_1/p^2 + 3b/q^2}.$$

In the disbalance case when $qb \neq a_1 p$, $s_0^{,,} = \frac{qb - a_1 p}{a_1 b}$, we have

$$K = c[a_0 I_0 \left(e^{\frac{(qb-a_1 p)V}{a_1 b}} - 1 \right) + a_1 \widehat{g}_0 \left(\frac{a_1 qb - a_1^2 p}{a_1 b} \right) I_1 (e^{\frac{(qb-a_1 p)V}{a_1 b}} - 1)$$

$$+ a_0 \widehat{g}_0 (a_1 s_0)(m_1 - I_1) e^{s_0 V}],$$

where

$$c = \left[I_0 e^{\frac{(qb-a_1 p)V}{a_1 b}} - I_1 \widehat{g}_0 \left(\frac{qb - a_1 p}{b} \right) - \frac{1}{p} + \frac{1}{q} \widehat{g}_0 \left(\frac{qb - a_1 p}{b} \right) e^{\frac{(qb-a_1 p)V}{a_1 b}} \right]^{-1},$$

$$\widehat{g}_0(\lambda) = \left(\frac{q}{q + \lambda} \right)^2$$

$$I_0 = \int_0^\infty (1 + qt) e^{\frac{-a_1 pt}{b}} = \frac{qb^2 + a_1 pb}{a_1^2 p^2},$$

$$I_1 = \int_0^\infty (1 + pt) e^{\frac{-bpt}{a_1}} = \frac{pa_1^2 + a_1 qb}{b^2 q^2}.$$

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Anatoly A. Pogorui, Zhitomir State University, Velyka Berdychivska str. 40, 10008 Zhitomir, Ukraine. *E-mail: pogor2@mail.ru*