EVOLUTION IN MULTIDIMENSIONAL SPACES

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Abstract. In this paper, we study evolutions in multidimensional spaces. By reducing multidimensional cases to one-dimensional, the properties of random evolutions in some semi-Markov media are studied.

Keywords. Random evolution, Brownian motion, Semi-Markov process..

2010 Mathematics Subject Classification. Primary 60J25; Secondary 60J65.

1 Introduction

The Goldstein-Kac telegraph process models a random motion of particles with constant velocity and it is usually proposed as an alternative to diffusion models. This basic telegrapher process was introduced by S.Goldstein and M.Kac in the mid-50s of the 20th century. The process has been extended in many manners since then [1]– [7] and references therein.

A great deal of research devoted to generalization of the telegraph process to higher-dimensional spaces $\mathbb{R}^n$, $n \geq 2$ [4], [6], [7] and other. The main difficulty in multidimensional cases is a continuum of possible directions. This difficulty can be remedied by discretization of possible directions and there are many papers devoted to random motions with a finite number of directions of particle motion [6]- [8] and the bibliography therein.

It should be noted that all papers on random motions in multidimensional spaces are devoted to analysis of models in which motions are driven by a homogeneous Poisson process, so their processes are Markovian [4], [6]- [8] and references therein.

In this paper, it is considered a non-Markovian generalization of the telegrapher's random process to higher dimensions where motion is driven by some semi-Markov processes. We investigate random motions with uniformly distributed on the unit sphere directions in higher dimension by reducing it to the one-dimensional random motion. In so doing, we reduce the study of random motion in multidimensional spaces to the study of well known renewal-reward processes.

2 Random motion in multidimensional spaces

Suppose $\theta_k > 0$, $k = 1, 2, ...$ are i.i.d random variables with a cumulative distribution function (cdf) $G(t)$ and there exists the probability density function (pdf) $g(t) = \frac{dG(t)}{dt}$. Let us put $\tau_m = \Sigma_{k=1}^{m} \theta_k$, $m = 1, 2, ...$, $\tau_0 = 0$ and consider the process $\xi(t) = \max\{m \geq 0 : \tau_m \leq t\}$.

We assume that the particle moves in n-dimensional space $\mathbb{R}^n$ in the following manner: Starting at the origin of coordinates $(0, 0, ..., 0) \in \mathbb{R}^n$ the particle continues its motion with constant velocity which norm is $v > 0$ and direction is a unit vector $\vec{\eta}_0^{(n)}$ with uniform distribution on the unit sphere

$$\Omega^n = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \ldots + x_n^2 = 1\}.$$

Then at each instant τ_m, $m \geq 1$ the particle takes on the new direction with uniform distribution on Ω^n.

Denote by $\vec{x}^{(n)}(t)$ the particle's position at time $t > 0$. Then $\vec{x}^{(n)}(t)$ in anytime t can be expressed as follows

$$\vec{x}^{(n)}(t) = v \sum_{i=1}^{\xi(t)} \vec{\eta}_i^{(n)} \theta_i + v \vec{\eta}_{\xi(t)}^{(n)} (t - \tau_{\xi(t)}) \tag{2.1}$$

Eq. (2.1) determines the random evolution of the particle in the semi-Markov media $\xi(t)$.

Our purpose is to investigate the distribution of the particle position $\vec{x}^{(n)}(t)$ at time $t > 0$.

Let us denote by $x^{(n)}(t)$ the length of the projection of $\vec{x}^{(n)}(t)$ on a fixed line, say the x_1-axis. The probabilistic properties of a random vector are completely determined by those of its projection on x_1-axis, and using the latter it is frequently possible to avoid analysis in n dimensions. It is important to know the relationship between the distribution function of $\vec{x}^{(n)}(t)$ and the cdf of $x^{(n)}(t)$.

It is shown in [9]that the probability density function $f_{\eta_i^{(n)}}(x)$ of the length $\eta_i^{(n)}$ of the projection of $\vec{\eta}_i^{(n)}$ on a fixed line is as follows

$$f_{\eta_i^{(n)}}(x) = \begin{cases} \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})}(1 - x^2)^{\frac{n-3}{2}}, & |x| \leq 1; \\ 0, & |x| > 1. \end{cases} \tag{2.2}$$

Therefore, the distribution function $F(t)$ of $\eta_i^{(n)} \theta_i$ is of the following form

$$F(t) = \begin{cases} \frac{1}{2} + \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int\limits_0^1 G(\frac{t}{x})(1-x^2)^{\frac{n-3}{2}} dx, & t \geq 0. \\ \frac{1}{2} - \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int\limits_0^1 G(-\frac{t}{x})(1-x^2)^{\frac{n-3}{2}} dx, & t < 0; \end{cases}$$

Let us denote by $\widehat{x}^{(n)}(t) = x^{(n)}(\tau_{\xi(t)})$ the particle position at the instant $\xi(t)$.

It is easy to see that $\widehat{x}^{(n)}(t)$ is a renewal-reward process of the following form

$$\widehat{x}^{(n)}(t) = v \sum_{i=1}^{\xi(t)} \eta_i^{(n)} \theta_i.$$

Therefore, the study of the multidimensional random motion at renewal epochs may be reduce to the investigation of renewal-reward processes [10]. These processes have been studied for a long time and there exists a great variety of works devoted to renewal-reward processes, say papers on large deviations [11] and references therein. Consequently, these results can be applied for the study of random motions.

3 Random motion in three-dimensional space

The cdf $F(t)$ of $\eta_i^{(3)}\theta_i$, $i \geq 1$. Since $\eta_i^{(3)}$ is uniformly distributed on $[-1, 1]$ it is easy to see that $F(t) = P(\eta_i^{(3)}\theta_i \leq t)$ has the following form

$$F(t) = \begin{cases} \frac{1}{2} + \frac{1}{2} \int\limits_0^1 G(\frac{t}{x})dx, & t \geq 0. \\ \frac{1}{2} - \frac{1}{2} \int\limits_0^1 G(-\frac{t}{x})dx, & t < 0; \end{cases}$$

For the pdf $f(t) = \frac{d}{dt}F(t)$ we have $f(t) = \frac{1}{2}\int_t^{+\infty} \frac{g(x)}{x}dx$, whence for $t \geq 0$

$$t\frac{d}{dt}f(t) = -\frac{1}{2}g(t), \tag{3.1}$$

and for $t < 0$

$$t\frac{d}{dt}f(t) = -\frac{1}{2}g(-t). \tag{3.2}$$

Suppose that θ_k, $k \geq 1$ are Erlang-2 distributed with pdf $g(t) = \lambda^2 t e^{-\lambda t}$, $\lambda > 0, t > 0$. Substituting it into Eqs.(3.1),(3.2), we get

$$f(t) = \frac{1}{2}\lambda e^{-\lambda|t|}.$$

Therefore, the cdf $F_{\widehat{x}}(y)$ of $\widehat{x}^{(3)}(t) = x^{(3)}(\tau_{\xi(t)})$ is of the following form

$$F_{\widehat{x}}(y) = P\left(v\sum_{i=0}^{\xi(t)}\eta_i\theta_i \leq y\right) = \sum_{k=0}^{\infty} P\left(v\sum_{i=0}^{k}\eta_i\theta_i \leq y\right) P(\xi(t) = k). \quad (3.3)$$

Let us denote by $\varphi(z,t) = Ee^{-iz\widehat{x}^{(3)}(t)} = \int\limits_{-\infty}^{\infty} e^{-izy}dF_{\widehat{x}}(y)$ the characteristic function of $\widehat{x}^{(3)}(t)$.

It is easy to see that

$$
\begin{aligned}
P(\xi(t) = k) &= P(\tau_k \leq t) - P(\tau_{k+1} \leq t) = G^{*(k)}(t) - G^{*(k+1)}(t) \\
&= e^{-\lambda t}\left(\frac{(\lambda t)^{2k}}{2k!} + \frac{(\lambda t)^{2k+1}}{(2k+1)!}\right),
\end{aligned}
$$

where $G(t) = 1 - e^{-\lambda t} - \lambda t e^{-\lambda t}$ and $G^{*(k)}(t)$ is the k-fold convolution of $G(t)$.

Whence, in view of Eq.(3.3), we have

$$
\begin{aligned}
\varphi(z,t) &= \sum_{k=0}^{\infty}\left(\frac{\lambda^2}{\lambda^2 + v^2 z^2}\right)^k P(\xi(t) = k) \\
&= e^{-\lambda t}\sum_{k=0}^{\infty}\left(\frac{\lambda^2}{\lambda^2 + v^2 z^2}\right)^k \left(\frac{(\lambda t)^{2k}}{2k!} + \frac{(\lambda t)^{2k+1}}{(2k+1)!}\right) \\
&= \frac{e^{-\lambda t}}{2}\left(e^{\frac{\lambda^2}{\sqrt{\lambda^2+v^2 z^2}}t} + e^{\frac{-\lambda^2}{\sqrt{\lambda^2+v^2 z^2}}t}\right) \\
&\quad + (\lambda^2 + v^2 z^2)\frac{e^{-\lambda t}}{2\lambda^2}\left(e^{\frac{\lambda^2}{\sqrt{\lambda^2+v^2 z^2}}t} - e^{\frac{-\lambda^2}{\sqrt{\lambda^2+v^2 z^2}}t}\right).
\end{aligned} \quad (3.4)
$$

Suppose $\lambda \to +\infty$ and $v \to +\infty$ such that $\frac{v^2}{\lambda} \to \sigma^2 > 0$ then it follows from Eq.(3.4) that

$$\varphi(z,t) \to e^{-\frac{1}{2}\sigma^2 t z^2}. \quad (3.5)$$

Therefore, $\widehat{x}^{(3)}(t)$ converges weakly to the Wiener process and consequently $\vec{x}^{(3)}(t)$ converges weakly to the Wiener process in $\mathbb{R}^3$.

Now suppose that θ_k, $k \geq 1$ have the Maxwell pdf $g(t) = \sqrt{\frac{2}{\pi}}t^2 e^{-\frac{t^2}{2}}$, then it follows from Eqs.(3.1),(3.2) that the random variable $\varsigma_i = \eta^{(3)}\theta_i$ has the normal distribution as follows

$$f(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}.$$

For this case the characteristic function $\varphi(\lambda, t) = Ee^{-\lambda i \hat{x}^{(3)}(t)}$ has the following form

$$\varphi(\lambda, t) = \sum_{k=0}^{\infty} e^{-\frac{k\lambda^2 v^2}{2}} (G^{*(k)}(t) - G^{*(k+1)}(t)).$$

Then passing to Fourier's transform of $\varphi(\lambda, t)$ w.r.t. t, we have

$$\varphi(\lambda, s) = \int_{-\infty}^{\infty} e^{-ist} \varphi(\lambda, t) dt = \sum_{k=0}^{\infty} e^{-\frac{k\lambda^2 v^2}{2}} \varphi^k(s)(1 - \varphi(s)),$$

where $\varphi(s)$ is the characteristic function of the Maxwell distribution

$$\varphi(s) = \frac{2}{\sqrt{\pi}} \sum_{l=0}^{\infty} \frac{(i\sqrt{2}s)^l}{l!} \Gamma\left(\frac{3+l}{2}\right)$$

4   Random motion in five-dimensional space

It follows from the lemma that the pdf $f_{\eta_i^{(5)}}(x)$ of $\eta_i^{(5)}$ has the following form

$$f_{\eta_i^{(5)}}(x) = \begin{cases} \frac{3}{4}(1 - x^2), & x \in [-1, 1]; \\ 0, & x \overline{\in} [-1, 1]. \end{cases}$$

The cdf of $\eta_i^{(5)} \theta_i$ is as follows

$$F(t) = P(\eta_i^{(5)} \theta_i \leq t) = \begin{cases} \frac{1}{2} + \frac{3}{4} \int_0^1 G(\frac{t}{x})(1 - x^2) dx, & t \geq 0; \\ \frac{1}{2} - \frac{3}{4} \int_0^1 G(-\frac{t}{x})(1 - x^2) dx, & t < 0. \end{cases}$$

Suppose that there exists the pdf $g(t) = \frac{dG(t)}{dt}$, then for $t \geq 0$ we have

$$f(t) = \frac{dF(t)}{dt} = \frac{3}{4} \int_0^1 g\left(\frac{t}{x}\right) \frac{1}{x}(1 - x^2) dx,$$

and for $t < 0$

$$f(t) = \frac{dF(t)}{dt} = \frac{3}{4} \int_0^1 g\left(-\frac{t}{x}\right) \frac{1}{x}(1 - x^2) dx.$$

Assume that θ_k, $k = 0, 1, 2, \ldots$ are Erlang distributed with the pdf $g(t) = \frac{1}{6}\lambda^4 t^3 e^{-\lambda t}$, $\lambda > 0$. It is easily verified that the pdf of the random variable $\eta_i^{(5)}\theta_i$ has the form

$$f(t) = \frac{1}{4}(\lambda + \lambda^2|t|)e^{-\lambda|t|}.$$

It may be proved that

$$f(t) = \frac{1}{4}(\lambda + \lambda^2|t|)e^{-\lambda|t|} = \frac{\lambda^2}{4}\int\limits_{-\infty}^{\infty} e^{-\lambda|t-u|}e^{-\lambda|u|}du.$$

Hence, $\eta_i^{(5)}\theta_i$ can be presented as follows $\eta_i^{(5)}\theta_i = \xi_i + \xi_i'$, where ξ_i, ξ_i' are i.i.d. random variables with the Laplace distribution $f_\xi(t) = \frac{1}{2}\lambda e^{-\lambda|t|}$.

Therefore, the cdf $F_{\widehat{x}}(y)$ of $\widehat{x}^{(5)}(t) = x^{(5)}(\tau_{\xi(t)})$ is of the form

$$F_{\widehat{x}}(y) = P\left(v\sum_{i=0}^{\xi(t)}\eta_i^{(5)}\theta_i \le y\right) = \sum_{k=0}^{\infty}P\left(v\sum_{i=0}^{k}\eta_i^{(5)}\theta_i \le y\right)P(\xi(t) = k).$$

$$(4.1)$$

It is easy to see that

$$P(\xi(t) = k) = e^{-\lambda t}\left(\frac{(\lambda t)^{4k}}{4k!} + \frac{(\lambda t)^{4k+1}}{(4k+1)!} + \frac{(\lambda t)^{4k+2}}{(4k+2)!} + \frac{(\lambda t)^{4k+3}}{(4k+3)!}\right). \quad (4.2)$$

Let us denote by $\varphi(z, t) = Ee^{-iz\widehat{x}^{(5)}(t)} = \int\limits_{-\infty}^{\infty} e^{-izy}dF_{\widehat{x}}(y)$ the characteristic function of $\widehat{x}^{(5)}(t)$.

Whence, taking into account Eq.(4.1),(4.2), we have

$$\varphi(z,t) = \sum_{k=0}^{\infty} \left(\frac{\lambda^2}{\lambda^2 + v^2 z^2} \right)^{2k} P(\xi(t) = k)$$

$$= e^{-\lambda t} \sum_{k=0}^{\infty} \left(\frac{(\lambda t)^{4k}}{4k!} + \frac{(\lambda t)^{4k+1}}{(4k+1)!} + \frac{(\lambda t)^{4k+2}}{(4k+2)!} + \frac{(\lambda t)^{4k+3}}{(4k+3)!} \right) \qquad (4.3)$$

$$= \frac{e^{-\lambda t}}{4} \left\{ e^{\frac{\lambda^2}{\sqrt{\lambda^2+v^2z^2}}t} + e^{\frac{-\lambda^2}{\sqrt{\lambda^2+v^2z^2}}t} + e^{\frac{i\lambda^2}{\sqrt{\lambda^2+v^2z^2}}t} + e^{\frac{-i\lambda^2}{\sqrt{\lambda^2+v^2z^2}}t} \right.$$

$$+ \frac{\lambda^2 + v^2 z^2}{\lambda^2} \left(e^{\frac{\lambda^2}{\sqrt{\lambda^2+v^2z^2}}t} - e^{\frac{-\lambda^2}{\sqrt{\lambda^2+v^2z^2}}t} - i\left(e^{\frac{i\lambda^2}{\sqrt{\lambda^2+v^2z^2}}t} - e^{\frac{-i\lambda^2}{\sqrt{\lambda^2+v^2z^2}}t} \right) \right)$$

$$+ \left(\frac{\lambda^2 + v^2 z^2}{\lambda^2} \right)^2 \left(e^{\frac{\lambda^2}{\sqrt{\lambda^2+v^2z^2}}t} + e^{\frac{-\lambda^2}{\sqrt{\lambda^2+v^2z^2}}t} - \left(e^{\frac{i\lambda^2}{\sqrt{\lambda^2+v^2z^2}}t} + e^{\frac{-i\lambda^2}{\sqrt{\lambda^2+v^2z^2}}t} \right) \right)$$

$$+ \left(\frac{\lambda^2 + v^2 z^2}{\lambda^2} \right)^3 \left(e^{\frac{\lambda^2}{\sqrt{\lambda^2+v^2z^2}}t} + e^{\frac{-\lambda^2}{\sqrt{\lambda^2+v^2z^2}}t} + i\left(e^{\frac{i\lambda^2}{\sqrt{\lambda^2+v^2z^2}}t} + e^{\frac{-i\lambda^2}{\sqrt{\lambda^2+v^2z^2}}t} \right) \right) \right\}$$

Similarly to Eq.(3.4),(3.5), it follows from Eq.(4.3) that if $\lambda \to +\infty$ and $v \to +\infty$ such that $\frac{v^2}{\lambda} \to \sigma^2 > 0$ then $\widehat{x}^{(5)}(t)$ converges weekly to the Wiener process and consequently $\vec{x}^{(5)}(t)$ converges weakly to the Wiener process in $\mathbb{R}^5$.

Analogously, we can study random motions in the Euclidean space $\mathbb{R}^n$ for all $n \geq 2$.

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