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One-dimensional semi-Markov evolutions with general Erlang sojourn times

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Abstract—In this paper we study a one-dimensional random motion by having a general Erlang distribution for the sojourn times and we obtain higher order hyperbolic equations for this case. We apply the methodology of random evolutions to find the partial differential equations governing the particle motion and we obtain a factorization of these equations. As a particular case we find the linear biwave equation for the symmetric motion case and 2-Erlang distributions for the sojourn times of a semi-Markov evolution.

Key words and phrases: Random evolutions; semi-Markov processes; Erlang distributions; random velocities, telegrapher's equation, biwave equation

1. INTRODUCTION

In this paper we study a one-dimensional random motion performed with two alternating velocities, where the random times separating consecutive changes of velocities perform an alternating semi-Markov process. The sojourn times of this process are random variables with general Erlang distributions.

Most of the papers on random motion are devoted to analysis of models in which motions are driven by a homogeneous Poisson process, so their processes are Markovian [1], [2], [3], and [4]. In paper [5], it is considered a non-Markovian generalization of the telegrapher's random process where motion is driven by an alternating semi-Markov process with Erlang distributed interrenewal times.

We assume that the particle moves on the line $\mathbb{R}$ in the following manner: At each instant it moves according to one of two velocities, namely $v_1 > 0$ or $v_2 < 0$. Starting at the position $x_0 \in \mathbb{R}$ the particle continues its motion with velocity $v_1 > 0$ during the random time $\tau_1 = \xi_{\lambda_1} + \cdots + \xi_{\lambda_n}$, where $n \geq 1$ and ξ_{λ_i} is an exponential random variable (r.v.) with parameter λ_i, then the particle moves with velocity $v_2 < 0$ during the random time $\tau_2 = \xi_{\mu_1} + \cdots + \xi_{\mu_m}$, where $m \geq 1$ and ξ_{μ_i} is an exponential r.v. with parameter μ_i. Moreover, the particle moves with velocity $v_1 > 0$ and so on.

We study this one-dimensional random motion by assuming a general Erlang distribution for the sojourn times. Meaning that τ_1, respectively τ_2, is general Erlang

distributed with parameters $\lambda_1, \cdots, \lambda_n$, respectively $\mu_1, \cdots, \mu_m$, with Laplace transforms

$$\hat{f}_{\tau_1}(s) := \int_0^{+\infty} e^{-st} f_{\tau_1}(t)\, dt = \prod_{i=1}^{n} \frac{1}{(s+\lambda_i)},\ s \geq 0 \tag{1}$$

$$\hat{f}_{\tau_2}(s) := \int_0^{+\infty} e^{-st} f_{\tau_2}(t)\, dt = \prod_{i=1}^{m} \frac{1}{(s+\mu_i)},\ s \geq 0. \tag{2}$$

2. MATHEMATICAL MODEL

Let us set the probability space $(\Omega, \mathcal{F}, \mathrm{P})$. On the phase space $\mathbb{T} = \{1, 2\}$ consider an alternating semi-Markov stochastic process $\{\eta(t),\, t \geq 0\}$, having the sojourn time τ_i corresponding to the state $x = i \in \mathbb{T}$, and transition probability matrix of the embedded Markov chain

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3}$$

Denote by $y(t)$, $t \geq 0$, the particle position at time t. Now, consider the function $C_1(\cdot)$ on $\mathbb{T}$ as

$$C_1(x) := \begin{cases} v_1, & \text{if state} \quad x = 1; \\ v_2, & \text{if state} \quad x = 2. \end{cases} \tag{4}$$

Then the position of the particle in anytime t can be expressed as

$$y(t) = x_0 + \int_0^t C_1(\eta(u))\, du. \tag{5}$$

Thus, Eq. (5) determines the random evolution of the particle in the semi-Markov media $\{\eta(t),\, t \geq 0\}$, [6], [7].

On the same probability space $(\Omega, \mathcal{F}, \mathrm{P})$ consider a Markov chain $\xi(t)$, $t \geq 0$, with the phase space

$$\mathbb{E} = \{1, 2, \ldots, n, n+1, \ldots, n+m\}$$

and the generating operator

$$Q = q[P - I],$$

where

$$q = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n, \mu_1, \mu_2, \ldots, \mu_m)$$

is the intensities matrix and the transition probability matrix is given by

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{6}$$

We will assume that the initial distribution is $\mathbf{P}\{\xi(0) = 1\} = 1$.

Now, let us introduce the following function $C(\cdot)$ on $\mathbb{E}$,

$$C(i) := \begin{cases} v_1, & \text{if state} \quad i = 1, \ldots, n; \\ v_2, & \text{if state} \quad i = n+1, \ldots, n+m. \end{cases} \tag{7}$$

Then the following equation

$$x(t) = x_0 + \int_0^t C(\xi(u))\, du \tag{8}$$

determines the random evolution in the Markov medium $\{\xi(t),\, t \geq 0\}$, [6], [7].

The random processes $y(t)$ and $x(t)$ are stochastically equivalent and they model the particle motion evolution with velocities v_1 and v_2, and the random times between velocities switching have probability density function $f_{\tau_1}(t)$ if the velocity is v_1, and $f_{\tau_2}(t)$ when the velocity of the particle is v_2. Basically, we have reduced the semi-Markov case given by Eq. (5) to the Markov case given by Eq. (8). So, we now pass to study the stochastic process given by Eq. (8).

3. PARTIAL DIFFERENTIAL EQUATIONS

Let us consider the two-component stochastic process $\zeta(t) = (\chi(t), \xi(t))$ with phase space $\mathbb{Z} = \mathbb{R} \times \mathbb{E}$, where $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{E} = \{1, \ldots, n, n+1, \ldots, n+m\}$. It is well-known that the generating operator of $\{\zeta(t)\}$ is of the following form [6], [7], and [8],

$$A\varphi(x,i) = C(x,i)\frac{\partial}{\partial x}\varphi(x,i) + q(P\varphi(x,i) - \varphi(x,i)), \tag{9}$$

where $\varphi \in D(A)$ = domain of the operator A, $x \in \mathbb{R}$, $i \in \mathbb{E}$, and $C(x,i) = C(i)$ in this case. Let us write the operator A in more detail

$$\begin{aligned}
A\varphi(x,1) &= v_1\frac{\partial}{\partial x}\varphi(x,i) + \lambda_1(\varphi(x,2) - \varphi(x,1)) \\
A\varphi(x,2) &= v_1\frac{\partial}{\partial x}\varphi(x,2) + \lambda_2(\varphi(x,3) - \varphi(x,2)) \\
&\vdots \\
A\varphi(x,n) &= v_1\frac{\partial}{\partial x}\varphi(x,n) + \lambda_n(\varphi(x,n+1) - \varphi(x,n)) \\
A\varphi(x,n+1) &= v_2\frac{\partial}{\partial x}\varphi(x,n+1) + \mu_1(\varphi(x,n+2) - \varphi(x,n+1)) \\
&\vdots \\
A\varphi(x,n+m) &= v_2\frac{\partial}{\partial x}\varphi(x,n+m) + \mu_m(\varphi(x,1) - \varphi(x,n+m)).
\end{aligned}$$

Now, let us consider the density function

$$f(t,x,n)\, dx = \mathbf{P}\{x \leq x(t) \leq x + dx,\, \zeta(t) = n\}, \quad n \in \mathbb{E}.$$

This function satisfies the first Kolmogorov equation, namely

$$\frac{\partial f(t,x,n)}{\partial t} = A\, f(t,x,n). \tag{10}$$

Eq. (10) can be written in more detail as follows

$$\begin{aligned}
\frac{\partial f(t,x,1)}{\partial t} &= v_1\frac{\partial}{\partial x}f(t,x,1)+\lambda_1(f(t,x,2)-f(t,x,1))\\
\frac{\partial f(t,x,2)}{\partial t} &= v_1\frac{\partial}{\partial x}f(t,x,2)+\lambda_2(f(t,x,3)-f(t,x,2))\\
&\vdots\\
\frac{\partial f(t,x,n)}{\partial t} &= v_1\frac{\partial f(t,x,n)}{\partial x}+\lambda_n(f(t,x,n+1)-f(t,x,n)) \qquad (11)\\
\frac{\partial f(t,x,n+1)}{\partial t} &= v_2\frac{\partial}{\partial x}f(t,x,n+1)+\mu_1(f(t,x,n+2)-f(t,x,n+1))\\
&\vdots\\
\frac{\partial f(t,x,n+m)}{\partial t} &= v_2\frac{\partial}{\partial x}f(t,x,n+m)+\mu_m(f(t,x,1)-f(t,x,n+m)).
\end{aligned}$$

The set of equations (11) can be represented as

$$Mf=0,$$

where

$$M=\begin{bmatrix}
\frac{\partial}{\partial t}-v_1\frac{\partial}{\partial x}+\lambda_1 & -\lambda_1 & 0 & \cdots & 0\\
0 & \frac{\partial}{\partial t}-v_1\frac{\partial}{\partial x}+\lambda_2 & -\lambda_2 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \cdots\\
0 & \cdots & \frac{\partial}{\partial t}-v_2\frac{\partial}{\partial x}+\mu_{m-1} & \cdots & -\mu_{n+m-1}\\
-\mu_{n+m} & \cdots & 0 & \cdots & \frac{\partial}{\partial t}-v_2\frac{\partial}{\partial x}+\mu_m
\end{bmatrix}. \qquad (12)$$

The probability density function of a particle position at time t is given by

$$f(t,x)=\sum_{i=1}^{n+m} f(t,x,i). \qquad (13)$$

It is well-known that $f(t,x)$ satisfies the following equation [3]

$$(\mathrm{Det}(M))f=0, \qquad (14)$$

where $\mathrm{Det}(M)$ is the determinant of matrix M.

It is easily verified that

$$\mathrm{Det}(M)=\prod_{i=1}^{n}\left(\frac{\partial}{\partial t}-v_1\frac{\partial}{\partial x}+\lambda_i\right)\prod_{j=1}^{m}\left(\frac{\partial}{\partial t}-v_2\frac{\partial}{\partial x}+\mu_j\right)-\prod_{i=1}^{n}\lambda_i\prod_{j=1}^{m}\mu_j. \qquad (15)$$

Now, let us consider $v_1=-v_2=v$, $\lambda_i=\lambda$, $\mu_j=\mu$, $n=2k$, and $m=2l$, where

$k, l \in \mathbb{N}$, then Eq. (15) has the factorization

$$\begin{aligned} &\left(\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x} + \lambda\right)^k \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \lambda\right)^l - \lambda^k \mu^l\right) \\ \times \quad &\left(\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x} + \lambda\right)^k \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \lambda\right)^l + \lambda^k \mu^l\right) f = 0. \end{aligned} \tag{16}$$

Furthermore, for the particular case of $n = m = 2$ and $\lambda = \mu$ we have

$$\operatorname{Det}(M) = \left[\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x} + \lambda\right)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \lambda\right)\right]^2 - \lambda^4. \tag{17}$$

Hence,

$$\left(\frac{\partial^2}{\partial t^2} - v^2\frac{\partial^2}{\partial x^2} + 2\lambda\frac{\partial}{\partial t} + 2\lambda^2\right)\left(\frac{\partial^2}{\partial t^2} - v^2\frac{\partial^2}{\partial x^2} + 2\lambda\frac{\partial}{\partial t}\right) f(t,x) = 0, \tag{18}$$

where

$$\left(\frac{\partial^2}{\partial t^2} - v^2\frac{\partial^2}{\partial x^2} + 2\lambda\frac{\partial}{\partial t} + 2\lambda^2\right) \quad \text{and} \quad \left(\frac{\partial^2}{\partial t^2} - v^2\frac{\partial^2}{\partial x^2} + 2\lambda\frac{\partial}{\partial t}\right)$$

are differential operators of the telegrapher's equations.

Let us apply the exponential transformation

$$f(t,x) = e^{-\lambda t} u(t,x)$$

to Eq. (18) and change the variable $y = x/v$, then Eq. (18) can be reduced to

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} + \lambda^2\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} - \lambda^2\right) u(t,y) = 0, \tag{19}$$

or equivalently

$$\Box^2 u(t,y) - \lambda^4 u(t,y) = 0, \tag{20}$$

where

$$\Box = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\right). \tag{21}$$

Eq. (20) was introduced in [10] and it is called the linear biwave equation. This equation is investigated in [9] as well.

Suppose that $u_-(t,y)$ is a solution of

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} - \lambda^2\right) u(t,y) = 0. \tag{22}$$

It will readily be seen that the particular solution of

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} + \lambda^2\right) u(t,y) = u_-(t,y) \tag{23}$$

is $\dfrac{1}{2\lambda^2}\, u_-(t,y)$. So, a solution of Eq. (23) is

$$u_+(t,y) + \frac{u_-(t,y)}{2\lambda^2},$$

where $u_+(t,y)$ is a solution of

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} + \lambda^2\right) u(t,y) = 0. \tag{24}$$

Thus, it is easy to see that any solution $u(t,y)$ of (20) is of the following form

$$u(t,y) = u_-(t,y) + u_+(t,y).$$

Eq. (22) is the telegrapher's equation [11], [12]. By changing the variables $y = t$, $t = y$, Eq.(24) can be reduced to the telegrapher's equation. The solution of the telegrapher's equation with initial conditions is well-known. For instance, the solution of Eq. (22) with initial conditions

$$u|_{t=0} = f(y), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(y) \tag{25}$$

is of the following form [11]

$$u(t,y) = \frac{f(y-t) + f(y+t)}{2} + \frac{1}{2}\int_{y-t}^{y+t} H(t,y,x)dx, \tag{26}$$

where

$$H(t,y,x) = F(x)J_0(\lambda\sqrt{(x-y)^2 - t^2}) + \lambda t f(x)\frac{J_0'(\lambda\sqrt{(x-y)^2 - t^2})}{\sqrt{(x-y)^2 - t^2}},$$

and $J_0(x)$ is the zero order Bessel function.

4. CONCLUSIONS

The random telegrapher's equation is closely related to the Poisson process, as it was noticed in the seminal work by Kac [1]. Since then there have been many works extending this basic model to more general setting, most of them Markovian [2], [3], and many others. In most cases the resulting higher order partial differential equations are very involved. In a recent work Di Crescenzo [5] analyzed a non-Markovian generalization of the telegrapher's random process on the real line. He considered an Erlang distribution for the time spending by the particle before a velocity reversal, and it might be interpreted as that the particle experiments n or r random collisions before reversing its direction. We also deal with this problem in this paper, however we consider a general Erlang distribution and we use the mathematical tool of random evolutions to find the corresponding higher order partial differential equations. It is interesting to notice that these equations can be factorized, and a particular case is the symmetric one when the particle spends a random time from a 2-Erlang distribution in each direction, and in such a case we found the biwave equation to model the particle dynamics.

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REFERENCES

1. Kac, M. A stochastic model related to the telegrapher's equation. *Rocky Mountain J. Math.* **4**, 497–509 (1974).
2. Orsingher, E. Hyperbolic equations arising in random models. *Stoch. Proc. Appl.* **21**, 93–106 (1985).
3. Turbin, A.F. and Kolesnik, A.D. Infinitesimal hyperbolic operator of the Markovian random evolution in $\mathbb{R}^n$. *Dokl. Acad. Nauka Ukrain.* **1**, 9–12 (1991). (in Russian).
4. Turbin, A.F. Mathematical model of Einstein, Wiener, Levy. *Fractal Analysis and Related Fields* **2**, 47–60 (1998). (in Russian).
5. Di Crescenzo, A. On random motions with velocities alternating at Erlang-distributed random times. *Adv. Appl. Prob.* **33**, 690–701 (2001).
6. Korolyuk, V.S. and Swishchuk, A.V. *Semi-Markov Random Evolutions*. Kluwer Academic Publishers (1995).
7. Korolyuk, V.S. and Korolyuk, V.V. *Stochastic Models of Systems*. Kluwer Academic Publishers (1999).
8. Korolyuk, V.S. and Turbin, A.F. *Mathematical Foundations of the State Lumping of Large Systems*. Kluwer Academic Publishers (1994).
9. Roman, O. V. Symmetry analysis of the multidimensional polywave equation. *Nonlinear Mathematical Physics* **3**, 435–440 (1996).
10. Fushchych, W. I., Roman, O.V., and Zdanov, R. Z. Symmetry reduction and exact solutions of nonlinear biwave equations. *Reports on Mathematical Physics* **37**, 267–281 (1996).
11. Koshlyakov N.S., Gliner E.B, Smirnov M.M. *Partial Differential Equations of Mathematical Physics*. Moscow, Vysshaya shkola (1970). (in Russian).
12. Farlow S.J. *Partial Differential Equations*. John Wiley and Sons (1982).