J Stat Phys
DOI 10.1007/s10955-011-0328-2

Isotropic Random Motion at Finite Speed with K-Erlang Distributed Direction Alternations

Anatoliy A. Pogorui · Ramón M. Rodríguez-Dagnino

Received: 10 June 2011 / Accepted: 18 August 2011

Abstract We study uniformly distributed direction of motion at finite speed where the direction alternations occur according to the renewal epochs of a K-Erlang pdf. At first sight, our generalizations of previous Markovian results appears to be a small step, however, it must be seen as an important non-Markovian case where we have found closed-form expressions for the pdf and the conditional characteristic function of this semi-Markov transport process. We present detailed calculations of a three-dimensional example for the 2-Erlang case, which is important not only from physical applications point of view but also to understand more general models. For instance, in principle the example of the 2-Erlang case can be extended to a K-Erlang case ($K = 3, 4, \dots$) but some of the mathematical expressions may be cumbersome.

Keywords Random evolutions · Semi-Markov processes · Erlang distributions

1 Introduction

The seminal derivation of the telegraph stochastic process describes the one-dimensional movement of a particle with alternating velocities and directions according to an underlying Poisson process [1, 2]. Many generalization have been proposed to the basic telegraph process. One-dimensional non-Markovian generalizations of the telegrapher's random process were obtained in [3, 4] with velocities alternating at K-Erlang-distributed sojourn times.

Grant CAT148 from Tecnológico de Monterrey.

A.A. Pogorui
Department of Mathematics, Zhytomyr State University, Valyka Berdychivska St., 40, 10008 Zhytomyr, Ukraine
e-mail: pogor@zu.edu.ua

R.M. Rodríguez-Dagnino (✉)
Electrical and Computer Engineering, Tecnológico de Monterrey, Av. Eugenio Garza Sada 2501 Sur, C.P. 64849, Monterrey, N.L., Mexico
e-mail: rmrodrig@itesm.mx

Published online: 09 September 2011

Uniformly distributed direction of motion or isotropic motion with motion driven by a homogeneous Poisson process has been studied by Pinsky [5] on a Riemannian manifold and by Orsingher and De Gregorio in higher dimensions [6], see also [7] and references therein. A recent study on multidimensional random motion shows that the evolutions are driven by hyperparabolic operators composed of the telegraph operator and their integer powers [8]. However, such a study is tied to the Markov assumption, or exponentially distributed interevents times. The exponential pdf is not the best option for many important applications in physics, biology, and engineering, since it gives higher probability to very short intervals, whereas the K-Erlang probability density function (pdf) has its maximum at $(K-1)/\lambda$, $\lambda > 0$, $K = 1, 2, 3, \ldots$. For instance, when we do not have many particles in a physical medium then the times between collisions of a particle traveling in this medium are not very short and they cannot be considered Poisson. The recent work of Le Caer [9, 10] departs from this Markovian trend since he is studying uniformly distributed orientation random motion with Pearson-Dirichlet distributed steps in a multidimensional random walk setting, and Beghin and Orsingher study an isotropic planar random motion of a particle of finite velocity at even-valued Poisson events, where they analyzed this problem based on properties of Bessel functions [11]. In this work, we consider multidimensional random motions with uniformly distributed directions with K-Erlang distributed steps. Our analysis is based on random evolutions on a semi-Markov media, and we have extended many of the results of [12].

Let $\{\xi(t),\ t \geq 0\}$ be an ordinary renewal counting process such that $\xi(t) = \max\{m \geq 0\colon\ \tau_m \leq t\}$, where $\tau_m = \sum_{k=1}^{m} \theta_k$, $\tau_0 = 0$ and $\theta_k \geq 0$, $k = 1, 2, \ldots$, are nonnegative random variables denoting the interarrival times. We assume that these random variables are independent and identically distributed (iid) with a cumulative distribution function (cdf) $G_\theta(t)$ and that there exists the pdf $g_\theta(t) = \frac{d}{dt} G_\theta(t)$.

We will study the random motion of a particle that starts from the coordinate origin $\mathbf{0} = (0, 0, \ldots, 0)$ of the space $\mathbb{R}^n$, at time $t = 0$, and continues its motion with a constant absolute velocity v along the direction $\boldsymbol{\eta}_0^{(n)}$, where $\boldsymbol{\eta}_0^{(n)} = (x_{01}, x_{02}, \ldots, x_{0n}) = (x_1, x_2, \ldots, x_n)$ is a random n-dimensional vector uniformly distributed on the unit sphere $\Omega_1^{n-1} = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}$.

At instant τ_1 the particle changes its direction to $\boldsymbol{\eta}_1^{(n)} = (x_{11}, x_{12}, \ldots, x_{1n})$, where $\boldsymbol{\eta}_0^{(n)}$ and $\boldsymbol{\eta}_1^{(n)}$ are iid random vectors on Ω_1^{n-1}. Then, at instant τ_2 the particle changes its direction to $\boldsymbol{\eta}_2^{(n)} = (x_{21}, x_{22}, \ldots, x_{2n})$, where $\boldsymbol{\eta}_2^{(n)}$ is also uniformly distributed on Ω_1^{n-1} and independent of $\boldsymbol{\eta}_0^{(n)}$ and $\boldsymbol{\eta}_1^{(n)}$, and so on.

Denote by $\mathbf{X}^{(n)}(t)$, $t \geq 0$, the particle position at time t. We have that

$$\mathbf{X}^{(n)}(t) = v \sum_{j=1}^{\xi(t)} \boldsymbol{\eta}_{j-1}^{(n)} (\tau_j - \tau_{j-1}) + v\, \boldsymbol{\eta}_{\xi(t)}^{(n)} (t - \tau_{\xi(t)}). \tag{1}$$

Basically, (1) determines the random evolution in the semi-Markov (or renewal) medium $\xi(t)$.

Now, let us denote as $\nu(t)$ the number of velocity alternations occurred in the interval $(0, t)$, and let us assume first a Poisson process for the epochs of these alternations. Then the random variables θ_k are exponentially distributed with parameter λ, hence $g_\theta(t) = \lambda e^{-\lambda t} I_{\{t \geq 0\}}$.

Furthermore, let us denote as $\mathbf{X}_m^{(n)}(t)$ the particle position at time $t > 0$ when $\nu(t) = m > 0$, i.e.,

$$P\left(\mathbf{X}_m^{(n)}(t) \leq y\right) = P\left(\mathbf{X}^{(n)}(t) \leq y \,\middle|\, \nu(t) = m\right).$$

It is not hard to see that $\mathbf{X}_m^{(n)}(t)$ is given by

$$\mathbf{X}_m^{(n)}(t) = v \sum_{j=1}^{m+1} \boldsymbol{\eta}_{j-1}^{(n)}(\tau_j - \tau_{j-1}), \tag{2}$$

where $\tau_{m+1} = t$.

The probabilistic properties of the random vector $\mathbf{X}_m^{(n)}(t)$ are completely determined by those of its projection $X_m^{(n)}(t) = v \sum_{j=1}^{m+1} \eta_{j-1}^{(n)}(\tau_j - \tau_{j-1})$ on a fixed line, where $\eta_j^{(n)}$ is the projection of $\boldsymbol{\eta}_j^{(n)}$ on the line.

Indeed, let us consider the conditional cdf

$$F_X(y \,|\, \nu(t) = m) = P\left(X_m^{(n)}(t) \le y \,\middle|\, \nu(t) = m\right).$$

Then, the conditional characteristic function (Fourier transform) $H_m(t,\alpha) = H_m(t)$ of $\mathbf{X}^{(n)}(t)$, where $\alpha = \|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}$, is given by

$$\begin{aligned} H_m(t) &= \mathbf{E}\left[\exp\left\{i\left(\boldsymbol{\alpha}, \mathbf{X}^{(n)}(t)\right)\right\} \,\middle|\, \nu(t) = m\right] = \mathbf{E}\exp\left\{i\,\left(\boldsymbol{\alpha}, \mathbf{X}_m^{(n)}(t)\right)\right\} \\ &= \mathbf{E}\exp\left\{i\,\|\boldsymbol{\alpha}\|\left(\mathbf{e}, \mathbf{X}_m^{(n)}(t)\right)\right\} = \mathbf{E}\exp\left\{i\,\|\boldsymbol{\alpha}\| X_m^{(n)}(t)\right\} \\ &= \int_0^\infty \exp\{i\,\|\boldsymbol{\alpha}\| y\}\, dF_X(y \mid \nu(t) = m), \end{aligned}$$

where $X_m^{(n)}(t)$ is the projection of $\mathbf{X}_m^{(n)}(t)$ onto the unit vector $\mathbf{e}$ and it has a conditional cdf $F_X(y \mid \nu(t) = m)$.

Let us denote by $f_\eta(x)$ the pdf of the projection $\eta_j^{(n)}$ of the vector $\boldsymbol{\eta}_j^{(n)}$ onto a fixed line. It is shown in [13] that $f_\eta(x)$ is of the following form

$$f_{\eta^{(n)}}(x) = \begin{cases} \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}\,(1-x^2)^{(n-3)/2}, & \text{if } x \in [-1,1], \\ 0, & \text{if } x \notin [-1,1]. \end{cases} \tag{3}$$

We can also consider the case $n = 1$ and it is easily seen that

$$f_{\eta^{(1)}}(x) = \frac{1}{2}\,\delta(|x| - 1).$$

It is also straightforward to verify that

$$\begin{aligned} H_m(t) &= \mathbf{E}\left[\exp\left\{i\,\alpha\, v \sum_{j=1}^{m+1} \eta_{j-1}^{(n)}\,(\tau_j - \tau_{j-1})\right\}\right] \\ &= \frac{m!}{t^m}\int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{m-1}}^t ds_m\, \mathbf{E}\left[\exp\left\{i\,\alpha\, v \sum_{j=1}^{m+1} \eta_{j-1}^{(n)}\,(s_j - s_{j-1})\right\}\right] \\ &= \frac{m!}{t^m}\int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{m-1}}^t ds_m \left\{\prod_{j=1}^{m+1} \varphi_\eta\left(\alpha\, v\,(s_j - s_{j-1})\right)\right\}, \end{aligned}$$

where $s_0 = 0$, $s_{m+1} = t$, and $\varphi_\eta(\omega) = \mathbf{E}e^{i\omega\eta_j^{(n)}} = \int_{-\infty}^{\infty} e^{i\omega x} f_{\eta^{(n)}}(x)dx$ is the characteristic function of $\eta_j^{(n)}$. The function $\varphi_\eta(\omega)$ will be denoted as $\varphi(\omega)$ in the rest of the paper. We should notice that $\varphi(\omega)$ was also obtained in [12] by using a different method.

2 A K-Erlang Distributed Direction Alternations

Let us assume a fixed integer $K \geq 2$. Now, suppose that the particle change its direction just at epochs τ_{Kj}, $j \geq 1$, i.e., for $m = 0, 1, 2, \ldots$

$$\mathbf{X}_m^{(n)}(t) = v \sum_{j=1}^{\lfloor \frac{m}{K} \rfloor} \boldsymbol{\eta}_{K(j-1)}^{(n)}(\tau_{Kj} - \tau_{K(j-1)}) + v\, \boldsymbol{\eta}_{K\lfloor \frac{m}{K} \rfloor}^{(n)}(\tau_{m+1} - \tau_{K\lfloor \frac{m}{K} \rfloor}), \tag{4}$$

where $\mathbf{X}_m^{(n)}(t)$ is the particle position at renewal epoch τ_m, $\tau_{m+1} = t$, and $\sum_{j=1}^{0} \boldsymbol{\eta}_{K(j-1)}^{(n)}(\tau_{Kj} - \tau_{K(j-1)}) = 0$.

Hence, defining $c = \alpha v$, we have for $m = \ell K + r$, $\ell = 0, 1, 2, \ldots$, $0 \leq r < K$

$$\begin{aligned} H_\ell^{(r)}(t) &= H_m(t) = \mathbf{E}\left[\exp\left(i\, c\mathbf{X}_m^{(n)}(t)\right)\right] \\ &= \frac{(\ell K + r)!}{t^{\ell K + r}} \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{\ell K + r - 1}}^t ds_{\ell K + r} \\ &\quad \times \left\{ \prod_{j=1}^{\ell} \varphi\left(c\,(s_{Kj} - s_{K(j-1)})\right) \varphi\left(c(t - s_{\ell K})\right) \right\}, \end{aligned}$$

where $\prod_{j=1}^{0} \varphi(c\,(s_{Kj} - s_{K(j-1)})) = 1$.

We should notice that $H_0^{(r)}(t) = \varphi(ct)$.

The random variables $\tau_{Kj} - \tau_{K(j-1)}$, $j \geq 1$, are K-Erlang distributed, and we may define the renewal process $\xi_K(t) = \max\{j \geq 0 : \tau_j^{(K)} \leq t\}$, $t \geq 0$, where $\tau_j^{(K)} = \sum_{i=0}^{j} \theta_i^{(K)}$, $\tau_0^{(K)} = 0$, and $\theta_i^{(K)}$ are iid interarrival times having a K-Erlang pdf $g_{\theta^{(K)}}(t) = \frac{\lambda^K t^{K-1}}{(K-1)!} e^{-\lambda t} I_{\{t \geq 0\}}$. Thus, the stochastic process $\mathbf{X}^{(n)}(t)$ can be considered as the random evolution of the particle in the non-Markovian K-Erlang medium $\xi_K(t)$.

The functions $H_m(t)$, $m = 0, 1, 2, \ldots$ are the conditional characteristic functions of $\mathbf{X}^{(n)}(t)$. We should remember that $\{\nu(t) = m\}$ represents that m Poisson events occurred in $[0, t)$, and the number of direction alternations $\nu_K(t)$ of $\mathbf{X}^{(n)}(t)$ is given by $\nu_K(t) = \lfloor \frac{\nu(t)}{K} \rfloor$.

Now, let us consider the following multiple integral

$$\begin{aligned} I_\ell^{(r)}(t) &= \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{\ell K + r - 1}}^t ds_{\ell K + r} \\ &\quad \times \left\{ \prod_{j=1}^{\ell} \varphi\left(c\,(s_{Kj} - s_{K(j-1)})\right) \varphi\left(c\,(t - s_{K\ell})\right) \right\} \end{aligned} \tag{5}$$

where $s_0 = 0$, and let us state the notation $\psi_K(t) = \frac{t^{K-1}}{(K-1)!} \varphi(ct)$.

Theorem 1 *For any $\ell \geq 0$ we have the following recursive relation*

$$I_{\ell+1}^{(r)}(t) = \int_0^t \frac{u^{K-1}}{(K-1)!}\varphi(cu) I_\ell^{(r)}(t-u)du = \psi_K * I_\ell^{(r)}(t) \tag{6}$$

where we have by assumption that $I_0^{(r)}(t) = \frac{t^r}{r!}\varphi(ct)$, $r = 0, 1, \ldots, K-1$.

Proof Let us prove (6) by induction arguments. First, suppose that $\ell = r = 0$, then

$$\begin{aligned}
I_1^{(0)}(t) &= \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{K-1}}^t ds_K \left[\varphi(c\, s_K) I_0^{(0)}\,(c(t-s_K))\right] \\
&= \left[s_1 \int_{s_1}^t ds_2 \cdots \int_{s_{K-1}}^t ds_K \left[\varphi(c\, s_K) I_0^{(0)}\,(c(t-s_K))\right]\right]_0^t \\
&\quad + \int_0^t s_1 \int_{s_1}^t ds_3 \cdots \int_{s_{K-1}}^t ds_K \left[\varphi(c\, s_K) I_0^{(0)}\,(c(t-s_K))\right] ds_1 \\
&= \int_0^t \frac{s_1^2}{2} \int_{s_1}^t ds_4 \cdots \int_{s_{K-1}}^t ds_K \left[\varphi(c\, s_K) I_0^{(0)}\,(c(t-s_K))\right] ds_1 = \cdots \\
&= \int_0^t \frac{s_1^{K-1}}{(K-1)!} \varphi(c\, s_K) I_0^{(0)}\,(c(t-s_1))\, ds_1 = \psi_K * I_0^{(0)}(t).
\end{aligned} \tag{7}$$

Now, suppose that (6) is valid for any $\ell \leq N-2$, for a fixed $N > 2$ and $r = 0$.

The integral $I_{\ell+1}^{(0)}(t)$ for $\ell = N-1$ can be represented in the following form

$$\begin{aligned}
I_N^{(0)} &= \int_0^t ds_1 \cdots \int_{s_{K-1}}^t ds_K \left\{\varphi(c\, s_K) \int_{s_K}^t ds_{K+1} \cdots \int_{s_{2K-1}}^t ds_{2K} \left\{\varphi(c(s_{2K} - s_K))\right.\right. \\
&\quad \times \int_{s_{2K}}^t ds_{2K+1} \cdots \left\{ \int_{s_{K(N-1)-1}}^t ds_{K(N-1)} \cdots \right. \\
&\quad \times \int_{s_{KN-1}}^t ds_{KN} \varphi(c(s_{KN} - s_{K(N-1)}))\varphi(c(t - s_{KN})) \underbrace{\left.\left.\left.\right\}\cdots\right\}}_{N \text{ times}}.
\end{aligned} \tag{8}$$

Let us consider the following interior integral in $I_N^{(0)}(t)$ (with respect to $s_{K(N-1)+1}$) in (8). Let us make the change of variables $\rho_j = s_{K(N-1)+j} - s_{K(N-1)}$, $j = 1, 2, \ldots, K$, and by taking into account (7), we obtain

$$\begin{aligned}
&\int_{s_{K(N-1)}}^t ds_{K(N-1)-1} \cdots \int_{s_{KN-1}}^t ds_{KN}\, \varphi(c(s_{KN} - s_{K(N-1)}))\varphi(c(t - s_{KN})) \\
&= \int_0^{t-s_{K(N-1)}} d\rho_1 \int_{\rho_1}^{t-s_{K(N-1)}} d\rho_2 \cdots \int_{\rho_{K-1}}^{t-s_{K(N-1)}} d\rho_K\, \varphi(c\rho_K)\varphi(c(t - s_{K(N-1)} - \rho_K)) \\
&= \psi_K * I_1^{(0)}(t - s_{K(N-1)}).
\end{aligned}$$

Now, let us deal with the next interior integral in (8). By defining the change of variables $\zeta_j = s_{K(N-2)+j} - s_{K(N-2)}$, $j = 1, 2, \ldots, K$, we obtain

$$\int_{s_{K(N-2)}}^{t} ds_{K(N-2)+1} \cdots \int_{s_{K(N-1)-1}}^{t} ds_{K(N-1)}\, \varphi(c(s_{K(N-1)} - s_{K(N-2)}))I_1^{(0)}(t - s_{K(N-1)})$$
$$= \int_0^{t-s_{K(N-1)}} d\zeta_1 \int_{\zeta_1}^{t-s_{K(N-1)}} d\zeta_2 \cdots \int_{\zeta_{K-1}}^{t-s_{K(N-1)}} d\zeta_K\, \varphi(c\zeta_K)I_1^{(0)}(c(t - s_{K(N-2)} - \zeta_K))$$
$$= \psi_K * I_1^{(0)}(t - s_{K(N-2)}),$$

which is in accordance with (7) and the induction assumption.

By continuing this procedure, we can obtain after the $(N-1)$th step,

$$I_N^{(0)}(t) = \psi_K * I_{N-1}^{(0)}(t).$$

Next, it can be easily verified that for any $0 \le r \le K-1$

$$\int_{s_K}^{t} ds_{K+1} \cdots \int_{s_{K+r-1}}^{t} ds_{K+r} = \frac{(t-s_K)^r}{r!}.$$

Hence, for any $0 < r \le K-1$ we have

$$I_1^{(r)}(t) = \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{K-1}}^t ds_K \,\{\varphi(c\,s_K)\varphi(c(t-s_K))\}$$
$$\times \int_{s_K}^t ds_{K+1} \cdots \int_{s_{K+r-1}}^t ds_{K+r}$$
$$= \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{K-1}}^t ds_K \left\{\varphi(c\,s_K)\,\frac{(t-s_K)^r}{r!}\,\varphi(c(t-s_K))\right\}$$
$$= \int_0^t \frac{s_1^{K-1}}{(K-1)!}\,\varphi(c\,s_1)\,I_0^{(r)}\,(c(t-s_K))\,ds_1 = \psi_K * I_0^{(r)}(t)$$

and by using this recursion (6) can be proved, for any $0 < r \le K-1$, in much the same manner as for the case $r = 0$. □

Corollary 1 *It follows from* (6) *that*

$$I_\ell^{(r)}(t) = \psi_K^{\ell *} * I_0^{(r)}(t),$$

where $\ell *$ *is the* ℓ*-fold convolution,* $\ell \ge 1$, $I_0^{(r)} = \frac{t^r}{r!}\varphi(ct)$, $r = 0, 1, \ldots, K-1$.

Corollary 2 *For any* $\ell \ge 0$ *the conditional characteristic functions satisfy the following equations*

$$H_{\ell+1}^{(r)}(t) = \frac{(K(\ell+1)+r)!}{t^{K(\ell+1)+r}(K\ell+r)!}\int_0^t u^{K\ell+r}\psi_K(u)H_\ell^{(r)}(t-u)du,$$

where $H_0^{(r)}(t) = \varphi(ct)$.

Proof It follows form (6) that

$$
\begin{aligned}
H_{\ell+1}^{(r)}(t) &= \frac{(K(\ell+1)+r)!}{t^{K(\ell+1)+r}} I_{\ell+1}^{(r)}(t) \\
&= \frac{(K(\ell+1)+r)!}{t^{K(\ell+1)+r}(K\ell+r)!} \int_0^t u^{K\ell+r} \psi_K(u) \left[\frac{(K\ell+r)!}{u^{K\ell+r}} I_\ell^{(r)}(t-u) \right] du \\
&= \frac{(K(\ell+1)+r)!}{t^{K(\ell+1)+r}(K\ell+r)!} \int_0^t u^{K\ell+r} \psi_K(u) H_\ell^{(r)}(t-u)\, du. \qquad \square
\end{aligned}
$$

3 A Three-Dimensional Example

Consider the three-dimensional case, i.e., $n=3$. For this case we have that (3) reduces to $f_{\eta^{(3)}}(x) = \frac{1}{2} I_{\{-1\le x\le 1\}}$. Hence, $\varphi(ct) = \mathbf{E}\, e^{ict\eta^{(3)}} = \frac{\sin ct}{ct}$, $H_0^{(0)}(t) = \varphi(\alpha v t) = \frac{\sin \alpha v t}{\alpha v t} = H_0^{(1)}(t)$, where $\alpha = \|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}$.

It is well-known that the function $4\pi R \frac{\sin \alpha R}{\alpha}$ is the Fourier transform of the simple layer $\delta_{S(R)}$ on the sphere $S(R)$ with radius R [14].

We should notice that $\widehat{H}_\ell(t) = H_\ell^{(0)}(t) P[\nu(t) = 2\ell] + H_\ell^{(1)}(t) P[\nu(t) = 2\ell+1]$ is the characteristic function of $X_\ell^{(n)}(t)$, which is the particle position at time t assuming that ℓ alternations of direction occurred.

Therefore, by applying the inverse Fourier transform to the function $\widehat{H}_0(t) = \frac{\sin \alpha v t}{\alpha v t} e^{-\lambda t} + \frac{\sin \alpha v t}{\alpha v t} \lambda t\, e^{-\lambda t}$, with respect to $\boldsymbol{\alpha}$ (or equivalently α) we have

$$
\begin{aligned}
&P[\mathbf{X}(t) \in d\mathbf{x} \,|\, \nu_2(t) = 0] P[\nu_2(t) = 0] \\
&= \mathcal{F}^{-1}(\widehat{H}_0(t))(\alpha) = \frac{\delta_{S(vt)}}{4\pi (vt)^2} \left(e^{-\lambda t} + \lambda t\, e^{-\lambda t} \right) dx_1 dx_2 dx_3,
\end{aligned}
$$

where $\mathbf{x} = (x_1, x_2, x_3)$.

The probability $P[\mathbf{X}(t) \in d\mathbf{x} \,|\, \nu_2(t) = 0]$ represents the conditional probability distribution of the particle position at time t when no random alternation of direction occurred.

Let us denote as $f(\mathbf{x} \,|\, \nu(t) = 2)$ the conditional probability distribution $f(\mathbf{x} \,|\, \nu(t) = 2)\, d\mathbf{x} = P[\mathbf{X}(t) \in d\mathbf{x} \,|\, \nu_2(t) = 2]$ giving that $\nu(t) = 2$, i.e., one random direction alternation occurred and no one Poisson event occurred after it.

We should calculate the particular case

$$
\begin{aligned}
H_1^{(0)}(t) &= \frac{2}{t^2} \int_0^t \psi(t-u) H_0^{(0)}(u)\, du \\
&= \frac{2}{(\alpha v t)^2} \int_0^t \frac{(t-u)\sin(\alpha v(t-u))}{(t-u)} \frac{\sin(\alpha v u)}{u}\, du \\
&= \frac{1}{(\alpha v t)^2} \big[\mathrm{Si}(2\alpha v t) \sin(\alpha v t) + \mathrm{Ci}(2\alpha v t) \cos(\alpha v t) \\
&\quad - (\ln(\alpha v t) + \ln(2e^\gamma)) \cos(\alpha v t) \big],
\end{aligned}
$$

where γ is the Euler's constant.

Let us denote as $\mathcal{F}^{-1}(\cdot)$ the inverse Fourier transform, then it can be verified that $f(\mathbf{x} \,|\, \nu(t) = 2) = \mathcal{F}^{-1}(H_1^{(0)}(t))$.

We can show, by using Formula 7.11, p. 76, of [15], that when $\|\mathbf{x}\| = vt$

$$\mathcal{F}^{-1}\left[\frac{1}{(\alpha v t)^2}\left(\ln(\alpha v t) + \ln(2e^{\gamma})\cos(\alpha v t)\right)\right](\mathbf{x})$$
$$= \frac{1}{2\pi^2}\int_0^{\infty}\left[\frac{1}{(\alpha v t)^2}\left(\ln(\alpha v t) + \ln(2e^{\gamma})\cos(\alpha v t)\right)\right]\alpha^2\frac{\sin(\alpha\|\mathbf{x}\|)}{\alpha\|\mathbf{x}\|}\,d\alpha = 0.$$

On the other hand, we could not find a closed-form expression for

$$\mathcal{F}^{-1}\left[\frac{1}{(\alpha v t)^2}\left(\ln(\alpha v t) + \ln(2e^{\gamma})\cos(\alpha v t)\right)\right](\mathbf{x})$$

when $\|\mathbf{x}\| < vt$.

It was obtained in [12] that

$$\mathcal{F}^{-1}\left[\frac{1}{(\alpha v t)^2}\left(\mathrm{Si}(2\alpha v t)\sin(\alpha v t) + \mathrm{Ci}(2\alpha v t)\cos(\alpha v t)\right)\right](\mathbf{x})$$
$$= \frac{1}{2\pi\|\mathbf{x}\|(vt)^2}\,\mathrm{arctanh}\left(\frac{\|\mathbf{x}\|}{vt}\right).$$

Therefore, for a fixed $t > 0$

$$f(\mathbf{x}\,|\,\nu(t) = 2) \sim \frac{1}{2\pi\|\mathbf{x}\|(vt)^2}\,\mathrm{arctanh}\left(\frac{\|\mathbf{x}\|}{vt}\right) \quad \text{as } \|\mathbf{x}\| \to vt.$$

Next, we should calculate

$$\begin{aligned}
H_1^{(1)}(t) &= \frac{6}{t^3}\int_0^t u\psi(t-u)H_0^{(1)}(u)\,du \\
&= \frac{6}{(\alpha v)^2 t^3}\int_0^t u\frac{(t-u)\sin(\alpha v(t-u))}{(t-u)}\frac{\sin(\alpha v u)}{u}\,du \\
&= \frac{6}{(\alpha v)^2 t^3}\int_0^t \sin(\alpha v(t-u))\sin(\alpha v u)\,du \\
&= 3\,\frac{\sin(\alpha v t) - \alpha v t\,\cos(\alpha v t)}{(\alpha v t)^3} = \chi(\alpha).
\end{aligned}$$

Now, denote by $f(\mathbf{x}\,|\,\nu(t) = 3)$ the conditional distribution $f(\mathbf{x}\,|\,\nu(t) = 3)\,d\mathbf{x} = P[\mathbf{X}(t) \in d\mathbf{x}\,|\,\nu(t) = 3]$, where we have assumed that the condition $\nu(t) = 3$ has occurred, i.e., one random direction alternation and one Poisson event occurred after it.

Since $\chi(\alpha)$ depends only on $\alpha = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$ then $f(\mathbf{x}\,|\,\nu(t) = 3)$ depends only on $\rho = \|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, or equivalently, $f(\mathbf{x}\,|\,\nu(t) = 3) = \phi_1^{(1)}(\rho)$, $\rho \le vt$.

Hence, by using Theorem 40, p. 69, of [15], we can express $\chi(\alpha)$ as

$$\chi(\alpha) = \frac{(2\pi)^{3/2}}{\sqrt{\alpha}}\int_0^{vt}\phi_1^{(1)}(\rho)\rho^{3/2}J_{1/2}(\rho\alpha)\,d\alpha = \frac{4\pi}{\alpha}\int_0^{vt}\phi_1^{(1)}(\rho)\,\rho\sin(\alpha\rho)\,d\rho. \qquad (9)$$

We should notice that when $\phi_1^{(1)}(\rho) = \frac{3}{4\pi(vt)^3} I_{\{\rho \le vt\}}$ we can obtain

$$\frac{4\pi}{\alpha} \int_0^{vt} \frac{3}{4\pi(vt)^3} \rho \sin(\alpha\rho)\, dr = \chi(\alpha).$$

Therefore, $f(\mathbf{x} \mid \nu(t) = 3) = \frac{3}{4\pi(vt)^3} I_{\{\rho \le vt\}}$ is the inverse Fourier transform of $\chi(\alpha)$.

Thus, after the occurrence of exactly three Poisson events the particle position is uniformly distributed in the ball $B(vt) = \{\|\mathbf{x}\| < vt\}$, i.e.,

$$f(\mathbf{x} \mid \nu(t) = 3) = \frac{3}{4\pi(vt)^3} I_{\{\|\mathbf{x}\| \le vt\}}.$$

4 Integral Equation for the Characteristic Function

The characteristic function of the random motion $\mathbf{X}^{(n)}(t)$ is given by the formal series

$$\begin{aligned} H(t) &= \exp\left\{i\left(\boldsymbol{\alpha}, \mathbf{X}^{(n)}(t)\right)\right\} \\ &= \sum_{\ell=0}^{\infty} \sum_{r=0}^{K-1} H_\ell^{(r)}(t) P(\nu(t) = K\ell + r) \\ &= \sum_{\ell=0}^{\infty} \sum_{r=0}^{K-1} H_\ell^{(r)}(t) \frac{(\lambda t)^{K\ell+r}}{(K\ell+r)!} e^{-\lambda t} = e^{-\lambda t} \sum_{\ell=0}^{\infty} \sum_{r=0}^{K-1} \lambda^{K\ell+r} I_\ell^{(r)}(t). \end{aligned}$$

Now, we need to show that for any $t > 0$ the series $\sum_{\ell=0}^{\infty} \sum_{r=0}^{K-1} \lambda^{K\ell+r} I_\ell^{(r)}(t)$ converges uniformly with respect to $c = \alpha v$.

Indeed, taking into account that $|\phi(\lambda)| = |\mathrm{E} e^{-i\lambda\eta_j^{(n)}}| \le 1$, it follows from (5) that

$$\left|I_\ell^{(r)}(t)\right| \le \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{K\ell+r-1}}^t ds_{K\ell+r} = \frac{t^{K\ell+r}}{(K\ell+r)!}.$$

By using this bound we can state

$$\begin{aligned} \left|\sum_{\ell=0}^{\infty} \sum_{r=0}^{K-1} \lambda^{K\ell+r} I_\ell^{(r)}(t)\right| &\le \sum_{j=0}^{\infty} \sum_{r=0}^{K-1} \lambda^{K\ell+r} \left|I_\ell^{(r)}(t)\right| \\ &\le \sum_{\ell=0}^{\infty} \lambda^{K\ell+r} \frac{t^{K\ell+r}}{(K\ell+r)!} = e^{\lambda t}. \end{aligned}$$

Therefore, $H(t)$ can be represented by the uniformly convergent series as follows

$$H(t) = e^{-\lambda t} \sum_{\ell=0}^{\infty} \sum_{r=0}^{K-1} \lambda^{K\ell+r} I_\ell^{(r)}(t) = e^{-\lambda t} \sum_{j=0}^{\infty} \sum_{r=0}^{K-1} \psi_K^{j*}(t) * I_0^{(r)}(t). \tag{10}$$

Theorem 2 *The characteristic function $H(t)$, $t \ge 0$, is a solution of the following Volterra integral equation*

$$H(t) = e^{-\lambda t} \sum_{r=0}^{K-1} \frac{t^r}{r!} \varphi(ct) + \lambda^K \int_0^t \psi_K(t-u)\, e^{-\lambda(t-u)} \varphi(t-u) H(u)\, du \tag{11}$$

or equivalently by using the convolution operator

$$H(t) = \varphi(ct)\,(1 - F_K(t)) + (f_K\varphi) * H(t),$$

where $F_K(t)$ is the K-Erlang cdf and $f_K(t)$ is its correspondent pdf.

Proof It follows from (10) that

$$\begin{aligned}
H(t) &= e^{-\lambda t} \sum_{\ell=0}^{\infty} \sum_{r=0}^{K-1} \lambda^{K\ell+r}\, I_\ell^{(r)}(t) \\
&= e^{-\lambda t} \left\{ \sum_{r=0}^{K-1} \frac{t^r}{r!}\varphi(ct) + \sum_{\ell=1}^{\infty} \sum_{r=0}^{K-1} \lambda^{K\ell+r}\, I_\ell^{(r)}(t) \right\} \\
&= e^{-\lambda t} \left\{ \sum_{r=0}^{K-1} \frac{t^r}{r!}\varphi(ct) + \int_0^t \psi_K(t-u) \left(\sum_{\ell=1}^{\infty} \sum_{r=0}^{K-1} \lambda^{K\ell+r}\, I_{\ell-1}^{(r)}(u) \right) du \right\} \\
&= e^{-\lambda t} \left\{ \sum_{r=0}^{K-1} \frac{t^r}{r!}\varphi(ct) + \lambda^K \int_0^t \psi_K(t-u) \left(\sum_{\ell=0}^{\infty} \sum_{r=0}^{K-1} \lambda^{K\ell+r}\, I_\ell^{(r)}(u) \right) du \right\} \\
&= e^{-\lambda t} \left\{ \sum_{r=0}^{K-1} \frac{t^r}{r!}\varphi(ct) + \lambda^K \int_0^t \psi_K(t-u)\, e^{\lambda u} H(u)\, du \right\},
\end{aligned}$$

proving (11). □

The function $H(t)$ is a unique solution of (11) in the class of continuous functions. This uniqueness stems from the theory of the second kind Volterra equations with continuous kernel.

We can apply the Laplace transform $\hat{f}(s) = \int_0^\infty f(t)e^{-st}dt$ to (11) and we obtain

$$\hat{H}(s) = \frac{\sum_{r=0}^{K-1} \hat{\psi}_r(s)}{1 - \hat{f}_K(s)},$$

where $\psi_r(t) = \frac{t^r}{r!} e^{-\lambda t}\, \varphi(ct)$.

References

1. Goldstein, S.: On diffusion by discontinuous movements and on the telegraph equation. Q. J. Mech. Appl. Math. **4**, 129–156 (1951)
2. Kac, M.: A stochastic model related to the telegrapher's equation. Rocky Mt. J. Math. **4**, 497–509 (1974)
3. Di Crescenzo, A.: On random motions with velocities alternating at Erlang-distributed random times. Adv. Appl. Probab. **61**, 690–701 (2001)
4. Pogorui, A.A., Rodriguez-Dagnino, R.M.: One-dimensional semi-Markov evolutions with general Erlang sojourn times. Random Oper. Stoch. Equ. **13**, 1720–1724 (2005)
5. Pinsky, M.: Isotropic transport process on a Riemann manifold. Trans. Am. Math. Soc. **218**, 353–360 (1976)
6. Orsingher, E., De Gregorio, A.: Random flights in higher spaces. J. Theor. Probab. **20**, 769–806 (2007)
7. Stadje, W.: Exact solution for non-correlated random walk models. J. Stat. Phys. **56**, 415–435 (1989)

8. Kolesnik, A.D., Pinsky, M.A.: Random evolutions are driven by the hyperparabolic operators. J. Stat. Phys. **142**, 828–846 (2011)
9. Le Caer, G.: A Pearson-Dirichlet random walk. J. Stat. Phys. **140**, 728–751 (2010)
10. Le Caer, G.: A new family of solvable Pearson-Dirichlet random walks. J. Stat. Phys. **144**, 23–45 (2011)
11. Beghin, L., Orsingher, E.: Moving randomly amid scattered obstacles. Stochastics **82**, 201–229 (2010)
12. Kolesnik, A.D.: Random motions at finite speed in higher dimensions. J. Stat. Phys. **131**, 1039–1065 (2008)
13. Pogorui, A.A.: Fading evolution in multidimensional spaces. Ukr. Math. J. **62**(11), 1828–1834 (2010)
14. Vladimirov, V.S., Zharinov, V.V.: Equations of Mathematical Physics. Physical and Mathematical Literature, Moscow (2000) (in Russian)
15. Bochner, S., Chandrasekharan, K.: Fourier Transforms. Annals of Mathematics Studies, vol. 19. Princeton University Press, Princeton (1949)