Complex Variables and Elliptic Equations
Vol. 52, No. 12, December 2007, 1155–1159

Hyperholomorphic functions on commutative algebras

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Communicated by R. P. Gilbert

(*Received 14 February 2007; in final form 4 July 2007*)

In this article, we study properties of hyperholomorphic functions on commutative finite-dimensional algebras. The Cauchy–Riemann type conditions for hyperholomorphic functions is investigated. We prove that a hyperholomorphic function on a commutative finite-dimensional algebra can be expanded in a Taylor series. We also present a technique for computing zeros of polynomials in commutative algebras.

Keywords: Hyperholomorphic function; Commutative algebra; Polynomial

AMS Subject Classifications: Primary 32A10; Secondary 13M10

1. Introduction

The development of hyperholomorphic function analysis has renewed interest in mathematics and physics because of fruitful applications. One of the most popular hypercomplex analysis is quaternionic analysis; however, noncommutativity of quaternion algebra causes many intractable problems, for instance, the problem of expansion of a hyperholomorphic quaternionic function in a Taylor series. In this regard, hyperholomorphic analysis on commutative unitary algebras is a natural extension of complex analysis, despite the fact that in these algebras we have the problem of zero divisors. There are many commutative generalizations of complex numbers, say, hyperbolic numbers, bicomplex algebra etc. [1]. In [2], it is proved that hyperholomorphic functions on bicomplex algebra can be expanded in a Taylor series. In this article, we generalize this result to any finite-dimensional commutative algebra.

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Complex Variables and Elliptic Equations
ISSN 1747-6933 print/ISSN 1747-6941 online © 2007 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/17476930701609437

2. Differentiation in finite-dimensional commutative algebra

Let **A** be a finite-dimensional commutative unitary algebra over $K = \mathbb{R}$ (or $\mathbb{C}$), a set of vectors $\vec{e}_0, \vec{e}_1, \ldots, \vec{e}_n$ be a basis of **A**, and $\vec{e}_0$ be algebra identity. Consider a function $\vec{f}: \mathbf{A} \to \mathbf{A}$ of the following form

$$\vec{f}(\vec{x}) = \sum_{k=0}^{n} \vec{e}_k u_k(\vec{x}),$$

where $u_k(\vec{x}) = u_k(x_0, x_1, \ldots, x_n)$ are real (or complex) functions of $n+1$ arguments.

Definition 2.1 $\vec{f}(\vec{x})$ is called **A**-differentiable at a point $\vec{x}_0 \in \mathbf{A}$ if there exists the function $\vec{f}': \mathbf{A} \to \mathbf{A}$ such that for any $\vec{h} \in \mathbf{A}$

$$\vec{h}\vec{f}'(\vec{x}_0) = \lim_{\varepsilon \to 0} \frac{\vec{f}\left(\vec{x}_0 + \varepsilon\vec{h}\right) - \vec{f}(\vec{x}_0)}{\varepsilon}, \tag{2.1}$$

where $\vec{f}'$ doesn't depend on $\vec{h}$.

A function $\vec{f}$ is said to be **A**-holomorphic if $\vec{f}$ is **A**-differentiable at every point of **A**.

THEOREM 2.2 *A function* $\vec{f}(\vec{x}) = \sum_{k=0}^{n} \vec{e}_k u_k(\vec{x})$ **A**-*holomorphic if and only if there exists the function* $\vec{f}': \mathbf{A} \to \mathbf{A}$ *such that for all* $k = 0, 1, \ldots, n$, *and* $\forall \vec{x} \in \mathbf{A}$

$$\vec{e}_k\vec{f}'(\vec{x}) = \lim_{\varepsilon \to 0} \frac{\vec{f}(\vec{x} + \varepsilon\vec{e}_k) - \vec{f}(\vec{x})}{\varepsilon}, \tag{2.2}$$

where $\vec{f}'$ *does not depend on* $\vec{e}_k$.

Proof Suppose that (2.2) is fulfilled, then it is easily verified that

$$\begin{aligned}
\vec{f}' &= \lim_{\varepsilon \to 0} \frac{\vec{f}(\vec{x} + \varepsilon\vec{e}_0) - \vec{f}(\vec{x})}{\varepsilon} = \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_0}, \\
\vec{e}_1\vec{f}' &= \lim_{\varepsilon \to 0} \frac{\vec{f}(\vec{x} + \varepsilon\vec{e}_1) - \vec{f}(\vec{x})}{\varepsilon} = \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_1} = \vec{e}_1 \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_0}, \\
&\vdots \\
\vec{e}_n\vec{f}' &= \lim_{\varepsilon \to 0} \frac{\vec{f}(\vec{x} + \varepsilon\vec{e}_n) - \vec{f}(\vec{x})}{\varepsilon} = \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_n} = \vec{e}_n \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_0}.
\end{aligned} \tag{2.3}$$

Consider $\vec{h} = \sum_{k=0}^{n} h_k \vec{e}_k$. It follows from equation (2.3) that

$$\begin{aligned}
h_0\vec{f}' &= h_0 \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_0}, \\
h_1\vec{e}_1\vec{f}' &= h_1 \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_1}, \\
&\vdots \\
h_n\vec{e}_n\vec{f}' &= h_n \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_n}.
\end{aligned}$$

This implies that

$$\begin{aligned}\vec{h}\vec{f}' &= h_0 \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_0} + h_1 \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_1} + \cdots + h_n \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_n} \\ &= \lim_{\varepsilon \to 0} \frac{\vec{f}\left(\vec{x}_0 + \varepsilon \vec{h}\right) - \vec{f}(\vec{x}_0)}{\varepsilon}.\end{aligned}$$

Furthermore, it follows from equation (2.3) that

$$\begin{aligned}&h_0 \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_0} + h_1 \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_1} + \cdots + h_n \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_n} \\ &\quad = h_0 \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_0} + h_1 \vec{e}_1 \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_0} + \cdots + h_n \vec{e}_n \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_0}.\end{aligned}$$

Therefore, for every $\vec{h} \in A$

$$\vec{h} \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_0} = \lim_{\varepsilon \to 0} \frac{\vec{f}\left(\vec{x}_0 + \varepsilon \vec{h}\right) - \vec{f}(\vec{x}_0)}{\varepsilon}$$

or

$$\vec{f}' = \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_0}. \tag{2.4}$$

■

By using equation (2.3), we have

$$\begin{aligned}\sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_1} &= \vec{e}_1 \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_0}, \\ \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_2} &= \vec{e}_2 \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_0}, \\ &\vdots \\ \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_n} &= \vec{e}_n \sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_0}.\end{aligned} \tag{2.5}$$

Equation (2.5) will be called the Cauchy-Riemann type conditions. It follows from Theorem 1.2 that if $\vec{f}(\vec{x}) = \sum_{k=0}^{n} \vec{e}_k u_k(\vec{x})$ satisfies (1.5) then $\vec{f}$ is **A**-holomorphic.

THEOREM 2.3 *If $\vec{f}$ is **A**-holomorphic and $u_k \in C^\infty, k = 1, \ldots, n$, then for all $l \geq 1$ there exists $\vec{f}^{(l)}$, which is **A**-holomorphic and $\vec{f}^{(l)} = \sum_{k=0}^{n} \vec{e}_k (\partial^l u_k / \partial x_0^l)$.*

Proof It is easy to see that functions $u_k' = (\partial u_k / \partial \vec{x}_0), k = 1, \ldots, n$, satisfy conditions (2.5) since $u_k \in C^\infty$. So $\vec{f}'$ is **A**-holomorphic and $\vec{f}' = \sum_{k=0}^{n} \vec{e}_k (\partial^2 u_k / \partial x_0^2)$ (see (2.4)). In complete analogy with this we can show that $\vec{f}^{(l)}$ is **A**-holomorphic and $\vec{f}^{(l)} = \sum_{k=0}^{n} \vec{e}_k (\partial^l u_k / \partial x_0^l)$. ■

THEOREM 2.4 *Suppose $\vec{f}$ is an* **A***-holomorphic function with $u_k \in C^\infty, k = 0, 1, \ldots, n$ and for fixed $\vec{x}, \vec{h} \in$* **A** *there exists $K > 0$ such that $\|\vec{f}^{(l)}\vec{h}\| K^l$ for $l = 1, 2, \ldots$. Then $\vec{f}$ can be expanded in a Taylor series*

$$\vec{f}\left(\vec{x}+\vec{h}\right) = \vec{f}(\vec{x}) + \vec{f}'(\vec{x})\vec{h} + \frac{1}{2!}\vec{f}'(\vec{x})\vec{h}^2 + \cdots$$

Proof Consider the function $\vec{F}(t) = \vec{f}(\vec{x} + t\vec{h})$. It is easily verified that $\mathrm{d}^l\vec{F}(0)/\mathrm{d}t^l = \vec{f}^l(\vec{x})\vec{h}^l$. Taking into account that $\|\vec{f}^{(l)}\vec{h}\| K^l$ for $> l = 1, 2, \ldots$, it follows that the function $\vec{F}(t)$ can be expanded in a Taylor series as follows

$$\vec{F}(t) = \vec{F}(0) + \sum_{l \geq 1} \frac{1}{l!} \frac{\mathrm{d}^l\vec{F}(0)}{\mathrm{d}t^l} t^l.$$

Putting $t = 1$, we get (2.6). ■

In the particular case where a bicomplex (or hyperbolic) function is hyperholomorphic and satisfies Theorem 2.4, it can be expanded in a Taylor series (2.6) [1, 2].

3. Zeros of polynomials in commutative algebras

Since on numerous occasions **A**-holomorphic function can be approximated by its Taylor polynomial of finite degree, zeros of such functions might be studied if we can calculate zeros of polynomials. Let $p_m(w) = a_m w^m + a_{m-1} w^{m-1} + \cdots + a_0$ be a polynomial in the algebra **A**. Our purpose is to investigate the structure of the set of zeros of the equation

$$p_m(w) = 0. \tag{3.1}$$

THEOREM 3.1 *If* **A** *has $n+1$ non trivial idempotents $i_0, i_1, \ldots, i_n$ such that $i_p i_r = 0$ for $p \neq r$, and $\sum_{l=0}^n i_l = 1$, then equation 3.1 can be reduced to the system of polynomial equations in the field K.*

Proof As a preliminary to the proof of the theorem, we shall prove several auxiliary lemmas. ■

LEMMA 3.2 *Idempotents $i_0, i_1, \ldots, i_n$ are linearly independent vectors.*

Proof Suppose the contrary, then there exist $k_0, k_1, \ldots, k_n \in K$ such that $\sum_{p=0}^n |k_p| > 0$ and $\sum_{p=0}^n k_p i_p = 0$. By using the properties of idempotents, we have $k_p i_p = 0$ for all $p = 0, 1, \ldots, n$, but this is impossible. Indeed, if $k_p i_p = 0$ for $k_p \neq 0$, then $k_p^{-1}(k_p i_p) = i_p = 0$. ■

Denote by $I_l = \{a i_l | a \in A\}$ the principal ideal generated by i_l, $l = 0, 1, \ldots, n$. It follows from Theorem 2.3 that the algebra **A** can be decomposed in the direct sum (the Pierce decomposition): $A = I_0 \oplus I_1 \oplus \cdots \oplus I_n$.

LEMMA 3.3 *If $a \in I_l$ then there exists $k \in K$ such that $a = k i_1$, i.e., the ideal I_l can be represented in the following form $I_l = \{k i_l | k \in K\}$.*

Proof For $a \in I_l$ there exists $b \in \mathbf{A}$ such that $a = bi_l$. Since $i_0, i_1, \ldots, i_n$ are linearly independent, there exist $k_0, k_1, \ldots, k_n \in K$ such that $b = \sum_{p=0}^{n} k_p i_p$. Thus, $a = bi_l = (\sum_{p=0}^{n} k_p i_p) i_l = k_l i_l$. ■

Let us consider decompositions

$$\begin{aligned} a_r &= a_r^{(0)} + \cdots + a_r^{(n)}, \quad r = 0, 1, \ldots m, \\ w &= w_0 + \cdots + w_n, \end{aligned} \tag{3.2}$$

where $a_r^{(p)}, w_p \in I_p$. Plugging (3.2) into (3.1), we obtain the following system of polynomial equations

$$\begin{gathered} a_m^{(0)} w_0^m + a_{m-1}^{(0)} w_0^{m-1} + \cdots + a_0^{(0)} = 0, \\ a_m^{(1)} w_1^m + a_{m-1}^{(1)} w_1^{m-1} + \cdots + a_0^{(1)} = 0, \\ \vdots \\ a_m^{(n)} w_n^m + a_{m-1}^{(n)} w_n^{m-1} + \cdots + a_0^{(n)} = 0. \end{gathered} \tag{3.3}$$

It follows from Lemma 1.3 that $a_r^{(s)} = k_r^{(s)} i_s, w_s^r = x i_s$, where $k_r^{(s)}, x \in K$.

Therefore, taking i_s out of the expression $a_m^{(s)} w_s^m + a_{m-1}^{(s)} w_s^{m-1} + \cdots + a_0^{(s)} = 0, s = 0, \ldots, n$, the system (3.3) can be reduced to the system of $n+1$ polynomial equations in K with coefficients $k_r^{(s)}$. ■

Example 3.4 Let $\mathbf{A}$ be the bicomplex algebra, i.e., $\mathbf{A} = \{c_0 + ec_1 | c_0, c_1 \in \mathbb{C}\}$, where $e^2 = 1$ and $\mathbf{A}$ is commutative. The bicomplex algebra has two idempotents $i_0 = (1+e)/2$ and $i_1 = (1-e)/2$. It is easy to see that $i_0 i_1 = 0$ and $i_0 + i_1 = 1$. Thus, in this case polynomial equation (3.1) can be reduced to the system of two polynomial equations in $\mathbb{C}$ [3]. Many properties of quarternionic polynomials have been presented in [4].

Example 3.5 Suppose $\mathbf{A}$ is the commutative algebra of the following form $A = \{a_0 + ea_1 + fa_2 + ga_3 | a_k \in \mathbb{R}\}$, where $e^2 = f^2 = g^2 = 1$ and $efg = 1$. This algebra has four idempotents: $i_0 = (1+e+f+g)/4$, $i_1 = (1-e-f+g)/4$, $i_2 = (1+e-f-g)/4$, $i_3 = (1-e+f-g)/4$. It is easy to see that $i_k i_l = 0$ for $k \neq l$, and $i_0 + i_1 + i_2 + i_3 = 1$. Therefore, in this case polynomial equation (3.1) can be reduced to the system of four polynomial equations in $\mathbb{R}$.

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