УДК 517.5

R. R. Salimov, E. A. Sevost'yanov

ACL AND DIFFERENTIABILITY OF OPEN DISCRETE RING (p, Q)-MAPPINGS

R. R. Salimov, E. A. Sevost'yanov. *ACL and differentiability of open discrete ring* (p, Q)*-mappings*, Mat. Stud. **35** (2011), 28–36.

We study the so–called (p, Q)-mappings which naturally generalize quasiregular mappings. It is proved that open discrete ring (p, Q)-mappings are differentiable almost everywhere as $p > n-1$ and locally integrable Q. Furthermore, we prove that open discrete (p, Q)-mappings belong to the class ACL in $\mathbb{R}^n$ and $f \in W^{1,1}_{\rm loc}$ the same conditions on p and Q.

Р. Р. Салимов, Е. А. Севостьянов. *ACL и дифференцируемость открытых дискретных* (p, Q)*-отображений* // Мат. Студiї. – 2011. – Т.35, №1. – C.28–36.

Изучаются так называемые (p, Q)-отображения, являющиеся естественным обобщением квазирегулярных отображений. Доказано, что открытые дискретные (p, Q)-отображения дифференцируемы почти всюду при $p > n-1$ и локально интегрируемой функции Q. Более того, доказано, что открытые дискретные (p, Q)-отображения принадлежат классу ACL в $\mathbb{R}^n$ и $f \in W^{1,1}_{\rm loc}$ при тех же условиях на p и Q.

1. Introduction. Recall that, given a family of paths Γ in $\mathbb{R}^n$, a Borel function $\rho\colon \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in {\rm adm}\Gamma$, if

$$\int_\gamma \rho ds \geq 1 \tag{1}$$

for all $\gamma \in \Gamma$. Given $p > 0$, the *p-modulus* of Γ is the quantity

$$M_p(\Gamma) = \inf_{\rho \in {\rm adm}\Gamma} \int_G \rho^p(x) dm(x). \tag{2}$$

Let D be a domain in $\mathbb{R}^n, n \geq 2$, and $f\colon D \to \mathbb{R}^n$ a Q-quasiconformal mapping. Then necessarily

$$M_n(f\Gamma) \leq \int_D K_I(x, f) \cdot \rho^n(x) dm(x) \tag{3}$$

for every family Γ of paths in D and every admissible function ρ for Γ, see e.g. [3], where $K_I(x, f)$ stands for the well–known inner dilatation of f at x.

Given a domain D and two sets E and F in $\overline{\mathbb{R}^n}$, $n \geq 2$, $\Gamma(E, F, D)$ denotes the family of all paths $\gamma\colon [a, b] \to \overline{\mathbb{R}^n}$ which join E and F in D, i.e., $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$. Let $r_0 = {\rm dist}(x_0, \partial D)$ and $Q\colon D \to [0, \infty]$ is a measurable function. Set

$$A(r_1, r_2, x_0) = \{x \in \mathbb{R}^n \colon r_1 < |x - x_0| < r_2\},$$
$$S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n \colon |x - x_0| = r_i\}, \quad i = 1, 2.$$

2010 *Mathematics Subject Classification:* 30C65, 30C75.

One can replace the above necessary condition (3) with the following, equivalent by Gehring's result [7], inequality

$$M_n \left(f \left(\Gamma \left(S_1, S_2, A \right) \right) \right) \leq \int\limits_{A(r_1, r_2, x_0)} K_I(x, f) \cdot \rho^n \left(|x - x_0| \right) dm(x) \tag{4}$$

for every point $x_0 \in D$ and every r_1, r_2, such that $0 < r_1 < r_2 < r_0 = \operatorname{dist}(x_0, \partial D)$. The above inequalities (3) and (4) together with the modulus technique are the powerful tools for the study of quasiconformal (quasiregular) mappings in the plane and in space, see e.g. [8], [15], [16] and [18]. In order to extend as much as possible the set of maps for the study of which the well developed modulus technique can be also applied, we replace in (3) (or in (4)) the dilatation $K_I(x, f)$ with a measurable function $Q(x)$, say of the class $L^1_{\mathrm{loc}}(D)$, and then declare the inequality

$$M_p(f\Gamma) \leq \int_D Q(x) \cdot \rho^p(x) dm(x) \tag{5}$$

or

$$M_p \left(f \left(\Gamma \left(S_1, S_2, A \right) \right) \right) \leq \int_A Q(x) \cdot \eta^p(|x - x_0|) dm(x) \tag{6}$$

for every measurable function $\eta \colon [0, \infty] \to [0, \infty]$ for which $\int_{r_1}^{r_2} \eta(t) dt \geq 1$. The above inequalities (5) and (6) are necessary conditions for the mapping $f \colon D \to \mathbb{R}^n$ to belong to the class of the Q-homeomorphisms as $p = n$, or the ring Q-homeomorphisms, respectively, see [12]. See also the conception of the weighted modulus, [1], applications of the Q-homeomorphisms, cf. [4] and [5]. The so-called (p, Q)-homeomorphisms, which defined as homeomorphisms satisfying the (5), were presented and studied by A. Golberg, see, e.g., [6].

A mapping $f \colon D \to \overline{\mathbb{R}^n}$ is said to be a (p, Q)-*mapping* if f satisfies (5) for every $\rho \in \operatorname{adm}\Gamma$. A mapping $f \colon D \to \overline{\mathbb{R}^n}$ is said to be a ring (p, Q)-*mapping* if f satisfies (6) for every $x_0 \in D$ and every measurable function $\eta \colon [0, \infty] \to [0, \infty]$ for which $\int_{r_1}^{r_2} \eta(t) dt \geq 1$. Note that by definition, every K-quasiconformal (or K-quasiregular) mapping satisfies (6) and (5) with $p = n$ and $Q(x) \equiv K$. Our paper is devoted to the study of mappings having unbounded $Q(x)$ in above definitions and $p \neq n$.

Recall that a mapping $f \colon D \to \mathbb{R}^n$ is said to be *absolutely continuous on lines*, write $f \in ACL$, if all coordinate functions $f = (f_1, \ldots, f_n)$ are absolutely continuous on almost all straight lines parallel to the coordinate axes for any n-dimensional parallelepiped P with edges parallel to the coordinate axes and such that $\overline{P} \subset D$.

It is well–known that quasiconformal and quasiregular mappings are absolutely continuous on lines, see e.g. Corollary 31.4 in [18], Lemma 4.11 and Theorem 4.13 in [11], and differentiable a.e., see e.g. Corollary 32.2 in [18], Theorem 2.1 Ch. I in [16] and Theorem 4 in [14]. Moreover, in the plane case, every ACL-homeomorphism is differentiable a.e., see [8]. However, above results did not give any information about differentiability (or ACL) for more general mappings having non–bounded dilatation.

The goal of the present paper is to prove the following:

I. Open discrete ring (p, Q)-mappings $f \colon D \to \overline{\mathbb{R}^n}$ with $Q \in L^1_{\mathrm{loc}}$ and $p > n - 1$ are differentiable a.e. in D and satisfy the inequality $\|f'(x)\|^p \leq C.|J(x, f)|^{1-n+p} Q^{n-1}(x)$ a.e. where a constant C depends only on n and p.

II. Open discrete (p, Q)-mappings $f \colon D \to \overline{\mathbb{R}^n}$ with $Q \in L^1_{\mathrm{loc}}$ and $p > n - 1$ belong to the class ACL in D and, consequently, to the Sobolev class $W^{1,1}_{\mathrm{loc}}$.

Above results were firstly stated by the authors in the paper [20] for the case $p = n$. Moreover, for every $p > n-1$, $p \neq n$, the ACL-property and differentiability a.e. of the homeomorphisms satisfying the (5) were stated by A. Golberg in [6]. In the present paper we prove the same properties of ACL and differentiability for more general mappings, admitting the branch points and satisfying the (5) or (6) as $p > n-1$, $p \neq n$, which are supposed to be open and discrete, only.

2. Preliminaries. Let D be a domain in $\mathbb{R}^n, n \geq 2$. A mapping $f\colon D \to \mathbb{R}^n$ is said to be *discrete* if the preimage $f^{-1}(y)$ of every point $y \in \mathbb{R}^n$ consists of isolated points, and an *open* if the image of every open set $U \subseteq D$ is open in $\mathbb{R}^n$. The notation $G \Subset D$ means that $\overline{G}$ is a compact subset of D. We suppose that $f\colon D \to \mathbb{R}^n$ is continuous and sense–preserving, i.e. a topological index $\mu(y, f, G) > 0$ for any $G \Subset D$ and $y \in f(G) \setminus f(\partial G)$. A *neighborhood* of a point x or a set A is called an open set B such that $x \in \mathrm{Int}B$ or $A \in \mathrm{Int}B$, correspondingly. Suppose that $x \in D$ has a connected neighborhood G such that $\overline{G} \cap f^{-1}(f(x)) = \{x\}$. Then $\mu(f(x), f, G)$ is well–defined and independent of the choice of G for discrete open f and denoted by $i(x, f)$. For $f\colon D \to \mathbb{R}^n$ and $E \subset D$, we use the *multiplicity functions*

$$N(y, f, E) = \mathrm{card}\,\{x \in E\colon f(x) = y\}\,,\ N(f, E) = \sup_{y \in \mathbb{R}^n} N(y, f, E).$$

In what follows, $B(x_0, r) = \{x \in \mathbb{R}^n\colon |x - x_0| < r\}$ and $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$. The above definitions can be extended in a natural way to mappings $f\colon D \to \overline{\mathbb{R}^n}$.

The following notion is motivated by the Gehring ring definition of quasiconformality, see [7], and generalizes a notion of ring Q-homeomorphism, see Ch. VII in [12]. In what follows $p \geq 1$.

A homeomorphism $f\colon D \to \overline{\mathbb{R}^n}$ is said to be a *ring (p, Q)-homeomorphism at a point $x_0 \in D$,* if

$$M_p\left(f\left(\Gamma\left(S_1, S_2, A\right)\right)\right) \leq \int_A Q(x) \cdot \eta^p(|x - x_0|)dm(x) \tag{7}$$

holds for every ring $A = A(r_1, r_2, x_0)$, $0 < r_1 < r_2 < r_0$ and every measurable function $\eta\colon (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r)dr \geq 1.$$

If (7) holds for every $x_0 \in D$, f is said to be a ring (p, Q)-homeomorphism. In general case, every (p, Q)-homeomorphism $f\colon D \to \overline{\mathbb{R}^n}$ is a ring (p, Q)-homeomorphism, but the inverse conclusion, generally speaking, is not true. In fact, it can be constructed some examples of ring (p, Q)-homeomorphisms as $p = n$ in a fixed point x_0 such that $Q(x) \in (0, 1)$ on some set for which x_0 is a density point, see e.g. Ch. XI in [12].

Let $D \subset \mathbb{R}^n$, $n \geq 2$, be a domain and $Q\colon D \to [0, \infty]$ be a measurable function. We say that a continuous sense–preserving mapping $f\colon D \to \overline{\mathbb{R}^n}$ is a *ring (p, Q)-mapping in D* if (7) holds for every $x_0 \in D$. Note that correspondingly to these definitions, the class of so–called (p, Q)-mappings which consists of the continuous sense–preserving mappings satisfying condition (5) is included in the class of ring (p, Q)-mappings.

Correspondingly to [11] or [16], a *condenser* is a pair $E = (A, C)$ where $A \subset \mathbb{R}^n$ is open and C is non–empty compact set contained in A . A condenser $E = (A, C)$ is said to be in a domain G if $A \subset G$. For a given condenser $E = (A, C)$, we set

$$\mathrm{cap}_p E = \mathrm{cap}_p\,(A, C) = \inf_{u \in W_0(E)} \int_A |\nabla u|^p dm(x) \tag{8}$$

where $W_0(E) = W_0(A, C)$ is the family of non–negative functions $u\colon A \to R^1$ such that (1) u is continuous and finite on A, (2) $u(x) \geq 1$ for $x \in C$, and (3) u is ACL. In the above formula

$$|\nabla u| = \left(\sum_{i=1}^{n} (\partial_i u)^2\right)^{1/2}.$$

The quantity $\mathrm{cap}_p E$ is called the *p-capacity* of the condenser E.

We say that a family of curves Γ_1 is minorized by a family Γ_2, denoted by $\Gamma_1 > \Gamma_2$, if for every curve $\gamma \in \Gamma_1$ there is a subcurve that belongs to the family Γ_2. It is known that $M_p(\Gamma_1) \leq M_p(\Gamma_2)$ as $\Gamma_1 > \Gamma_2$, see Theorem 6.4 in [18].

3. Differentiability. Let $f\colon D \to \mathbb{R}^n$ be a discrete open mapping. Let $\beta\colon [a, b) \to \mathbb{R}^n$ be a path and $x \in f^{-1}(\beta(a))$. A path $\alpha\colon [a, c) \to D$ is called a *maximal f-lifting* of β starting at x if (1) $\alpha(a) = x$; (2) $f \circ \alpha = \beta|_{[a,c)}$; (3) if $c < c' \leq b$, then there is no path $\alpha'\colon [a, c') \to D$ such that $\alpha = \alpha'|_{[a,c)}$ and $f \circ \alpha' = \beta|_{[a,c')}$. If f is a discrete open mapping, then every path β with $x \in f^{-1}(\beta(a))$ has a maximal f-lifting starting at the point x, see Corollary 3.3, Ch. II in [16]. We need the following statement, see Proposition 10.2, Ch. II in [16].

Lemma 1. *Let $E = (A, C)$ be a condenser in $\mathbb{R}^n$ and let Γ_E be the family of all paths of the form $\gamma\colon [a, b) \to A$ with $\gamma(a) \in C$ and $|\gamma| \cap (A \setminus F) \neq \varnothing$ for every compact $F \subset A$. Then $\mathrm{cap}_p E = M_p(\Gamma_E)$.*

Theorem 1. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and $f\colon D \to \overline{\mathbb{R}^n}$ be a ring (p, Q)-mapping with $Q \in L^1_{\mathrm{loc}}$ and $p > n - 1$. Suppose that f is discrete and open. Then f is differentiable a.e. in D.*

Proof. Without loss of generality we may assume that $\infty \notin D' = f(D)$. Let us consider the set function $\Phi(B) = m(f(B))$ defined over the algebra of all the Borel sets B in D. By 2.2, 2.3 and 2.12 in [11]

$$\varphi(x) = \limsup_{\varepsilon \to 0} \frac{\Phi(B(x, \varepsilon))}{\Omega_n \varepsilon^n} < \infty \tag{9}$$

for a.e. $x \in D$. Consider the spherical ring $R_\varepsilon(x) = \{y\colon \varepsilon < |x - y| < 2\varepsilon\}$, $x \in D$, with $\varepsilon > 0$ such that $B(x, 2\varepsilon) \subset D$. Note that $E = (B(x, 2\varepsilon), \overline{B(x, \varepsilon)})$ is a condenser in D and

$$f(E) = (f(B(x, 2\varepsilon)), f(\overline{B(x, \varepsilon)}))$$

is a condenser in D'. Let Γ_E and $\Gamma_{f(E)}$ be path families from Lemma 1. Then

$$\mathrm{cap}_p(f(B(x, 2\varepsilon), f(\overline{B(x, \varepsilon)})) = M_p(\Gamma_{f(E)}). \tag{10}$$

Let Γ^* be a family of maximal f-liftings of $\Gamma_{f(E)}$ starting at $\overline{B(x, \varepsilon)}$. We show that $\Gamma^* \subset \Gamma_E$. Suppose the contrary. Then there is a path $\beta\colon [a, b) \to \mathbb{R}^n$ of $\Gamma_{f(E)}$ such that the corresponding maximal f-lifting $\alpha\colon [a, c) \to B(x, 2\varepsilon)$ is contained in some compact K inside of $B(x, 2\varepsilon)$. Thus, $\overline{\alpha}$ is a compactum in $B(x, 2\varepsilon)$, see Theorem 2, §45 in [10]. Remark that $c \neq b$. Indeed, in the contrary case $\overline{\beta}$ is a compactum in $f(A)$ that contradicts the condition $\beta \in \Gamma_{f(E)}$. Consider the set

$$G = \left\{x \in \mathbb{R}^n\colon x = \lim_{k \to \infty} \alpha(t_k)\right\},\ t_k \in [a, c),\ \lim_{k \to \infty} t_k = c.$$

Without loss of generality we may assume that t_k is a monotone sequence. By continuity of f, for $x \in G$, $f(\alpha(t_k)) \to f(x)$ as $k \to \infty$ where $t_k \in [a, c), t_k \to c$ as $k \to \infty$. However, $f(\alpha(t_k)) = \beta(t_k) \to \beta(c)$ as $k \to \infty$. Thus, f is a constant in $G \subset B(x, 2\varepsilon)$. On the other hand, from the Cantor condition on the compact $\overline{\alpha}$,

$$G = \bigcap_{k=1}^{\infty} \overline{\alpha([t_k, c))} = \limsup_{k\to\infty} \alpha([t_k, c)) = \liminf_{k\to\infty} \alpha([t_k, c)) \neq \varnothing$$

by monotonicity of the sequences of connected sets $\alpha([t_k, c))$, see [10]. Thus, G is connected by relation (9.12) of Ch. I in [19]. Consequently, G is a single point by discreteness of f. So, a path $\alpha\colon [a, c) \to B(x, 2\varepsilon)$ can be extended to $\alpha\colon [a, c] \to K \subset B(x, 2\varepsilon)$ and $f(\alpha(c)) = \beta(c)$. By Corollary 3.3 Ch. II in [16], we can construct a maximal f-lifting α' of $\beta|_{[c,b)}$ started at $\alpha(c)$. United the liftings α and α', we have a new f-lifting α'' of β defined on $[a, c')$, $c' \in (c, b)$, that contradicts the maximality of f-lifting α. Thus, $\Gamma^* \subset \Gamma_E$. Remark that $\Gamma_{f(E)} > f\Gamma^*$ and consequently, $M_p\left(\Gamma_{f(E)}\right) \leq M_p(f\Gamma^*) \leq M_p(f\Gamma_E)$.

Let $\{r_i\}_{i=1}^{\infty}$ be an arbitrary sequence of numbers with $\varepsilon < r_i < 2\varepsilon$ such that $r_i \to 2\varepsilon - 0$. Denote by Γ_i the family of paths joining the spheres $|x - y| = \varepsilon$ and $|x - y| = r_i$ in the ring $\varepsilon < |x - y| < r_i$. Then $\Gamma_E > \Gamma_i$ for every $i \in \mathbb{N}$. Consider the family of functions

$$\eta_{i,\varepsilon}(t) = \begin{cases} \frac{1}{r_i - \varepsilon}, & t \in (\varepsilon, r_i), \\ 0, & t \in \mathbb{R} \setminus (\varepsilon, r_i). \end{cases}$$

By the definition of a ring Q-mapping,

$$M_p(f\Gamma_E) \leq M_p(f\Gamma_i) \leq \frac{1}{(r_i - \varepsilon)^p} \int\limits_{\varepsilon<|x-y|<r_i} Q(y)dm(y) \leq \frac{1}{(r_i - \varepsilon)^p} \int\limits_{B(x,2\varepsilon)} Q(y)dm(y). \quad (11)$$

Letting to the limit in (11) as $i \to \infty$, we obtain

$$M_p(f\Gamma_E) \leq \frac{1}{\varepsilon^p} \int\limits_{B(x,2\varepsilon)} Q(y)dm(y). \quad (12)$$

From (10) and (12)

$$\operatorname{cap}_p\left(f(B(x, 2\varepsilon)), f\left(\overline{B(x, \varepsilon)}\right)\right) \leq \frac{1}{\varepsilon^p} \int\limits_{B(x,2\varepsilon)} Q(y)dm(y). \quad (13)$$

On the other hand, by Proposition 6 in [9]

$$\operatorname{cap}_p\left(f(B(x, 2\varepsilon)), f\left(\overline{B(x, \varepsilon)}\right)\right) \geq \left(c_1 \frac{d^p(f(B(x, \varepsilon)))}{[m(f(B(x, 2\varepsilon)))]^{1-n+p}}\right)^{\frac{1}{n-1}} \quad (14)$$

where c_1 depends only on n and p, $d(A)$ is the diameter and $m(A)$ is the Lebesgue measure of A in $\mathbb{R}^n$.

Combining (13) and (14), we obtain that

$$\frac{d(f(B(x, \varepsilon)))}{\varepsilon} \leq c_2 \Big(\frac{m(f(B(x, 2\varepsilon)))}{m(B(x, 2\varepsilon))}\Big)^{\frac{1-n+p}{p}} \Big(\frac{1}{m(B(x, 2\varepsilon))} \int\limits_{B(x,2\varepsilon)} Q(y)dm(y)\Big)^{\frac{n-1}{p}}$$

and hence,

$$L(x,f) \le \limsup_{\varepsilon\to 0} \frac{d(f\,(B(x,\varepsilon)))}{\varepsilon} \le c_2 \varphi^{\frac{1-n+p}{p}}(x) Q^{\frac{n-1}{p}}(x)$$

where

$$L(x,f) = \limsup_{y\to x} \frac{|f(y)-f(x)|}{|y-x|}. \tag{15}$$

Thus, $L(x,f) < \infty$ a.e. in D. Finally, applying the Rademacher–Stepanov theorem, see e.g. [17], p. 311, we conclude that f is differentiable a.e. in D. □

Corollary 1. *Let D be a domain in $\mathbb{R}^n$, $n \ge 2$, and $f\colon D \to \overline{\mathbb{R}^n}$ be a ring (p,Q)-mapping with $Q \in L^1_{\mathrm{loc}}$ and $p > n-1$. Suppose that f is discrete and open. Then the partial derivatives of f are locally integrable.*

Proof. Given a compact set $V \subset D$, we have

$$\int_V L(x,f)dm(x) \le c_2 \int_V \varphi^{\frac{1-n+p}{p}}(x) Q^{\frac{n-1}{p}}(x) dm(x).$$

Applying the Hölder inequality, see (17.3) in [2] and taking into account that $p > n-1$, we obtain

$$\int_V \varphi^{\frac{1-n+p}{p}}(x) Q^{\frac{n-1}{p}}(x)dm(x) \le \Big(\int_V \varphi(x)dm(x)\Big)^{\frac{1-n+p}{p}} \Big(\int_V Q(x)dm(x)\Big)^{\frac{n-1}{p}}$$

and since $Q \in L^1_{\mathrm{loc}}$

$$\int_V L(x,f)dm(x) \le c_2 \left(N(f,V)\right)^{2(1-n+p)/p} \cdot \Big(\int_V Q(x)dm(x)\Big)^{\frac{n-1}{p}} < \infty,$$

see Lemma 2.3 in [11]. □

Corollary 2. *Let D be a domain in $\mathbb{R}^n$, $n \ge 2$, and $f\colon D \to \overline{\mathbb{R}^n}$ be a ring (p,Q)-mapping with $Q \in L^1_{\mathrm{loc}}$ and $p > n-1$. Suppose that f is discrete and open. Then*

$$\|f'(x)\|^p \le C \cdot |J(x,f)|^{1-n+p} Q^{n-1}(x)$$

a.e. where the constant C depends on n and p only.

4. On the ACL property of discrete open (p, Q)-mappings.

Theorem 2. *Let D be a domain in $\mathbb{R}^n$, $n \ge 2$, and $f\colon D \to \overline{\mathbb{R}^n}$ be a ring (p,Q)-mapping with $Q \in L^1_{\mathrm{loc}}$ and $p > n-1$. Suppose that f is discrete and open. Then $f \in ACL$.*

Proof. Without loss of generality we may assume that $\infty \notin D' = f(D)$. Let $I = \{x \in \mathbb{R}^n\colon a_i < x_i < b_i,\ i = 1,\dots,n\}$ be an n-dimensional interval in $\mathbb{R}^n$ such that $\overline{I} \subset D$. Then $I = I_0 \times J$ where I_0 is an $(n-1)$-dimensional interval in $\mathbb{R}^{n-1}$ and J is an open segment of the axis x_n, $J = (a,b)$. Next we identify $\mathbb{R}^{n-1} \times \mathbb{R}$ with $\mathbb{R}^n$. We prove that for almost all segments $J_z = \{z\} \times J, z \in I_0$, the mapping $f|_{J_z}$ is absolutely continuous.

Consider the set function $\Phi(B) = m\,(f(B \times J))$ defined over the algebra of all Borel sets B in I_0. By 2.2, 2.3 and 2.12 in [11]

$$\varphi(z) = \limsup_{r\to 0} \frac{\Phi(B(z,r))}{\Omega_{n-1} r^{n-1}} < \infty \tag{16}$$

for a.e. $z \in I_0$ where $B(z,r)$ is the ball in $\mathbb{R}^{n-1}$ centered at a point $z \in I_0$ of radius r and Ω_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$.

Let $\Delta_i, i = 1, 2, ...,$ be some enumeration S of all intervals in J such that $\overline{\Delta_i} \subset J$ and the endpoints of Δ_i are rational numbers. Set

$$\varphi_i(z) := \int_{\Delta_i} Q(z, x_n) dx_n.$$

Then by the Fubini theorem, see e.g. III. 8.1 in [17], the functions $\varphi_i(z)$ are finite a.e. and integrable in $z \in I_0$. In addition, by the Lebesgue theorem on the differentiability of the indefinite integral, there is a finite a.e. limit

$$\lim_{r \to 0} \frac{\Phi_i(B(z,r))}{\Omega_{n-1} r^{n-1}} = \varphi_i(z) \tag{17}$$

where Φ_i for a fixed $i = 1, 2, \dots$ is the set function

$$\Phi_i(B) = \int_B \varphi_i(\zeta) d\zeta$$

defined on the algebra of all Borel sets B in I_0.

Let us show that the mapping f is absolutely continuous on each segment $J_z, z \in I_0$, where the finite limits (16) and (17) exist. Fix one of such a point z. We have to prove that the sum of diameters of the images of an arbitrary finite collection of mutually disjoint segments in $J_z = \{z\} \times J$ tends to zero together with the total length of the segments. In view of the continuity of the mapping f, it is sufficient to verify this fact for mutually disjoint segments with rational endpoints in J_z only. So, let $\Delta_i^* = \{z\} \times \overline{\Delta_i} \subset J_z$ where $\Delta_i \in S, i = 1, ..., k$ under the corresponding re-enumeration of S, are mutually disjoint intervals. Without loss of generality, we may assume that $\overline{\Delta_i}, i = 1, ..., k$ are also mutually disjoint.

Let $\delta > 0$ be an arbitrary rational number which is less than a half of the minimum of the distances between $\Delta_i^*, i = 1, ..., k$, and also less than their distances to the endpoints of the interval J_z. Let $\Delta_i^* = \{z\} \times [\alpha_i, \beta_i]$ and $A_i = A_i(r) = B(z,r) \times (\alpha_i - \delta, \beta_i + \delta)$, $i = 1, ..., k$, where $B(z,r)$ is the open ball in $I_0 \subset \mathbb{R}^{n-1}$ centered at a point z of radius $r > 0$.

For small $r > 0$, $E_i = (A_i, \Delta_i^*), i = 1, ..., k$ are condensers in I and hence, $f(E_i) = (f(A_i), f(\Delta_i^*)), i = 1, ..., k$ are condensers in D'. By Lemma 1, $\mathrm{cap}_p (f(A_i), f(\Delta_i^*)) = M_p(\Gamma_{f(E_i)})$. Denoting through $\Gamma_{E_i^*}$ a family of maximal f-liftings of $\Gamma_{f(E_i)}$ starting at Δ_i^*, we obtain $\Gamma_{E_i^*} \subset \Gamma_{E_i}$ and

$$\mathrm{cap}_p (f(A_i), f(\Delta_i^*)) \le M_p(f\Gamma_{E_i}). \tag{18}$$

Note that the function

$$\rho_i(x) = \begin{cases} \frac{1}{r}, & x \in A_i, \\ 0, & x \notin A_i \end{cases}$$

is admissible for the path's families E_i for $r < \delta$, thus from (5) and by the definition of Q-mapping we obtain

$$\mathrm{cap}_p (f(A_i), f(\Delta_i^*)) \le \frac{1}{r^p} \int_{A_i} Q(x) dm(x). \tag{19}$$

On the other hand, by Proposition 6 of §1 in [9],

$$\mathrm{cap}_p (f(A_i), f(\Delta_i^*)) \ge \left(c \frac{d_i^p}{m_i^{1-n+p}} \right)^{\frac{1}{n-1}} \tag{20}$$

where d_i is the diameter of the set $f(\Delta_i^*)$, m_i is the volume of $f(A_i)$ and c is a constant depending on n and p only.

Combining (19) and (20), we have

$$\Big(\frac{d_i^p}{m_i^{1-n+p}}\Big)^{\frac{1}{n-1}} \le \frac{c_1}{r^p}\int\limits_{A_i} Q(x)dm(x) \tag{21}$$

with a constant c_1 depending only on n, p and all $i=1,...,k$.

By the discrete Hölder inequality see e.g. (17.3) in [2], we obtain

$$\sum_{i=1}^k d_i \le \Big(\sum_{i=1}^k\Big(\frac{d_i^p}{m_i^{1-n+p}}\Big)^{\frac{1}{n-1}}\Big)^{\frac{n-1}{p}}\Big(\sum_{i=1}^k m_i\Big)^{\frac{1-n+p}{p}},$$

i.e.

$$\Big(\sum_{i=1}^k d_i\Big)^p \le \Big(\sum_{i=1}^k\Big(\frac{d_i^p}{m_i^{1-n+p}}\Big)^{\frac{1}{n-1}}\Big)^{n-1}[\Phi(B(z,r))]^{1-n+p}.$$

By (21)

$$\Big(\sum_{i=1}^k d_i\Big)^p \le c_2\left[\frac{\Phi(B(z,r))}{\Omega_{n-1}r^{n-1}}\right]^{1-n+p}\left(\sum_{i=1}^k\frac{\int_{A_i}Q(x)dm(x)}{\Omega_{n-1}r^{n-1}}\right)^{n-1}$$

where c_2 depends on n and p only. Passing to the limit first as $r\to 0$ and then as $\delta\to 0$, we obtain

$$\Big(\sum_{i=1}^k d_i\Big)^p \le c_2[\varphi(z)]^{1-n+p}\Big(\sum_{i=1}^k\varphi_i(z)\Big)^{n-1}. \tag{22}$$

Finally, in view of (22), the absolute continuity of the indefinite integral of Q over the segment J_z implies the absolute continuity of the mapping f over the same segment. Hence, $f\in ACL$. □

Combining Theorem 2 with Corollary 1, we obtain the following conclusion, see also [13].

Corollary 3. *Let D be a domain in $\mathbb{R}^n$, $n\ge 2$, and $f\colon D\to\overline{\mathbb{R}^n}$ be a ring (p,Q)-mapping with $Q\in L^1_{\rm loc}$ and $p>n-1$. Suppose that f is discrete and open. Then $f\in W^{1,1}_{\rm loc}$.*

REFERENCES

1. C. Andrean Cazacu, *Moduli inequalities for quasiregular mappings*, Ann. Acad. Sci. Fenn. Math. **2** (1976), 17–28.
2. E. F. Beckenbach, R. Bellman, Inequalities, Springer-Verlag, New York, 1965.
3. C. J. Bishop, V. Ya. Gutlyanskii, O. Martio, M. Vuorinen, *On conformal dilatation in space*, Intern. Journ. Math. and Math. Scie. **22** (2003), 1397–1420.
4. B. V. Bojarski, V. Ya. Gutlyanskii, V.I. Ryazanov, *General Beltrami equations and BMO*, Ukr. Math. Bull. **5** (2008), №3, 305–326.
5. M. Cristea, *Open discrete mappings having local ACL^n inverses*, Compl. Var. and Ellipt. Equat. **55** (2010), №1-3, 61–90.

6. A. Golberg, Differential properties of (α, Q)–homeomorphisms, Further Progress in Analysis, World Scientific Publ. (2009), 218–228.
7. F. W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–393.
8. F. W. Gehring, O. Lehto, *On the total differentiability of functions of a complex variable*, Ann. Acad. Sci. Fenn. Math. AI. **272** (1959), 1–9.
9. V. Kruglikov, *Capacity of condensers a nd spatial mappings quasiconformal in the mean*, Mat. Sb. **130** (1986), №2, 185–206. (In Russian)
10. K. Kuratowski, Topology, V.1, Academic Press, New York and London, 1966.
11. O. Martio, S. Rickman, J. Väisälä, *Definitions for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A1 **448** (1969), 1–40.
12. O. Martio, V. Ryazanov, U. Srebro, E. Yakubov, Moduli in modern mapping theory, Springer Science + Business Media, LLC, New York, 2009.
13. V. Maz'ya, Sobolev classes, Springer, Berlin – New York, 1985.
14. Yu. G. Reshetnyak, *Generalized derivatives and differentiability a.e.* (In Russian), Mat. Sb. **75** (1968), №3, 323–334. (In Russian)
15. Yu. G. Reshetnyak, Space mappings with bounded distortion, Transl. Math. Monographs 73, Amer. Math. Soc., Providence, RI, 1989.
16. S. Rickman, Quasiregular Mappings, Springer, Berlin etc., 1993.
17. S. Saks, Theory of the Integral, Dover Publ. Inc., New York, 1964.
18. J. Väisälä, Lectures on n–dimensional quasiconformal mappings, Lecture Notes in Math. 229., Springer–Verlag, Berlin etc., 1971.
19. G. T. Whyburn, Analytic topology, American Mathematical Society, Rhode Island, 1942.
20. R. R. Salimov, E. A. Sevost'yanov, *ACL and differentiability of the open discrete ring mappings*, Complex Variables and Elliptic Equations **55** (2010), №1–3, 49–59.

Institute of Applied Mathematics and Mechanics,
Donetsk, Ukraine
salimov@iamm.ac.donetsk.ua, brusin2006@rambler.ru

Received 25.06.2010