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On Quasilinear Beltrami-Type Equations with Degeneration

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Abstract—We consider the solvability problem for the equation $f_{\overline{z}} = \nu(z, f(z))f_z$, where the function $\nu(z, w)$ of two variables may be close to unity. Such equations are called quasilinear Beltrami-type equations with ellipticity degeneration. We prove that, under some rather general conditions on $\nu(z, w)$, the above equation has a regular homeomorphic solution in the Sobolev class $W^{1,1}_{\mathrm{loc}}$. Moreover, such solutions f satisfy the inclusion $f^{-1} \in W^{1,2}_{\mathrm{loc}}$.

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1. INTRODUCTION

One of the first most informative works concerned with quasilinear Beltrami-type equations belongs to the outstanding scientist, Academician Boyarskii; see [1]. In that paper, already classical in this field of research, the author formulated several fundamental theorems, and a particular case of one of these theorems is considered below. For a complex-valued function $f\colon D \to \mathbb{C}$ defined on the domain $D \subset \mathbb{C}$ and having partial derivatives with respect to x and y for almost all $z = x + iy$, we set

$$\overline{\partial} f = f_{\overline{z}} = \frac{f_x + if_y}{2} \qquad \text{and} \qquad \partial f = f_z = \frac{f_x - if_y}{2}.$$

A function $\nu = \nu(z, w)\colon D \times \mathbb{C} \to \mathbb{D}$ is said to satisfy the *Carathéodory conditions* if ν is measurable in $z \in D$ for each fixed $w \in \mathbb{C}$ and continuous in $w \in \mathbb{C}$ for almost all $z \in D$. We consider the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and, in it, we study the equation

$$f_{\overline{z}} = \nu(z, f(z))f_z, \tag{1.1}$$

which will be called a *quasilinear Beltrami-type equation* below. We assume that the function $\nu(z, w)$ satisfies the Carathéodory conditions and

$$|\nu(z, w)| \le k < 1 \tag{1.2}$$

for almost all $z \in \mathbb{D}$ for each fixed $w \in \mathbb{C}$. Then Eq. (1.1) has a homeomorphic solution $f\colon \mathbb{D} \to \mathbb{C}$ satisfying the normalization conditions $f(0) = 0$ and $f(1) = 1$ (see Theorem 8.2 in [1]). From now on, a *solution* of Eq. (1.1) is understood as a mapping $f\colon \mathbb{D} \to \mathbb{C}$ of class ACL satisfying Eq. (1.1) for almost all $z \in \mathbb{D}$. We recall that a mapping $f\colon D \to \mathbb{C}$, $D \subset \mathbb{C}$, is said to be *absolutely continuous on the lines $f \in ACL$* if, in any rectangle P whose edges are parallel to the coordinate axes and $\overline{P} \subset D$, the function f is absolutely continuous on almost all (almost all) segments in P that are parallel to the coordinate axes. It is well known that

$$ACP^p_{\mathrm{loc}} = W^{1,p}_{\mathrm{loc}}, \qquad 1 \le p < \infty,$$

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where $ACP^p_{\rm loc}$ denotes the class of all ACL-mappings whose first-order partial derivatives raised to the corresponding power p are locally integrable (see, e.g., [2, p. 8]. In particular, $W^{1,1}_{\rm loc}\subset ACL$. Any homeomorphism $f\colon D\to\mathbb{C}$ of class ACL is differentiable a.e. (see, e.g., [3, p. 128]). Thus, for homeomorphisms of class ACL, the notation (1.1) is meaningful; in what follows, we shall consider solutions of Eq. (1.1) in the above sense only in the class ACL of homeomorphisms.

The main goal of the present paper is to prove existence theorems for Eq. (1.1) without using conditions of the form (1.2). As far as we know, degenerate Beltrami-type equations, i.e., equations where the complex coefficient ν may be close to unity, have been studied rather intensively (see, e.g., [4]–[8], etc.). At the same time, it is well known that, in the context of the solvability of Eq. (1.1) in the class ACL of homeomorphisms and even in the simple case where ν depends only on z, the condition that the left-hand side of (1.2) is bounded cannot be replaced, for example, by the condition that ν raised to an arbitrarily large power $p\ge 1$ is locally integrable. The conditions for the existence of solutions to the equation mentioned above require a more precise analysis related, in particular, to functions of *bounded mean oscillation*, see [9], and to more general function classes that will be discussed later.

2. MAIN DEFINITIONS

From now on, D is a domain in the complex plane $\mathbb{C}$, $Q\colon D\to[0,\infty]$ is a Lebesgue measurable function, and m is the Lebesgue measure in $\mathbb{C}$; for a set $A\subset\mathbb{C}$, the notation $m(A)$ stands for the Lebesgue measure in $\mathbb{C}$, and $\operatorname{dist}(A,B)$ is the Euclidean distance between the sets $A,B\subset\mathbb{C}$. The notation $f\colon D\to\mathbb{C}$ means that the mapping f is continuous. We also assume that the mapping $f\colon D\to\mathbb{C}$ *preserves orientation*, i.e., the topological index $\mu(y,f,G)$ is positive for an arbitrary domain $G\subset D$ such that $\overline{G}\subset D$ and for an arbitrary $y\in f(G)\setminus f(\partial G)$. The Jacobian of an (orientation preserving) homeomorphism $f\colon D\to\mathbb{C}$ of class ACL is nonnegative a.e.:

$$J_f(z)=|f_z|^2-|f_{\overline{z}}|^2\ge 0, \tag{2.1}$$

see [3, p. 10]. A homeomorphism $f\colon D\to\mathbb{C}$ of class ACL is said to be *regular* if the inequality for this homeomorphism in (2.1) is strict. Similarly, a solution $f\colon\mathbb{D}\to\mathbb{C}$ of Eq. (1.1) is said to be *regular* if, for this solution, $J_f(z)$ is positive a.e. in $\mathbb{D}$. The *complex dilatation* of a homeomorphism $f\colon D\to\mathbb{C}$ of class ACL at a point z is defined as

$$\mu(z)=\mu_f(z)=\frac{f_{\overline{z}}}{f_z}\qquad\text{if}\quad f_z\ne 0$$

and $\mu(z)=0$ otherwise. The *maximum dilatation* is defined as

$$K_\mu(z)=\frac{1+|\mu(z)|}{1-|\mu(z)|}\,.$$

We note that condition (2.1) always implies that $|\mu(z)|\le 1$ a.e. and $K_\mu\ge 1$ a.e. Moreover, we note that any homeomorphism of class ACL satisfies Eq. (1.1), where $\nu(z,f(z))=\mu_f(z)$. We shall say that a homeomorphism $f\colon D\to\overline{\mathbb{C}}$ of class ACL is *$Q(z)$-quasiconformal* if $K_\mu(z)\le Q(z)$ for almost all $z\in D$. In what follows, in the extended space $\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$, we use the *spherical* (*chordal*) *metric*

$$h(x,y)=|\pi(x)-\pi(y)|,$$

where π is the stereographic projection of $\overline{\mathbb{C}}$ on the sphere $S^3(e_3/2,1/2)$, $e_3=(0,0,1)$, in $\mathbb{R}^3$:

$$h(x,\infty)=\frac{1}{\sqrt{1+|x|^2}}\,,\quad h(x,y)=\frac{|x-y|}{\sqrt{1+|x|^2}\,\sqrt{1+|y|^2}}\,,\qquad x\ne\infty\ne y.$$

We let $h(E)=\sup_{x,y\in E}h(x,y)$ be the "chordal" (spherical) diameter of the set $E\subset\overline{\mathbb{C}}$. We recall that a Borel function $\rho\colon\mathbb{C}\to[0,\infty]$ is said to be *admissible* for a family Γ of curves γ in $\mathbb{C}$ if $\int_\gamma\rho(z)\,|dz|\ge 1$ for all curves $\gamma\in\Gamma$. In this case, we write $\rho\in\operatorname{adm}\Gamma$. The *modulus* of a family Γ of curves is

$$M(\Gamma)=\inf_{\rho\in\operatorname{adm}\Gamma}\int_D\rho^2(z)\,dm(z).$$

Let $E, F \subset \overline{\mathbb{C}}$ be arbitrary sets. By $\Gamma(E,F,D)$ we denote the family of all curves $\gamma\colon [a,b] \to \overline{\mathbb{C}}$ connecting E and F in D, i.e.,

$$\gamma(a) \in E, \quad \gamma(b) \in F, \qquad \text{and} \qquad \gamma(t) \in D \quad \text{for } t \in (a,b).$$

Let $r_0 = \operatorname{dist}(z_0, \partial D)$, and let

$$A(r_1, r_2, z_0) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}, \qquad S_i = S(z_0, r_i) = \{z \in \mathbb{C} : |z - z_0| = r_i\}.$$

A homeomorphism $f\colon D \to \overline{\mathbb{C}}$ is said to be an *annulus Q-homeomorphism at a point* $z_0 \in D$ (see, e.g., Secs. 7 and 11 in [6], also see [8] and [10]) if the relation

$$M(f(\Gamma(S_1, S_2, A))) \le \int_A Q(z) \cdot \eta^2(|z - z_0|)\, dm(z) \tag{2.2}$$

holds for any annulus

$$A = A(r_1, r_2, z_0), \qquad 0 < r_1 < r_2 < r_0,$$

and for each measurable function $\eta\colon (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r)\, dr \ge 1. \tag{2.3}$$

Moreover, the homeomorphism $f\colon D \to \overline{\mathbb{C}}$ is called a *annulus Q-homeomorphism in the domain D* if relation (2.2) holds at each point $z_0 \in D$ and for each function η in (2.3). We need one more definition. Let $Q(z)\colon D \to [1, \infty]$ be a real-valued function. A homeomorphism $f\colon D \to \overline{\mathbb{C}}$ is called a *Q-homeomorphism* if

$$M(f(\Gamma)) \le \int_D Q(z) \cdot \rho^2(z)\, dm(z) \tag{2.4}$$

for any family Γ of paths γ in D and for each admissible function $\rho \in \operatorname{adm}\Gamma$. We note that relations (2.2) and (2.4) are a part of the definition of q-quasiconformal mappings if $Q(z) \equiv q = \text{const}$. In the general case, these "weighted" relations with weight $Q(z)$ are not equivalent (see, e.g., [6]) and we need them as an apparatus for studying equations of the form (1.1).

Proposition 1. *Let $f_m\colon D \to \overline{\mathbb{C}}$ be a sequence of $Q(z)$-homeomorphisms in D, where $Q(z)$ is a fixed function, which is the same for all m. Assume that the sequence f_m converges to a mapping f in D locally uniformly. Then f is either an annulus Q-homeomorphism or a constant in D under the condition that $Q(z) \in L^1_{\mathrm{loc}}(D)$ (see Theorem 7.7 in [6]).*

Remark 1. We note that, in the statement of Proposition 1, we generally cannot use a sequence of "annulus" homeomorphisms, because the above statement is not generally proved in this case. Similarly, we cannot state that the limit mapping f in Proposition 1 is a Q-homeomorphism (for more details, see Sec. 7 in [6]).

Proposition 2. *Let $f\colon D \to \overline{\mathbb{C}}$ be an annulus Q-homeomorphism at a point $z_0 \in D$ such that*

$$h(\overline{\mathbb{C}} \setminus f(D)) \ge \delta > 0.$$

If, for some positive ε_0 not greater than $\operatorname{dist}(z_0, \partial D)$,

$$\int_{\varepsilon < |z - z_0| < \varepsilon_0} Q(z) \cdot \psi^2(|z - z_0|)\, dm(z) \le c \cdot I^p(\varepsilon) \qquad \textit{for all} \quad \varepsilon \in (0, \varepsilon_0), \tag{2.5}$$

where $p \le 2$ and $\psi(t)$ is a nonnegative measurable function on $(0, \infty)$ such that

$$0 < I(\varepsilon) = \int_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt < \infty \qquad \textit{for all} \quad \varepsilon \in (0, \varepsilon_0),$$

then, for $z \in B(z_0, \varepsilon_0)$,

$$h(f(z), f(z_0)) \le \frac{\alpha}{\delta} \exp\{-\beta I^{\gamma_p}(|z - z_0|)\}, \tag{2.6}$$

where α, β, and γ_p are some constants, the first two of which are absolute constants and the last depends only on p (see Lemma 7.6 *in* [6]).

Remark 2. In particular, by the well-known Arzela–Ascoli theorem, an estimate of the form (2.6) implies that the class of all annulus Q-homeomorphisms $\mathfrak{H}_{Q,\delta} = \{f\colon D \to \overline{\mathbb{C}}\}$ in D satisfying the inequality

$$h(\overline{\mathbb{C}} \setminus f(D)) \geq \delta > 0 \tag{2.7}$$

and condition (2.5) for Q is a normal family of mappings with respect to the metric h provided $I(\varepsilon) \to \infty$ as $\varepsilon \to 0$, i.e., from any sequence $f_m \in \mathfrak{H}_{Q,\delta}$, it is possible to choose a subsequence f_{m_k} such that

$$\sup_{z \in E} h(f_{m_k}(z), f(z)) \to 0 \qquad \text{as} \quad k \to \infty$$

for a continuous mapping $f\colon D \to \overline{\mathbb{C}}$ and any compact set $E \subset D$ (for more details, see Sec. 7.5 in [6]). Finally, it is easy to see that the conclusion that the corresponding family of homeomorphisms is equicontinuous and normal remains valid if the condition of the form (2.7) in Proposition 2 is replaced by the condition that $f(z_1) = y_1$ and $f(z_2) = y_2$ for some fixed (independent of f) elements z_1, y_1, z_2, and y_2.

Proposition 3. *Let $D \subset \mathbb{C}$, and let $f_n\colon D \to \mathbb{C}$ be a sequence of homeomorphisms of class ACL that have complex dilatations $\mu_n(z)$ satisfying the condition*

$$\frac{1 + |\mu_n(z)|}{1 - |\mu_n(z)|} \leq Q(z) \in L^1_{\mathrm{loc}} \qquad \textit{for all} \quad n = 1, 2, \dots .$$

If $f_n \to f$ is locally uniform in D as $n \to \infty$ and f is a homeomorphism in D, then $f \in ACL$ and the sequences ∂f_n and $\overline{\partial} f_n$ converge weakly in L^1_{loc} to ∂f and $\overline{\partial} f$, respectively. In this case, the mapping f is $Q(z)$-quasiconformal. Moreover, if $\mu_n \to \mu$ as $n \to \infty$, then $\overline{\partial} f = \mu \partial f$ a.e. (see, e.g., Theorem 3.1 *and Remark* 3.1 *in* [8]).

3. MAIN LEMMA

Lemma 1. *Assume that a function $\nu = \nu(z, w)\colon \mathbb{D} \times \mathbb{C} \to \mathbb{D}$ satisfies the Carathéodory conditions and*

$$K_\nu(z, w) := \frac{1 + |\nu(z, w)|}{1 - |\nu(z, w)|} \leq Q(z) \in L^1_{\mathrm{loc}}(\mathbb{D}) \tag{3.1}$$

for almost all $z \in \mathbb{D}$ and for any $w \in \mathbb{C}$. Assume also that, for any $z_0 \in \mathbb{D}$ and some number $\varepsilon_0 < \operatorname{dist}(z_0, \partial\mathbb{D})$,

$$\int_{\varepsilon < |z - z_0| < \varepsilon_0} Q(z) \cdot \psi^2(|z - z_0|)\, dm(z) \leq c \cdot I^p(\varepsilon), \tag{3.2}$$

where $I(\varepsilon) \to \infty$ as $\varepsilon \to 0$, p is a constant such that $0 < p \leq 2$, and $\psi(t)$ is a nonnegative measurable function on $(0, \infty)$ such that

$$0 < I(\varepsilon) = \int_\varepsilon^{\varepsilon_0} \psi(t)\, dt < \infty, \qquad \varepsilon \in (0, \varepsilon_0).$$

Then Eq. (1.1) *has a regular homeomorphic solution f of class $W^{1,1}_{\mathrm{loc}}$ in $\mathbb{D}$ such that $f^{-1} \in W^{1,2}_{\mathrm{loc}}(f(\mathbb{D}))$.*

Proof. We consider the sequence of functions

$$\nu_n(z, w) = \begin{cases} \nu(z, w), & Q(z) \leq n, w \in \mathbb{C}, \\ 0, & Q(z) > n, w \in \mathbb{C}. \end{cases} \tag{3.3}$$

We note that $K_{\nu_n}(z,w) \le n$ for almost all $z \in \mathbb{D}$ and for all $w \in \mathbb{C}$. Therefore, we have

$$\nu_n(z,w) \le \frac{n-1}{n+1} < 1,$$

and hence Eq. (1.1), where ν in the right-hand side is replaced by $\nu := \nu_n$ and ν_n is defined by relations (3.3), has a homeomorphic solution $f_n\colon \mathbb{D} \to \mathbb{C}$ satisfying the normalization conditions $f_n(0) = 0$ and $f_n(1) = 1$. This solution is n-quasiconformal in $\mathbb{D}$ (see Theorem 8.2 in [1]). At the same time, the f_n are $Q(z)$-quasiconformal because of relation (3.1) and the fact that

$$K_{\nu_n}(z,w) \le K_\nu(z,w).$$

Therefore, according to relation (6.6) in Chap. V in [3], each f_n is a Q-homeomorphism and hence an annulus Q-homeomorphism. By Proposition 2, Remark 2, and relation (3.2), we see that the sequence $\{f_n\}_{n=1}^{\infty}$ has a subsequence f_{n_k} that locally uniformly converges to a mapping f. By Propositions 1 and 3 and the normalization conditions $f_n(0) = 0$ and $f_n(1) = 1$, the limit mapping f is $Q(z)$-quasiconformal. We note that, for almost all $z \in \mathbb{D}$, there exists a number $k_0 = k_0(z)$ such that

$$\nu_{n_k}(z,w) = \nu(z,w) \qquad \text{for} \quad n_k \ge n_{k_0}(z) \quad \text{and all} \quad w \in \mathbb{C}.$$

Therefore, for almost all z,

$$\mu_{n_k}(z) = \nu_{n_k}(z, f_{n_k}(z)) \to \nu(z, f(z)) \qquad \text{as} \quad k \to \infty.$$

Let $\mu_f(z)$ be a characteristic of the limit mapping f. Again, it follows from Proposition 3 that $\nu(z, f(z)) = \mu_f(z)$ a.e. But this just means that the mapping f is a solution of the initial equation (1.1).

It remains to show that the mapping is f-regular and $f^{-1} \in W^{1,2}_{\mathrm{loc}}$. Since f is a homeomorphism, f_n^{-1} converges to f^{-1} locally uniformly as $n \to \infty$. We write $g_n = f_n^{-1}$. Now we note that the complex characteristic of the inverse mapping $g = f^{-1}$ is related to the characteristic of f as $\mu_g = -\mu_f \circ g$, (see, e.g., relation 4 in Sec. C in Chap. I in [11]). Then, for sufficiently large n, we have

$$\begin{aligned}\int_B |\partial g_n(w)|^2 \, dm(w) &= \int_B (|\partial g_n(w)|^2 - |\overline{\partial} g_n(w)|^2) \cdot \frac{|\partial g_n(w)|^2 \, dm(w)}{(|\partial g_n(w)|^2 - |\overline{\partial} g_n(w)|^2)} \\ &= \int_B J_{g_n}(w) \cdot \frac{1}{1 - |\overline{\partial} g_n(w)/\partial g_n(w)|^2} \, dm(w) = \int_{g_n(B)} \frac{dm(z)}{1 - |\mu_n(z)|^2} \\ &\le \int_{B^*} Q(z) \, dm(z) < \infty,\end{aligned}$$

where B and B^* are relatively compact domains in $\mathbb{D}$ and $f(\mathbb{D})$, respectively, and satisfy the condition $g(\overline{B}) \subset B^*$. The change of variables in the integrals is valid, because $g_n, f_n \in W^{1,2}_{\mathrm{loc}}$. It follows from the last estimate that $f^{-1} \in W^{1,2}_{\mathrm{loc}}(f(\mathbb{D}))$ (see [12, Chap. III, Lemma 3.5]). This implies that f has the (N^{-1})-property (see Remark 8.4 (3) in Sec. 8.4 in [6]), which, in turn, is equivalent to the fact that $J_f(z) \ne 0$ a.e. (see [13]). Finally, for an arbitrary compact set $C \subset \mathbb{D}$, it follows from the Schwartz inequality that the norm of the derivatives ∂f and $\overline{\partial} f$ in $L^1(C)$ can be estimated as follows:

$$\|\overline{\partial} f\| \le \|\partial f\| \le \|Q(z)\|^{1/2} \cdot \|J_f(z)\|^{1/2} \le \|Q(z)\|^{1/2} \cdot m(f(C)),$$

which implies that $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{D})$ (see [2, p. 8]). The proof of Lemma 1 is complete. □

4. IMPORTANT CONSEQUENCES

Lemma 1 formulated and proved above is one of the most important tools that now allows us to state the main results of the present paper. First, we formulate sufficient conditions for the existence of homeomorphic solutions to Eq. (1.1) on the basis of the condition that a certain integral diverges. Such a condition is considered not at random. The divergence conditions for integrals of the form $\int \frac{dt}{tK(t)}$, where $K(t)$ is a certain function, were considered by many prominent scientists, for example, by Shabat and Zorich (see, e.g., [14], [15]; also see Secs. 11.4–11.6 in [6]). By $q_{z_0}(r)$ we denote the mean value of a function $Q(z)$ over the circle $\{|z - z_0| = r\}$:

$$q_{z_0}(r) = \frac{1}{2\pi} \int_0^{2\pi} Q(z_0 + re^{i\theta})\, d\theta.$$

Theorem 1. *Assume that a function $\nu(z, w)\colon \mathbb{D} \times \mathbb{C} \to \mathbb{D}$ satisfies the Carathéodory conditions and $K_\nu(z, w) \le Q(z) \in L^1_{\mathrm{loc}}(\mathbb{D})$. Assume also that*

$$\int_0^{\delta(z_0)} \frac{dr}{rq_{z_0}(r)} = \infty,$$

where $\delta(z_0)$ is a positive number and $\delta(z_0) < \operatorname{dist}(z_0, \partial\mathbb{D})$. Then Eq. (1.1) has a regular homeomorphic solution f of class $W^{1,1}_{\mathrm{loc}}$ in $\mathbb{D}$ such that $f^{-1} \in W^{1,2}_{\mathrm{loc}}(f(\mathbb{D}))$.

Proof. We note that $Q(z)$ is no less than 1, because, by definition, $K_\nu(z, w) \ge 1$ for almost all z and each fixed w. Therefore, we have $q_{z_0}(t) \ge 1$ for almost all t. We set

$$\psi(t) = \begin{cases} \dfrac{1}{tq_{z_0}(t)}, & t \in (0, \delta(z_0)), \\ 0, & t \notin (0, \delta(z_0)), \end{cases}$$

and note that $\int_\varepsilon^{\delta(z_0)} \psi(t)\, dt > 0$ for all $\varepsilon \in (0, \delta(z_0))$, because, otherwise, $q_{z_0}(t) = \infty$ for almost all $t \in (0, \delta(z_0))$, which is impossible, because $Q(z) \in L^1_{\mathrm{loc}}(\mathbb{D})$ by the assumptions of the theorem. Moreover,

$$\int_\varepsilon^{\delta(z_0)} \psi(t)\, dt \le \int_\varepsilon^{\delta(z_0)} \frac{dt}{t} < \infty \qquad \text{for all} \quad \varepsilon \in (0, \delta(z_0)).$$

Thus, it is possible to apply Lemma 1 to the function ψ mentioned above and to obtain the desired conclusion. □

In the next important section of the present paper, we consider some functions of a special form. We recall that a function $\varphi\colon D \to \mathbb{R}$, $\varphi \in L^1_{\mathrm{loc}}(D)$, is of *bounded mean oscillation* in the domain D, which can be written as $\varphi \in BMO$, if

$$\|\varphi\|_* = \sup_{B \subset D} \frac{1}{m(B)} \int_B |\varphi(z) - \varphi_B|\, dm(z) < \infty, \tag{4.1}$$

where sup is taken over all circles $B \subset D$ and

$$\varphi_B = \frac{1}{m(B)} \int_B \varphi(z)\, dm(z)$$

is the mean value of the function φ over the circle B (see, e.g., [9]). It is well known that

$$L^\infty(D) \subset BMO(D) \subset L^p_{\mathrm{loc}}(D)$$

(see, e.g., [9]). We consider the following definition generalizing the notion of bounded mean oscillation to the case in which the variable in the sup symbol in (4.1) is not "uniform" over a given domain (see, e.g., Sec. 6.1 in [6]). We shall say that a function $\varphi\colon D \to \mathbb{R}$ is of *finite mean oscillation at a point* $z_0 \in D$ and write $\varphi \in FMO(z_0)$ if

$$\limsup_{\varepsilon \to 0} \frac{1}{\pi\varepsilon^2} \int_{B(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon|\, dm(z) < \infty, \tag{4.2}$$

where

$$\overline{\varphi}_\varepsilon = \frac{1}{\pi\varepsilon^2} \int_{B(z_0,\varepsilon)} \varphi(z)\, dm(z).$$

We note that if condition (4.2) is satisfied, then it is possible that $\overline{\varphi}_\varepsilon \to \infty$ as $\varepsilon \to 0$. We shall also say that $\varphi\colon D \to \mathbb{R}$ is a *function of finite mean oscillation in the domain* D and write $\varphi \in FMO(D)$ if φ is of finite mean oscillation at each point $z_0 \in D$. Obviously, $BMO \subset FMO$. We note that $FMO \neq BMO_{\mathrm{loc}}$ (see, e.g., Sec. 11.2 in [6]).

Proposition 4. *Assume that* $0 \in D \subset \mathbb{C}$ *and* $\varphi\colon D \to \mathbb{R}$ *is a nonnegative function of finite mean oscillation at a point* $z_0 = 0$. *Then*

$$\int_{\varepsilon<|z|<\varepsilon_0} \frac{\varphi(z)\,dm(z)}{(|z|\log(1/|z|))^2} = O\left(\log\log\frac{1}{\varepsilon}\right)$$

as $\varepsilon \to 0$ *and for some* $\varepsilon_0 \le \operatorname{dist}(0, \partial D)$ (*see Corollary* 6.3 *in* [6]).

In particular, we note that each constant function $\varphi(z) \equiv c$ always satisfies a relation of the form (4.2). At the same time, functions of finite mean oscillation are generally integrable only in the first power; it is possible to construct an example of a function of class FMO that is locally integrable in the first power and is not locally integrable in any power $p > 1$ (see, e.g., Sec. 11.3 in [6]).

Theorem 2. *Assume that a function* $\nu(z, w)\colon \mathbb{D}\times\mathbb{C} \to \mathbb{D}$ *satisfies the Carathéodory conditions. Assume also that* $K_\nu(z, w) \le Q(z) \in FMO(\mathbb{D})$. *Then Eq.* (1.1) *has a regular homeomorphic solution* f *of class* $W^{1,1}_{\mathrm{loc}}$ *in* $\mathbb{D}$ *such that* $f^{-1} \in W^{1,2}_{\mathrm{loc}}(f(\mathbb{D}))$.

Proof. Let $z_0 \in \mathbb{D}$, and let $\varepsilon_0 < \min\{\operatorname{dist}(z_0, \partial\mathbb{D}),\ e^{-1}\}$. By Proposition 4, for the function $\psi(t)$,

$$0 < \psi(t) = \frac{1}{t\log(1/t)}\,,$$

we have

$$\int_{\varepsilon<|z-z_0|<\varepsilon_0} Q(z)\cdot\psi^2(|z-z_0|)\,dm(z) = O\left(\log\log\frac{1}{\varepsilon}\right).$$

We also note that

$$I(\varepsilon, \varepsilon_0) := \int_\varepsilon^{\varepsilon_0} \psi(t)\,dt = \log\left(\frac{\log(1/\varepsilon)}{\log(1/\varepsilon_0)}\right).$$

Now the statement of Theorem 2 follows from Lemma 1. $\square$

Corollary 1. *In particular, if*

$$\limsup_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} \int_{B(z_0,\varepsilon)} Q(z)\,dm(z) < \infty$$

at each point $z_0 \in \mathbb{D}$, *then Eq.* (1.1) *has a regular homeomorphic solution* f *of class* $W^{1,1}_{\mathrm{loc}}$ *in* $\mathbb{D}$ *such that* $f^{-1} \in W^{1,2}_{\mathrm{loc}}(f(\mathbb{D}))$.

Theorem 3. *Assume that a function* $\nu(z, w)\colon \mathbb{D}\times\mathbb{C} \to \mathbb{D}$ *satisfies the Carathéodory conditions. Assume also that* $K_\nu(z, w) \le Q(z)$, *where*

$$q_{z_0}(r) = O\left(\log\frac{1}{r}\right) \qquad \text{for all} \quad z_0 \in \mathbb{D} \quad \text{as } r \to 0. \tag{4.3}$$

Then Eq. (1.1) *has a regular homeomorphic solution* f *of class* $W^{1,1}_{\mathrm{loc}}$ *in* $\mathbb{D}$ *such that* $f^{-1} \in W^{1,2}_{\mathrm{loc}}(f(\mathbb{D}))$.

Proof. To prove the theorem, it suffices to choose an arbitrary $\varepsilon_0 < \operatorname{dist}(z_0, \partial\mathbb{D})$ and the function $\psi(t) = 1(t\log(1/t))$ in Lemma 1. We note that

$$\int_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{Q(z)\,dm(z)}{(|z-z_0|\log(1/|z-z_0|))^2} = \int_{\varepsilon}^{\varepsilon_0} \left(\int_{|z-z_0|=r} \frac{Q(z)\,dm(z)}{(|z-z_0|\log(1/|z-z_0|))^2}\,dS\right) dr \le 2\pi \int_{\varepsilon}^{\varepsilon_0} \frac{dr}{r\log(1/r)} = 2\pi \log\frac{\log(1/\varepsilon)}{\log(1/\varepsilon_0)} = 2\pi\cdot I(\varepsilon,\varepsilon_0),$$

where $I(\varepsilon,\varepsilon_0) := \int_{\varepsilon}^{\varepsilon_0} \psi(t)\,dt$. The statement of the theorem now follows from Lemma 1. □

Corollary 2. *Condition* (4.3) *and the statement of Theorem* 3 *are satisfied if it is required that, at each point* $z_0 \in \mathbb{D}$,

$$Q(z) \le C\cdot \log\frac{1}{|z-z_0|}$$

for some constant C (*that can depend on* z_0) *as* $z \to z_0$.

Remark 3. We note that the solutions of the quasilinear Beltrami equation studied in the present paper are generally not unique. The uniqueness of solutions of the Beltrami equation requires a separate study based on the use of different methods that are not related to the theory of convergence of mappings.

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