CONTENTS

ON CONVERGENCE AND COMPACTNESS
OF SPATIAL HOMEOMORPHISMS

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Various theorems on convergence of general space homeomorphisms are proved and, on this basis, theorems on convergence and compactness for classes of the so-called ring Q-homeomorphisms are obtained. In particular, it was established by us that a family of all ring Q-homeomorphisms f in $\mathbb{R}^n$ fixing two points is compact provided that the function Q is of finite mean oscillation. These results will have broad applications to Sobolev's mappings.

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1. **INTRODUCTION**

We give here the foundations of the convergence theory for general homeomorphisms in space and then develope the compactness theory for the so-called ring Q-homeomorphisms. The ring Q-homeomorphisms have been introduced first in the plane in the connection with the study of the degenerate Beltrami equations, see e.g. the papers [28]–[33] and the monographs [12] and [20]. The theory of ring Q-homeomorphisms is applied to various classes of mappings with finite distortion intensively investigated in many recent works, see e.g. [17] and [20] and further references therein. The present paper is a natural continuation of our previous works [25] and [36].

Given a family Γ of locally rectifiable paths γ in $\mathbb{R}^n$, $n \geq 2$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm} \Gamma$, if

$$\int_\gamma \rho(x) \, |\mathrm{d}x| \geq 1 \quad \forall \, \gamma \in \Gamma \,.$$

The *modulus* of Γ is the quantity

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm} \Gamma} \int_{\mathbb{R}^n} \rho^n(x) \, \mathrm{d}m(x) \,.$$

Given a domain D and two sets E and F in $\overline{\mathbb{R}^n}$, $n \geq 2$, $\Gamma(E, F, D)$ denotes the family of all rectifiable paths $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ which join E and F in D, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$. We set $\Gamma(E, F) = \Gamma(E, F, \mathbb{R}^n)$ if $D = \overline{\mathbb{R}^n}$. A *ring domain*, or shortly a *ring* in $\overline{\mathbb{R}^n}$, is a domain R in $\overline{\mathbb{R}^n}$ whose complement has two connected components. Let R be a ring in $\overline{\mathbb{R}^n}$. If C_1 and C_2 are the connected components of $\overline{\mathbb{R}^n} \setminus R$, we write $R = R(C_1, C_2)$. The *capacity* of the ring R can be defined by the equality

$$\operatorname{cap} R(C_1, C_2) \;=\; M(\Gamma(C_1, C_2, R)) \;,$$

see e.g. 5.49 in [36]. Note also that

$$M(\Gamma(C_1, C_2, R)) \;=\; M(\Gamma(C_1, C_2)) \;,$$

see e.g. Theorem 11.3 in [35]. A *conformal modulus* of a ring $R(C_0, C_1)$ is defined by

$$\operatorname{mod} R(C_0, C_1) = \left(\frac{\omega_{n-1}}{M(\Gamma(C_0, C_1))} \right)^{1/(n-1)} ,$$

where ω_{n-1} denotes the area of the unit sphere in $\mathbb{R}^n$, see e.g. (5.50) in [36].

Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $Q : D \to (0, \infty)$ be a (Lebesgue) measurable function. Set

$$A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\} \;,$$

$$S(x_0, r_i) = \{x \in \mathbb{R}^n : |x - x_0| = r_i\} \;, \quad i = 1, 2.$$

We say, see [25] for the spatial case, that a homeomorphism f of D into $\overline{\mathbb{R}^n}$ is a *ring Q-homeomorphism at a point* $x_0 \in D$ if

$$(1) \qquad M\left(\Gamma\left(f(S_1), f(S_2)\right)\right) \;\leq\; \int\limits_A Q(x) \cdot \eta^n(|x - x_0|) \, \mathrm{d}m(x)$$

for every ring $A = A(x_0, r_1, r_2)$, $0 < r_1 < r_2 < r_0 = \operatorname{dist}(x_0, \partial D)$, $S_i = S(x_0, r_i)$, $i = 1, 2$, and for every Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r) \, \mathrm{d}r \;\geq\; 1 \;.$$

If the condition (1) holds at every point $x_0 \in D$, then we also say that f is a ring Q-homeomorphism in the domain D.

The notion of ring Q-homeomorphisms was motivated by the ring definition of Gehring for quasiconformal mappings, see e.g. [11], and it is closely related to the concept of moduli with weights essentially due to Andreian Cazacu, see e.g. [1]–[3], cf. also the recent work [8].

2. ON BMO AND FMO FUNCTIONS

Recall that a real valued function $\varphi \in \mathrm{L}^1_{\mathrm{loc}}(D)$ given a domain $D \subset \mathbb{R}^n$ is said to be of *bounded mean oscillation* by John and Nierenberg, abbr. $\varphi \in$ BMO(D) or simply $\varphi \in BMO$, if

$$\|\varphi\|_* = \sup_{B \subset D} \fint_B |\varphi(x) - \varphi_B| \, \mathrm{d}m(x) \; < \; \infty$$

where the supremum is taken over all balls B in D and

$$\varphi_B = \fint_B \varphi(x) \, \mathrm{d}m(x) := \frac{1}{|B|} \int_B \varphi(x) \, \mathrm{d}m(x)$$

is the average of the function φ over B. For connections of BMO functions with quasiconformal and quasiregular mappings, see e.g. [4], [5], [15], [21] and [23].

Following [13], we say that a function $\varphi : D \to \mathbb{R}$ has *finite mean oscillation* at a point $x_0 \in D$ if

$$(2) \qquad \overline{\lim_{\varepsilon \to 0}} \fint_{B(x_0, \varepsilon)} |\varphi(x) - \widetilde{\varphi}_\varepsilon| \, \mathrm{d}m(x) \; < \; \infty$$

where

$$\widetilde{\varphi}_\varepsilon = \fint_{B(x_0, \varepsilon)} \varphi(x) \, \mathrm{d}m(x)$$

is the average of the function $\varphi(x)$ over the ball $B(x_0, \varepsilon) = \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon\}$. Note that under (2) it is possible that $\widetilde{\varphi}_\varepsilon \to \infty$ as $\varepsilon \to 0$.

We also say that a function $\varphi : D \to \mathbb{R}$ is of finite mean oscillation in the domain D, abbr. $\varphi \in \mathrm{FMO}(D)$ or simply $\varphi \in FMO$, if φ has finite mean oscillation at every point $x \in D$. Note that FMO is not $\mathrm{BMO}_{\mathrm{loc}}$, see examples in [20], p. 211. It is well-known that $L^\infty(D) \subset \mathrm{BMO}(D) \subset L^p_{\mathrm{loc}}(D)$ for all $1 \leq p < \infty$, see e.g. [14] and [23], but $\mathrm{FMO}(D) \not\subset L^p_{\mathrm{loc}}(D)$ for any $p > 1$.

Recall some facts on finite mean oscillation from [13], see also 6.2 in [20].

PROPOSITION 2.1. *If, for some numbers* $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$,

$$\overline{\lim_{\varepsilon \to 0}} \fint_{B(x_0, \varepsilon)} |\varphi(x) - \varphi_\varepsilon| \, \mathrm{d}m(x) < \infty \, ,$$

then φ *has finite mean oscillation at* x_0.

COROLLARY 2.1. *If, for a point* $x_0 \in D$,

$$\overline{\lim_{\varepsilon \to 0}} \fint_{B(x_0, \varepsilon)} |\varphi(x)| \, \mathrm{d}m(x) < \infty \, ,$$

then φ *has finite mean oscillation at* x_0.

LEMMA 2.1. *Let* $\varphi : D \to \mathbb{R}, n \geq 2$, *be a nonnegative function with a finite mean oscillation at* $0 \in D$. *Then*

$$\int\limits_{\varepsilon < |x| < \varepsilon_0} \frac{\varphi(x)\,\mathrm{d}m(x)}{(|x| \log \frac{1}{|x|})^n} = O\left(\log \log \frac{1}{\varepsilon} \right)$$

as $\varepsilon \to 0$ *for a positive* $\varepsilon_0 \leq \mathrm{dist}\,(0, \partial D)$.

This lemma takes an important part in many applications to the mapping theory as well as to the theory of the Beltrami equations, see e.g. the monographs [12] and [20].

3. CONVERGENCE OF GENERAL HOMEOMORPHISMS

In what follows, we use in $\overline{\mathbb{R}^n} = \mathbb{R}^n \bigcup \{\infty\}$ the *spherical (chordal) metric* $h(x, y) = |\pi(x) - \pi(y)|$ where π is the stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2} e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$, i.e.

$$h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}, \quad x \neq \infty \neq y,$$

$$h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}\;.$$

It is clear that $\overline{\mathbb{R}^n}$ is homeomorphic to the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$.

The *spherical (chordal) diameter* of a set $E \subset \overline{\mathbb{R}^n}$ is

$$h(E) = \sup_{x, y \in E} h(x, y)\,.$$

We also define $h(z, E)$ for $z \in \overline{\mathbb{R}^n}$ and $E \subseteq \overline{\mathbb{R}^n}$ as a infimum of $h(z, y)$ over all $y \in E$ and $h(F, E)$ for $F \subseteq \overline{\mathbb{R}^n}$ and $E \subseteq \overline{\mathbb{R}^n}$ as a infimum of $h(z, y)$ over all $z \in F$ and $y \in E$. Later on, we also use the notation $B^*(x_0, \rho)$, $x_0 \in \overline{\mathbb{R}^n}$, $\rho \in (0, 1)$, for the balls $\{x \in \overline{\mathbb{R}^n} : h(x, x_0) < \rho\}$ with respect to the spherical metric.

Let us start from the simple consequence of the well-known Brouwer theorem on invariance of domains.

COROLLARY 3.1. *Let* U *be an open set in* $\overline{\mathbb{R}^n}$ *and let* $f : U \to \overline{\mathbb{R}^n}$ *be continuous and injective. Then* f *is a homeomorphism* U *onto* $V = f(U)$.

Proof. Let $y_0 \in f(D)$ and $x_0 := f^{-1}(y_0)$. Set $B = B^*(x_0, \varepsilon_0)$ where $0 < \varepsilon_0 < h(x_0, \partial D)$. Then $\overline{B} \subset D$. Note that a mapping $f_0 := f|_{\overline{B}}$ is injective and continuous and maps the compactum $\overline{B}$ into the Hausdorff topological

space $\mathbb{R}^n$. Consequently, f_0 is a homeomorphism of $\overline{B}$ onto the topological space $f_0(\overline{B})$ with the topology induced by topology of $\mathbb{R}^n$ (see Theorem 41.III.3 in [19]). By the Brouwer theorem on invariance domains (see e.g. Theorem 4.7.16 in [34]), f maps the ball B onto a domain in $\overline{\mathbb{R}^n}$ as a homeomorphism. Hence, it follows that the mapping $f^{-1}(y)$ is continuous at the point y_0. Thus, $f : D \to \overline{\mathbb{R}^n}$ is a homeomorphism. $\square$

The *kernel of a sequence of open sets* $\Omega_l \subset \overline{\mathbb{R}^n}$, $l = 1, 2, \ldots$ is the open set

$$\Omega_0 \ = \ \text{Kern } \Omega_l \ : \ = \ \bigcup_{m=1}^{\infty} \ \text{Int} \left(\bigcap_{l=m}^{\infty} \Omega_l \right),$$

where Int A denotes the set consisting of all inner points of A; in other words, Int A is the union of all open balls in A with respect to the spherical distance.

The following statement for the plane case can be found in [6], see also Proposition 2.7 in [12].

PROPOSITION 3.1. *Let* $g_l : D \to D'_l$, $D'_l := g_l(D)$, *be a sequence of homeomorphisms given in a domain* $D \subset \overline{\mathbb{R}^n}$. *Suppose that* g_l *converge as* $l \to \infty$ *locally uniformly with respect to the spherical (chordal) metric to a mapping* $g : D \to D' := g(D) \subset \overline{\mathbb{R}^n}$ *which is injective. Then* g *is a homeomorphism and* $D' \subset \text{Kern } D'_l$.

Proof. First of all, the mapping g is continuous as a locally uniform limit of continuous mappings, see e.g. Theorem 13.VI.3 in [18]. Thus, by Corollary 3.1 g is a homeomorphism.

Now, let y_0 be a point in D'. Consider the spherical ball $B^*(z_0, \rho)$ where $z_0 := g^{-1}(y_0) \in D$ and $\rho < h(z_0, \partial D)$. Then

$$r_0 \ : \ = \ \min_{z \in \partial B^*(z_0, \rho)} \ h(y_0, g(z)) \ > \ 0 \, .$$

There is an integer N large enough such that $g_l(z_0) \in B^*(y_0, r_0/2)$ for all $l \geq N$ and simultaneously

$$B^*(y_0, r_0/2) \ \cap \ g_l(\partial B^*(z_0, \rho)) \ = \ B^*(y_0, r_0/2) \ \cap \ \partial g_l(B^*(z_0, \rho)) \ = \ \varnothing$$

because $g_l \to g$ uniformly on the compact set $\partial B^*(z_0, \rho)$. Hence, by the connectedness of balls

$$B^*(y_0, r_0/2) \ \subset \ g_l(B^*(z_0, \rho)) \qquad \forall \, l \geq N \, ,$$

see e.g. Theorem 46.I.1 in [19]. Consequently, $y_0 \in \text{Kern } D'_l$, i.e. $D' \subset \text{Kern } D'_l$ by arbitrariness of y_0. $\square$

Remark 3.1. In particular, Proposition 3.1 implies that $D' := g(D) \subset \mathbb{R}^n$ if $D'_l := g_l(D) \subset \mathbb{R}^n$ for all $l = 1, 2, \ldots$.

The following statement for the plane case can be found in the paper [16], see also Lemma 2.16 in the monograph [12].

LEMMA 3.1. *Let D be a domain in $\overline{\mathbb{R}^n}$, $l = 1, 2, \ldots$, and let f_l be a sequence of homeomorphisms from D into $\overline{\mathbb{R}^n}$ such that f_l converge as $l \to \infty$ locally uniformly with respect to the spherical metric to a homeomorphism f from D into $\overline{\mathbb{R}^n}$. Then $f_l^{-1} \to f^{-1}$ locally uniformly in $f(D)$, too.*

Proof. Given a compactum $C \subset f(D)$, we have by Proposition 3.1 that $C \subset f_l(D)$ for all $l \geq l_0 = l_0(C)$. Set $g_l = f_l^{-1}$ and $g = f^{-1}$. The locally uniform convergence $g_l \to g$ is equivalent to the so-called continuous convergence, meaning that $g_l(u_l) \to g(u_0)$ for every convergent sequence $u_l \to u_0$ in $f(D)$; see e.g. [9], p. 268 or Theorems 20.VIII.2 and 21.X.4 in [18]. So, let $u_l \in f(D)$, $l = 0, 1, 2, \ldots$ and $u_l \to u_0$ as $l \to \infty$. Let us show that $z_l := g(u_l) \to z_0 := g(u_0)$ as $l \to \infty$.

It is known that every metric space is $\mathcal{L}^*$-space, i.e. a space with a convergence (see e.g. Theorem 21.II.1 in [18]), and the Uhryson axiom in compact spaces says that $z_l \to z_0$ as $l \to \infty$ if and only if, for every convergent subsequence $z_{l_k} \to z_*$, the equality $z_* = z_0$ holds; see e.g. the definition 20.I.3 in [18]. Hence, it suffices to prove that the equality $z_* = z_0$ holds for every convergent subsequence $z_{l_k} \to z_*$ as $k \to \infty$. Let D_0 be a subdomain of D such that $z_0 \in D_0$ and $\overline{D_0}$ is a compact subset of D. Then, by Proposition 3.1, $f(D_0) \subset \text{Kern} f_l(D_0)$ and hence, u_0 together with its neighborhood belongs to $f_{l_k}(D_0)$ for all $k \geq K$. Thus, with no loss of generality we may assume that $u_{l_k} \in f_{l_k}(D_0)$, i.e. $z_{l_k} \in D_0$ for all $k = 1, 2, \ldots$, and, consequently, $z_* \in D$. Then, by the continuous convergence $f_l \to f$, we have that $f_{l_k}(z_{l_k}) \to f(z_*)$, i.e. $f_{l_k}(g_{l_k}(u_{l_k})) = u_{l_k} \to f(z_*)$. The latter implies that $u_0 = f(z_*)$, i.e. $f(z_0) = f(z_*)$ and hence $z_* = z_0$. The proof is complete. $\square$

The following statement for the plane case can be found in the paper [31], see also Proposition 2.6 in the monograph [12].

THEOREM 3.1. *Let D be a domain in $\overline{\mathbb{R}^n}$, $n \geq 2$, and let f_m, $m = 1, 2, \ldots$, be a sequence of homeomorphisms of D into $\overline{\mathbb{R}^n}$ converging locally uniformly to a discrete mapping $f : D \to \overline{\mathbb{R}^n}$ with respect to the spherical metric. Then f is a homeomorphism of D into $\overline{\mathbb{R}^n}$.*

Proof. First of all, let us show by contradiction that f is injective. Indeed, let us assume that there exist $x_1, x_2 \in D$, $x_1 \neq x_2$, with $f(x_1) = f(x_2)$ and that $x_1 \neq \infty$. Set $B_t = B(x_1, t)$. Let t_0 be such that $\overline{B_t} \subset D$ and $x_2 \notin \overline{B_t}$ for every $t \in (0, t_0]$. By the Jordan–Brower theorem, see e.g. Theorem 4.8.15 in [34], $f_m(\partial B_t) = \partial f_m(B_t)$ splits $\overline{\mathbb{R}^n}$ into two components

$$C_m := f_m(B_t), \qquad C_m^* = \overline{\mathbb{R}^n} \setminus \overline{C_m}.$$

By construction $y_m := f_m(x_1) \in C_m$ and $z_m := f_m(x_2) \in C_m^*$. Remark that the ball $B^*(y_m, h(y_m, \partial C_m))$ is contained inside of C_m and, consequently, its closure is inside of $\overline{C_m}$. Hence,

$$(3) \qquad\qquad h(y_m, \partial C_m) < h(y_m, z_m), \qquad m = 1, 2, \dots .$$

By compactness of $\partial C_m = f_m(\partial B_t)$, there is $x_{m,t} \in \partial B_t$ such that

$$(4) \qquad\qquad h(y_m, \partial C_m) = h(y_m, f_m(x_{m,t})), \qquad m = 1, 2, \dots .$$

By compactness of ∂B_t, for every $t \in (0, t_0]$, there is $x_t \in \partial B_t$ such that $h(x_{m_k,t}, x_t) \to 0$ as $k \to \infty$ for some subsequence m_k. Since the locally uniform convergence of continuous functions in a metric space implies the continuous convergence (see [9], p. 268 or Theorem 21.X.3 in [18]), we have that

$$h(f_{m_k}(x_{m_k,t}), f(x_t)) \to 0$$

as $k \to \infty$. Consequently, from (3) and (4) we obtain that

$$h(f(x_1), f(x_t)) \leq h(f(x_1), f(x_2)) \qquad \forall \quad t \in (0, t_0].$$

However, by the above assumption $f(x_1) = f(x_2)$ and we have $f(x_t) = f(x_1)$ for every $t \in (0, t_0]$. The latter contradicts to the discreteness of f. Thus, f is injective.

It remains to show that f and f^{-1} are continuous. A mapping f is continuous as a locally uniform limit of continuous mappings, see e.g. Theorem 13.VI.3 in [18]. Finally, f^{-1} is continuous by Corollary 3.1. $\square$

4. CONVERGENCE OF HOMEOMORPHISMS WITH MODULAR CONDITIONS

Later on, the following lemma plays a very important role. Its plane analog can be found in the paper [7], see also supplement A1 in the monograph [21].

LEMMA 4.1. *Let f_m, $m = 1, 2, \dots$, be a sequence of homeomorphisms of a domain $D \subseteq \mathbb{R}^n$ into $\mathbb{R}^n$, $n \geq 2$, converging to a mapping f uniformly on every compact set in D with respect to the spherical metric in $\overline{\mathbb{R}^n}$. Suppose that for every $x_0 \in D$ there exist sequences $R_k > 0$ and $r_k \in (0, R_k)$, $k = 1, 2, \dots$, such that $R_k \to 0$ as $k \to \infty$ and $\operatorname{mod} f_m(A(x_0, r_k, R_k)) \to \infty$ as $k \to \infty$ uniformly with respect to $m = 1, 2, \dots$. Then the mapping f is either a constant in $\overline{\mathbb{R}^n}$ or a homeomorphism of D into $\mathbb{R}^n$.*

Proof. Assume that f is not constant. Let us consider the open set V consisting of all points in D which have neighborhoods where f is a constant and show that $f(x) \neq f(x_0)$ for every $x_0 \in D \setminus V$ and $x \neq x_0$. Without loss of

generality, we may assume that $f(x_0) \neq \infty$. Now, let us fix a point $x_* \neq x_0$ in $D \setminus V$ and choose $k = 1, 2, \ldots$ such that $R := R_k < |x_* - x_0|$ and

$$(5) \qquad \operatorname{mod} f_m \left(A(x_0, r, R) \right) > \left(\omega_{n-1}/\tau_n(1) \right)^{1/(n-1)}$$

for $r = r_k$ where $\tau_n(s)$ denotes the capacity of the Teichmüller ring $R_{T,n}(s) := [\mathbb{R}^n \setminus \{te_1 : t \geq s\}, [-e_1, 0]]$, $s \in (0, \infty)$.

Let $c_m \in f_m(S(x_0, R))$ and $b_m \in f_m(S(x_0, r))$ be such that

$$\min_{w \in f_m(S(x_0, R))} |w - f_m(x_0)| = |c_m - f_m(x_0)|,$$

$$\max_{w \in f_m(S(x_0, r))} |w - f_m(x_0)| = |b_m - f_m(x_0)|.$$

Since f_m is a homeomorphism, the set $f_m(A(x_0, r, R))$ is a ring domain $\mathfrak{R}_m = (C_m^1, C_m^2)$, where $a_m := f_m(x_0)$ and $b_m \in C_m^1$, c_m and $\infty \in C_m^2$. Applying Lemma 7.34 in [36] with $a = a_m$, $b = b_m$ and $c = c_m$, we obtain that

$$(6) \qquad \operatorname{cap} \mathfrak{R}_m = M(\Gamma(C_m^1, C_m^2)) \geq \tau_n \left(\frac{|a_m - c_m|}{|a_m - b_m|} \right).$$

Note that the function $\tau_n(s)$ is strictly decreasing (see Lemma 7.20 in [36]). Thus, it follows from (5) and (6) that

$$\frac{|a_m - c_m|}{|a_m - b_m|} \geq \tau_n^{-1} \left(\operatorname{cap} \mathfrak{R}_m \right) > \tau_n^{-1}(\tau_n(1)) = 1.$$

Hence, there is a spherical ring $A_m = \{y \in \mathbb{R}^n : \rho_m < |y - f_m(x_0)| < \rho_m^*\}$ in the ring domain $\mathfrak{R}_m$ for every $m = 1, 2, \ldots$. Since f is not locally constant at x_0, we can find a point x' in the ball $|x - x_0| < r$ with $f(x_0) \neq f(x')$. The ring A_m separates $f_m(x_0)$ and $f_m(x')$ from $f_m(x_*)$ and, thus, $|f_m(x') - f_m(x_0)| \leq \rho_m$ and $|f_m(x_*) - f_m(x_0)| \geq \rho_m^*$. Consequently, $|f_m(x') - f_m(x_0)| \leq |f_m(x_*) - f_m(x_0)|$ for all $m = 1, 2, \ldots$. Under $m \to \infty$ we have then $0 < |f(x') - f(x_0)| \leq |f(x_*) - f(x_0)|$ and hence $f(x_*) \neq f(x_0)$.

It remains to show that the set V is empty. Let us assume that V has a nonempty component V_0. Then $f(x) \equiv z$ for every $x \in V_0$ and some $z \in \overline{\mathbb{R}^n}$. Note that $\partial V_0 \cap D \neq \varnothing$ by connectedness of D because $f \not\equiv const$ in D and the set $D \setminus \overline{V_0}$ is also open. If $x_0 \in \partial V_0 \cap D$, then by continuity $f(x_0) = z$ contradicting the first part of the proof because $x_0 \in D \setminus V$.

Thus, we have proved that the mapping f is injective if f is not constant. But f is continuous as a locally uniform limit of continuous mappings f_m, see Theorem 13.VI.3 in [18], and then by Corollary 3.1 f is a homeomorphism. Finally, by Remark 3.1 $f(D) \subset \mathbb{R}^n$ and the proof is complete. $\square$

LEMMA 4.2. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $Q_m : D \to (0, \infty)$ be measurable functions, f_m, $m = 1, 2, \ldots$, be a sequence of ring Q_m-homeomor-*

phisms of D into $\mathbb{R}^n$ converging locally uniformly to a mapping f. Suppose

$$(7) \qquad \int\limits_{\varepsilon<|x-x_0|<\varepsilon_0} Q_m(x) \cdot \psi^n(|x-x_0|) \; \mathrm{d}m(x) \; = \; o(I^n(\varepsilon,\varepsilon_0)) \quad \forall \, x_0 \in D$$

where $o(I^n(\varepsilon,\varepsilon_0))/I^n(\varepsilon,\varepsilon_0) \to 0$ as $\varepsilon \to 0$ uniformly with respect to m for $\varepsilon_0 < \mathrm{dist}\,(x_0,\partial D)$ and a measurable function $\psi(t) : (0,\varepsilon_0) \to [0,\infty]$ such that

$$(8) \qquad\qquad 0 < I(\varepsilon,\varepsilon_0) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t) \; \mathrm{d}t < \infty \quad \forall \, \varepsilon \in (0,\varepsilon_0) \; .$$

Then the mapping f is either a constant in $\overline{\mathbb{R}^n}$ or a homeomorphism into $\mathbb{R}^n$.

   Remark 4.1. In particular, the conclusion of Lemma 4.2 holds for Q-homeomorphisms f_m with a measurable function $Q : D \to (0,\infty)$ such that

$$(9) \qquad \int\limits_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x) \cdot \psi^n(|x-x_0|) \; \mathrm{d}m(x) = o(I^n(\varepsilon,\varepsilon_0)) \qquad \forall \, x_0 \in D \; .$$

   Proof. By Lusin theorem there exists a Borel function $\psi_*(t)$ such that $\psi(t) = \psi_*(t)$ for a.e. $t \in (0,\varepsilon_0)$, see e.g. 2.3.6 in [10]. Since $Q_m(x) > 0$ for all $x \in D$ we have from (7) that $I(\varepsilon,a) \to \infty$ for every fixed $a \in (0,\varepsilon_0)$ and, in particular, $I(\varepsilon,a) > 0$ for every $\varepsilon \in (0,b)$ and some $b = b(a) \in (0,a)$. Given $x_0 \in D$ and a sequence of such numbers $b = \varepsilon_k \to 0$ as $k \to \infty$, $k = 1, 2, \ldots$, consider a sequence of the Borel measurable functions $\rho_{\varepsilon,k}$ defined as

$$\rho_{\varepsilon,k}(x) = \begin{cases} \psi_*(|x-x_0|)/I(\varepsilon,\varepsilon_k), & \varepsilon < |x-x_0| < \varepsilon_k, \\ 0, & \text{otherwise}\,. \end{cases}$$

Note that the function $\rho_{\varepsilon,k}(x)$ is admissible for

$$\Gamma_{\varepsilon,k} := \Gamma(S(x_0,\varepsilon), S(x_0,\varepsilon_k), A(x_0,\varepsilon,\varepsilon_k))$$

because

$$\int\limits_{\gamma} \rho_{\varepsilon,k}(x)|\mathrm{d}x| \geq \frac{1}{I(\varepsilon,\varepsilon_k)} \int\limits_{\varepsilon}^{\varepsilon_k} \psi(t)\mathrm{d}t = 1$$

for all (locally rectifiable) curves $\gamma \in \Gamma_{\varepsilon,k}$ (see Theorem 5.7 in [35]). Then by definition of ring Q-homeomorphisms

$$(10) \qquad M(f_m(\Gamma_{\varepsilon,k})) \; \leq \; \frac{1}{I^n(\varepsilon,\varepsilon_k)} \int\limits_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x) \cdot \psi^n(|x-x_0|) \, \mathrm{d}m(x)$$

for all $m \in \mathbb{N}$. Note that $\frac{1}{I^n(\varepsilon,\varepsilon_k)} = \alpha_{\varepsilon,k} \cdot \frac{1}{I^n(\varepsilon,\varepsilon_0)}$, where $\alpha_{\varepsilon,k} := \left(1 + \frac{I(\varepsilon_k,\varepsilon_0)}{I(\varepsilon,\varepsilon_k)}\right)^n$ is independent on m and bounded as $\varepsilon \to 0$. Then, it follows from (7) and (10) that there exists $\varepsilon_k^* \in (0,\varepsilon_k)$ such that for all

$$M(f_m(\Gamma_{\varepsilon_k^*,k})) \quad \leq \quad \frac{1}{2^k} \qquad \forall\, m \in \mathbb{N}.$$

Applying Lemma 4.1 we obtain a desired conclusion. $\square$

The following important statements follow just from Lemma 4.2.

THEOREM 4.1. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $Q : D \to (0,\infty)$ a Lebesgue measurable function and let f_m, $m = 1,2,\ldots$, be a sequence of ring Q-homeomorphisms of D into $\mathbb{R}^n$ converging locally uniformly to a mapping f. Suppose that $Q \in FMO$. Then the mapping f is either a constant in $\overline{\mathbb{R}^n}$ or a homeomorphism into $\mathbb{R}^n$.*

Proof. Let $x_0 \in D$. We may consider further that $x_0 = 0 \in D$. Choosing a positive $\varepsilon_0 < \min\left\{\mathrm{dist}\,(0, \partial D),\, e^{-1}\right\}$, we obtain by Lemma 2.1 for the function $\psi(t) = \frac{1}{t \log \frac{1}{t}}$ that

$$\int\limits_{\varepsilon < |x| < \varepsilon_0} Q(x) \cdot \psi^n(|x|)\, \mathrm{d}m(x) \ = \ O\left(\log\log \frac{1}{\varepsilon}\right).$$

Note that $I(\varepsilon, \varepsilon_0) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, \mathrm{d}t \ = \ \log \frac{\log \frac{1}{\varepsilon}}{\log \frac{1}{\varepsilon_0}}$. Now, the desired conclusion follows from Lemma 4.2. $\square$

The following conclusions can be obtained on the basis of Theorem 4.1, Proposition 2.1 and Corollary 2.1.

COROLLARY 4.1. *In particular, the limit mapping f is either a constant in $\overline{\mathbb{R}^n}$ or a homeomorphism of D into $\mathbb{R}^n$ whenever*

$$\varlimsup_{\varepsilon \to 0} \ \fint_{B(x_0,\varepsilon)} Q(x) \ \mathrm{d}m(x) < \infty \qquad \forall\ x_0 \in D$$

or whenever every $x_0 \in D$ is a Lebesgue point of Q.

THEOREM 4.2. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $Q : D \to (0,\infty)$ be a measurable function such that*

(11)
$$\int\limits_{0}^{\varepsilon(x_0)} \frac{\mathrm{d}r}{r q_{x_0}^{\frac{1}{n-1}}(r)} = \infty \qquad \forall\, x_0 \in D$$

for a positive $\varepsilon(x_0) < \mathrm{dist}\,(x_0, \partial D)$ where $q_{x_0}(r)$ denotes the average of $Q(x)$ over the sphere $|x - x_0| = r$. Suppose that f_m, $m = 1,2,\ldots$, is a sequence of ring Q-homeomorphisms from D into $\mathbb{R}^n$ converging locally uniformly to a mapping f. Then the mapping f is either a constant in $\overline{\mathbb{R}^n}$ or a homeomorphism into $\mathbb{R}^n$.

Proof. Fix $x_0 \in D$ and set $I = I(\varepsilon, \varepsilon_0) = \int_{\varepsilon}^{\varepsilon_0} \psi(t)\, \mathrm{d}t$, $\varepsilon \in (0, \varepsilon_0)$, where

$$\psi(t) \quad = \quad \begin{cases} 1/[tq_{x_0}^{\frac{1}{n-1}}(t)]\,, & t \in (\varepsilon, \varepsilon_0)\,, \\ 0\,, & t \notin (\varepsilon, \varepsilon_0)\,. \end{cases}$$

Note that $I(\varepsilon, \varepsilon_0) < \infty$ for every $\varepsilon \in (0, \varepsilon_0)$. Indeed, by Theorem 3.15 in [25] on the criterion of ring Q–homeomorphisms, we have that

$$(12) \qquad M(f(\Gamma(S(x_0, \varepsilon), S(x_0, \varepsilon_0), A(x_0, \varepsilon, \varepsilon_0)))) \leq \frac{\omega_{n-1}}{I^{n-1}}\,.$$

On the other hand, by Lemma 1.15 in [22], we see that

$$M(\Gamma(f(S(x_0, \varepsilon)), f(S(x_0, \varepsilon_0)), f(A(x_0, \varepsilon, \varepsilon_0)))) > 0\,.$$

Then, it follows from (12) that $I < \infty$ for every $\varepsilon \in (0, \varepsilon_0)$. In view of (11), we obtain that $I(\varepsilon, \varepsilon_*) > 0$ for all $\varepsilon \in (0, \varepsilon_*)$ with some $\varepsilon_* \in (0, \varepsilon_0)$. Finally, simple calculations show that (9) holds, in fact,

$$\int\limits_{\varepsilon < |x - x_0| < \varepsilon_*} Q(x) \cdot \psi^n(|x - x_0|)\, \mathrm{d}m(x) \quad = \quad \omega_{n-1} \cdot I(\varepsilon, \varepsilon_*)$$

and $I(\varepsilon, \varepsilon_*) = o\left(I^n(\varepsilon, \varepsilon_*)\right)$ by (11). The rest follows by Lemma 4.2. $\quad\square$

COROLLARY 4.2. *In particular, the conclusion of Theorem 4.2 holds if*

$$q_{x_0}(r) = O\left(\log^{n-1} \frac{1}{r}\right) \qquad \forall\, x_0 \in D\,.$$

COROLLARY 4.3. *Under assumptions of Theorem 4.2, the mapping f is either a constant in $\overline{\mathbb{R}^n}$ or a homeomorphism into $\mathbb{R}^n$ provided $Q(x)$ has singularities only of the logarithmic type of the order which is not more than $n - 1$ at every point $x_0 \in D$.*

THEOREM 4.3. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and $Q : D \to (0, \infty)$ be a measurable function such that*

$$(13) \qquad \int\limits_{\varepsilon < |x - x_0| < \varepsilon_0} \frac{Q(x)}{|x - x_0|^n}\, \mathrm{d}m(x) = o\left(\log^n \frac{1}{\varepsilon}\right) \qquad \forall\, x_0 \in D$$

as $\varepsilon \to 0$ for some positive number $\varepsilon_0 = \varepsilon(x_0) < \mathrm{dist}\,(x_0, \partial D)$. Suppose that f_m, $m = 1, 2, \ldots$, is a sequence of ring Q-homeomorphisms from D into $\overline{\mathbb{R}^n}$ converging locally uniformly to a mapping f. Then, the limit mapping f is either a constant in $\overline{\mathbb{R}^n}$ or a homeomorphism into $\mathbb{R}^n$.

Proof. The conclusion follows from Lemma 4.2 by the choice $\psi(t) = \frac{1}{t}$. $\square$

For every nondecreasing function $\Phi : [0, \infty] \to [0, \infty]$, the *inverse function* $\Phi^{-1} : [0, \infty] \to [0, \infty]$ can be well defined by setting

$$\Phi^{-1}(\tau) \;=\; \inf_{\Phi(t) \geq \tau} t \;.$$

As usual, here inf is equal to ∞ if the set of $t \in [0, \infty]$ such that $\Phi(t) \geq \tau$ is empty. Note that the function Φ^{-1} is nondecreasing, too. Note also that if $h : [0, \infty] \to [0, \infty]$ is a sense-preserving homeomorphism and $\varphi : [0, \infty] \to [0, \infty]$ is a nondecreasing function, then

$$(14) \qquad\qquad (\varphi \circ h)^{-1} \;=\; h^{-1} \circ \varphi^{-1} \;.$$

THEOREM 4.4. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, let $Q : D \to (0, \infty)$ be a measurable function and $\Phi : [0, \infty] \to [0, \infty]$ be a nondecreasing convex function. Suppose that*

$$(15) \qquad \int\limits_D \Phi\left(Q(x)\right) \frac{\mathrm{d}m(x)}{(1 + |x|^2)^n} \;\leq\; M < \infty$$

and

$$(16) \qquad \int\limits_\delta^\infty \frac{\mathrm{d}\tau}{\tau \left[\Phi^{-1}(\tau)\right]^{\frac{1}{n-1}}} \;=\; \infty$$

for some $\delta > \Phi(0)$. Suppose that f_m, $m = 1, 2, \ldots$, is a sequence of ring Q-homeomorphisms of D into $\mathbb{R}^n$ converging locally uniformly to a mapping f. Then, the mapping f is either a constant in $\overline{\mathbb{R}^n}$ or a homeomorphism into $\mathbb{R}^n$.

Proof. It follows from (15)–(16) and Theorem 3.1 in [26] that the integral in (11) is divergent for some positive $\varepsilon(x_0) < \mathrm{dist}\,(x_0, \partial D)$. The rest follows by Theorem 4.2. $\square$

Remark 4.2. We may assume in Theorem 4.4 that the function $\Phi(t)$ is not convex on the whole segment $[0, \infty]$ but only on the segment $[t_*, \infty]$ where $t_* = \Phi^{-1}(\delta)$. Indeed, every non-decreasing function $\Phi : [0, \infty] \to [0, \infty]$ which is convex on the segment $[t_*, \infty]$ can be replaced by a non-decreasing convex function $\Phi_* : [0, \infty] \to [0, \infty]$ in the following way. Set $\Phi_*(t) \equiv 0$ for $t \in [0, t_*]$, $\Phi(t) = \varphi(t)$ for $t \in [t_*, T_*]$ and $\Phi_* \equiv \Phi(t)$ for $t \in [T_*, \infty]$, where $\tau = \varphi(t)$ is the line passing through the point $(0, t_*)$ and touching the graph of the function $\tau = \Phi(t)$ at a point $(T_*, \Phi(T_*))$, $T_* \in (t_*, \infty)$. By the construction, we have that $\Phi_*(t) \leq \Phi(t)$ for all $t \in [0, \infty]$ and $\Phi_*(t) = \Phi(t)$ for all $t \geq T_*$ and, consequently, the conditions (15) and (16) hold for Φ_* under the same M and every $\delta > 0$.

Furthermore, by the same reasons it is sufficient to assume that the function Φ is only minorized by a nondecreasing convex function Ψ on a segment

$[T, \infty]$ such that

(17)
$$\int_{\delta}^{\infty} \frac{d\tau}{\tau \left[\Psi^{-1}(\tau) \right]^{\frac{1}{n-1}}} = \infty$$

for some $T \in [0, \infty)$ and $\delta > \Psi(T)$. Note that the condition (17) can be written in terms of the function $\psi(t) = \log \Psi(t)$:

(18)
$$\int_{\Delta}^{\infty} \psi(t) \, \frac{dt}{t^{n'}} = \infty$$

for some $\Delta > t_0 \in [T, \infty)$ where $t_0 := \sup_{\psi(t) = -\infty} t$, $t_0 = T$ if $\psi(T) > -\infty$, and where $\frac{1}{n'} + \frac{1}{n} = 1$, i.e., $n' = 2$ for $n = 2$, n' is decreasing in n and $n' = n/(n-1) \to 1$ as $n \to \infty$, see Proposition 2.3 in [26]. It is clear that if the function ψ is nondecreasing and convex, then the function $\Phi = e^{\psi}$ is so but the inverse conclusion generally speaking is not true. However, the conclusion of Theorem 4.4 is valid if $\psi : [0, \infty] \to [0, \infty]$ and $\psi^m(t)$, $t \in [T, \infty]$, is convex and (18) holds for ψ^m under some $m \in \mathbb{N}$ because $e^{\tau} \geq \tau^m / m!$ for all $m \in \mathbb{N}$.

COROLLARY 4.4. *In particular, the conclusion of Theorem 4.4 is valid if, for some $\alpha > 0$,*

(19)
$$\int_{D} e^{\alpha Q^{\frac{1}{n-1}}(x)} \frac{dm(x)}{(1 + |x|^2)^n} \leq M < \infty .$$

The same is true for any function $\Phi = e^{\psi}$ where $\psi(t)$ is a finite product of the function αt^{β}, $\alpha > 0$, $\beta \geq 1/(n-1)$, and some of the functions $[\log(A_1 + t)]^{\alpha_1}$, $[\log \log(A_2 + t)]^{\alpha_2}$, ... , $\alpha_m \geq -1$, $A_m \in \mathbb{R}$, $m \in \mathbb{N}$, $t \in [T, \infty]$, $\psi(t) \equiv \psi(T)$, $t \in [0, T]$ for a large enough T.

Remark 4.3. For further applications, the integral conditions (15) and (16) for Q and Φ can be written in other forms that are more convenient for some cases. Namely, by (14) with $h(t) = t^{\frac{1}{n-1}}$ and $\varphi(t) = \Phi(t^{n-1})$, $\Phi = \varphi \circ h$, the couple of conditions (15) and (16) is equivalent to the following couple

(20)
$$\int_{D} \varphi \left(Q^{\frac{1}{n-1}}(x) \right) \frac{dm(x)}{(1 + |x|^2)^n} \leq M < \infty$$

and

(21)
$$\int_{\delta}^{\infty} \frac{d\tau}{\tau \varphi^{-1}(\tau)} = \infty$$

for some $\delta > \varphi(0)$. Moreover, by Theorem 2.1 in [32] the couple of the conditions (20) and (21) is in turn equivalent to the next couple

$$(22) \qquad \int\limits_{D} e^{\psi\left(Q^{\frac{1}{n-1}}(x)\right)} \frac{\mathrm{d}m(x)}{(1+|x|^2)^n} \ \leq \ M < \infty$$

and

$$(23) \qquad \int\limits_{\Delta}^{\infty} \psi(t) \, \frac{\mathrm{d}t}{t^2} \ = \ \infty$$

for some $\Delta > t_0$ where $t_0 := \sup\limits_{\psi(t)=-\infty} t$, $t_0 = 0$ if $\psi(0) > -\infty$, $\psi(t) = \log \varphi$.

Finally, as it follows from Lemma 4.2 all the results of this section are valid if f_m are Q_m-homeomorphisms and the above conditions on Q hold for Q_m uniformly with respect to the parameter $m = 1, 2, \ldots$.

5. ON COMPLETENESS OF RING HOMEOMORPHISMS

The following result for the plane case can be found in the paper [31], Theorem 4.1, see also the monograph [12], Theorem 6.2.

THEOREM 5.1. *Let* $f_m : D \to \overline{\mathbb{R}^n}$, $m = 1, 2, \ldots$, *be a sequence of ring* Q-*homeomorphisms at a point* $x_0 \in D$. *If* f_m *converges locally uniformly to a homeomorphism* $f : D \to \overline{\mathbb{R}^n}$, *then* f *is also a ring* Q-*homeomorphism at* x_0.

Proof. Note first that every point $w_0 \in D' = f(D)$ belongs to $D'_m = f_m(D)$ for all $m \geq N$ together with $\overline{B^*(w_0, \varepsilon)}$, where $B^*(w_0, \varepsilon) = \{w \in \overline{\mathbb{R}^n} : h(w, w_0) < \varepsilon\}$ for some $\varepsilon > 0$ (see Proposition 3.1).

Now, we note that $D' = \bigcup_{l=1}^{\infty} C_l$ where $C_l = \overline{D_l^*}$, and D_l^* is a connected component of the open set $\Omega_l = \{w \in D' : h(w, \partial D') > 1/l\}$, $l = 1, 2, \ldots$, including a fixed point $w_0 \in D'$. Indeed, every point $w \in D'$ can be joined with w_0 by a path γ in D'. Because the locus $|\gamma|$ is compact we have that $h(|\gamma|, \partial D') > 0$ and, consequently, $|\gamma| \subset D_l^*$ for large enough $l = 1, 2, \ldots$.

Next, take an arbitrary pair of continua E and F in D which belong to the different connected components of the complement of a ring $A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$, $x_0 \in D$, $0 < r_1 < r_2 < r_0 = \mathrm{dist}\,(x_0, \partial D)$. For $l \geq l_0$, continua $f(E)$ and $f(F)$ belong to D_l^*. Then, the continua $f_m(E)$ and $f_m(F)$ also belong to D_l^* for large enough m. Fix one such m. It is known that

$$M(\Gamma(f_m(E), f_m(F), D_l^*)) \ \to \ M(\Gamma(f(E), f(F), D_l^*))$$

as $m \to \infty$, see e.g. Theorem A.12 in [20]. However, $D_l^* \subset f_m(D)$ for large enough m and hence

$$M(\Gamma(f_m(E), f_m(F), D_l^*)) \leq M(\Gamma(f_m(E), f_m(F), f_m(D)))$$

and, thus, by definition of ring Q-homeomorphisms

$$M(\Gamma(f(E), f(F), D_l^*)) \leq \int\limits_A Q(x) \cdot \eta^n(|x - x_0|) \, \mathrm{d}m(x) \ ,$$

for every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that $\int \eta(r) \, \mathrm{d}r \geq 1$. Finally, since $\Gamma = \bigcup\limits_{l=l_0}^{\infty} \Gamma_l$ where $\Gamma = \Gamma(f(E), f(F), f(D))$, $\Gamma_l = \Gamma(f(E), f(F), D_l^*)$ is increasing in $l = 1, 2, \ldots$, we obtain that $M(\Gamma) = \lim\limits_{l \to \infty} M(\Gamma_l)$, see e.g. Theorems A.7 and A.25 in [20]. Thus,

$$M(\Gamma(f(E), f(F), f(D))) \leq \int\limits_A Q(x) \cdot \eta^n(|x - x_0|) \, \mathrm{d}m(x) \ ,$$

i.e., f is a ring Q-homeomorphism at x_0. $\square$

6. NORMAL CLASSES OF RING Q-HOMEOMORPHISMS

Given a domain D in $\mathbb{R}^n$, $n \geq 2$, a measurable function $Q : D \to (0, \infty)$, and $\Delta > 0$, denote by $\mathfrak{F}_{Q,\Delta}$ the family of all ring Q-homeomorphisms f of D into $\overline{\mathbb{R}^n}$ such that $h(\overline{\mathbb{R}^n} \setminus f(D)) \geq \Delta$. Recall that a class of mappings is called normal if every sequence of mappings in the class contains a subsequence that converges locally uniformly.

LEMMA 6.1. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $Q : D \to (0, \infty)$ be a measurable function. If the conditions (8)–(9) hold, then the class $\mathfrak{F}_{Q,\Delta}$ forms a normal family for all $\Delta > 0$.*

Proof. By Lemma 7.5 in [20], cf. also Lemma 4.1 in [25], for $y \in B(x_0, r_0)$, $r_0 < \mathrm{dist}\,(x_0, \partial D)$, $S_0 = \{x \in \mathbb{R}^n : |x - x_0| = r_0\}$ and $S = \{x \in \mathbb{R}^n : |x - x_0| = |y - x_0|\}$ we have that

$$(24) \quad h(f(y), f(x_0)) \ \leq \ \frac{\alpha_n}{\Delta} \cdot \exp\left(-\left\{ \frac{\omega_{n-1}}{M\left(\Gamma\left(f(S), f(S_0), f(D)\right)\right)} \right\}^{1/n-1} \right),$$

where ω_{n-1} is the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, $\alpha_n = 2\lambda_n^2$ with $\lambda_n \in [4, 2e^{n-1})$, $\lambda_2 = 4$ and $\lambda_n^{\frac{1}{n}} \to e$ as $n \to \infty$. We may consider that ψ is a Borel function, because, by 2.3.4 and 2.3.6 in [10] there exists a Borel function $\psi_*(t)$ with $\psi(t) = \psi_*(t)$ for a.e. $t \in (0, \varepsilon_0)$. Given $\varepsilon \in (0, \varepsilon_0)$, consider a Borel measurable function ρ_ε defined as

$$\rho_\varepsilon(x) = \begin{cases} \psi_*(|x - x_0|)/I(\varepsilon, \varepsilon_0), & \varepsilon < |x - x_0| < \varepsilon_0, \\ 0, & \text{otherwise}. \end{cases}$$

Note that $\rho_\varepsilon(x)$ is admissible for $\Gamma_\varepsilon := \Gamma(S(x_0, \varepsilon), S(x_0, \varepsilon_0), A(x_0, \varepsilon, \varepsilon_0))$ because

$$\int\limits_{\gamma} \rho_\varepsilon(x)|\mathrm{d}x| \geq \frac{1}{I(\varepsilon,\varepsilon_0)} \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\mathrm{d}t = 1$$

for all (locally rectifiable) curves $\gamma \in \Gamma_\varepsilon$ (see Theorem 5.7 in [35]). Then, by definition of ring Q-homeomorphism at the point x_0

$$(25) \quad M(f(\Gamma_\varepsilon)) \quad \leq \quad \mathcal{J}(\varepsilon) := \frac{1}{I^n(\varepsilon,\varepsilon_0)} \int\limits_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x)\cdot\psi^n(|x-x_0|)\,\mathrm{d}m(x) \ ,$$

for all $f \in \mathfrak{F}_{Q,\Delta}$. It follows from (9) that, given $\sigma > 0$, there exists $\delta = \delta(\sigma)$ such that $\mathcal{J}(\varepsilon) < \sigma$ for all $\varepsilon \in (0,\delta)$. Then, from (24) and (25) we have that

$$(26) \quad\quad h(f(x),f(x_0)) \ \leq \ \frac{\alpha_n}{\Delta}\cdot\exp\left(-\left\{\frac{\omega_{n-1}}{\sigma}\right\}^{1/n-1}\right)$$

provided $|x - x_0| < \delta$. In view of arbitrariness of $\sigma > 0$ the equicontinuity of $\mathfrak{F}_{Q,\Delta}$ follows from (26). $\square$

Remark 6.1. In particular, the conclusion of Lemma 6.1 holds if at least one of the conditions on Q in Theorems 4.1–4.4 and Corollary 4.1–4.4 holds. The corresponding normality results have been formulated in [25] and [26] and hence, we will not repeat them in the explicit form here.

Furthermore, as it follows from the analysis of the proof of Lemma 6.1, its conlusion is valid for a more wide class $\mathfrak{F}_\Delta$, $\Delta > 0$, consisting of all ring Q-homeomorphisms f of D into $\overline{\mathbb{R}^n}$ such that $h\left(\overline{\mathbb{R}^n} \setminus f(D)\right) \geq \Delta$ satisfying the uniform condition (9) for the variable Q but with a fixed function ψ in (8). Thus, the conclusion of Lemma 6.1 is also valid if at least one of the conditions on Q in Theorems 4.1–4.4 and Corollary 4.1–4.4 is uniform with respect to the variable functional parameter Q.

All notes in Remarks 4.2 and 4.3 are also valid for the normality results.

7. ON COMPACT CLASSES OF RING Q-HOMEOMORPHISMS

Given a domain D in $\mathbb{R}^n$, $n \geq 2$, a measurable function $Q : D \to (0,\infty)$, $x_1, x_2 \in D$, $y_1, y_2 \in \mathbb{R}^n$, $x_1 \neq x_2$, $y_1 \neq y_2$, set $\mathfrak{R}_Q$ the class of all ring Q-homeomorphisms from D into $\mathbb{R}^n$, $n \geq 2$, satisfying the normalization conditions $f(x_1) = y_1$, $f(x_2) = y_2$.

Recall that a class of mappings is called compact if it is normal and closed. Combining the above results on normality and closure, we obtain the following results on compactness for the classes of ring Q-homeomorphisms.

THEOREM 7.1. *If $Q \in FMO$, then the class $\mathfrak{R}_Q$ is compact.*

COROLLARY 7.1. *The class $\mathfrak{R}_Q$ is compact if*

$$\overline{\lim_{\varepsilon \to 0}} \fint_{B(x_0, \varepsilon)} Q(x) \ \mathrm{d}m(x) < \infty, \qquad \forall \ x_0 \in D.$$

COROLLARY 7.2. *The class $\mathfrak{R}_Q$ is compact if every $x_0 \in D$ is a Lebesgue point of Q.*

THEOREM 7.2. *Let Q satisfy the condition*

$$\int_0^{\varepsilon(x_0)} \frac{\mathrm{d}r}{r q_{x_0}^{\frac{1}{n-1}}(r)} = \infty, \qquad \forall \, x_0 \in D,$$

for some $\varepsilon(x_0) < \mathrm{dist}\,(x_0, \partial D)$ where $q_{x_0}(r)$ denotes the average of $Q(x)$ over the sphere $|x - x_0| = r$. Then, the class $\mathfrak{R}_Q$ is compact.

COROLLARY 7.3. *The class $\mathfrak{R}_Q$ is compact if $Q(x)$ has singularities only of the logarithmic type of the order which is not more than $n - 1$ at every point $x_0 \in D$.*

THEOREM 7.3. *The class $\mathfrak{R}_Q$ is compact if*

$$\int_{\varepsilon < |x - x_0| < \varepsilon_0} \frac{Q(x)}{|x - x_0|^n} \, \mathrm{d}m(x) = o\left(\log^n \frac{1}{\varepsilon}\right) \qquad \forall \, x_0 \in D$$

as $\varepsilon \to 0$ for some $\varepsilon_0 = \varepsilon(x_0) < \mathrm{dist}\,(x_0, \partial D)$.

THEOREM 7.4. *The class $\mathfrak{R}_Q$ is compact if*

$$(27) \qquad \int_D \Phi\left(Q(x)\right) \frac{\mathrm{d}m(x)}{(1 + |x|^2)^n} \ \leq \ M \ < \ \infty,$$

for a nondecreasing convex function $\Phi : [0, \infty] \to [0, \infty]$ such that

$$(28) \qquad \int_\delta^\infty \frac{\mathrm{d}\tau}{\tau \left[\Phi^{-1}(\tau)\right]^{\frac{1}{n-1}}} \ = \ \infty,$$

for some $\delta > \Phi(0)$.

COROLLARY 7.4. *In particular, the conclusion of Theorem 7.4 is valid if, for some $\alpha > 0$,*

$$(29) \qquad \int_D e^{\alpha Q^{\frac{1}{n-1}}(x)} \frac{\mathrm{d}m(x)}{(1 + |x|^2)^n} \ \leq \ M < \infty \ .$$

The same is true for any function $\Phi = e^\psi$ where $\psi(t)$ is a finite product of the function αt^β, $\alpha > 0$, $\beta \geq 1/(n - 1)$, and some of the functions $[\log(A_1 + t)]^{\alpha_1}$,

$[\log\log(A_2 + t)]^{\alpha_2}$, ..., $\alpha_m \geq -1$, $A_m \in \mathbb{R}$, $m \in \mathbb{N}$, $t \in [T, \infty]$, $\psi(t) \equiv \psi(T)$, $t \in [0, T]$ *with a large enough* $T \in (0, \infty)$.

Remark 7.1. Note that the condition (28) is not only sufficient but also necessary for the compactness of the classes $\mathfrak{R}_Q$ with integral constraints of the type (27) on Q, see Theorem 5.1 in [26], and (28) is equivalent to the following condition

$$(30) \qquad \int\limits_{\Delta}^{\infty} \log \Phi(t) \, \frac{\mathrm{d}t}{t^{n'}} = +\infty,$$

for all $\Delta > t_0$ where $t_0 := \sup\limits_{\Phi(t)=0} t$, $t_0 = 0$ if $\Phi(0) > 0$, and where $\frac{1}{n'} + \frac{1}{n} = 1$, i.e., $n' = 2$ for $n = 2$, n' is strictly decreasing in n and $n' = n/(n-1) \to 1$ as $n \to \infty$, see Remark 4.2 in [26].

Finally, all the notes in Remarks 4.2, 4.3 and 6.1 above are also related to the compactness results in this section.

These results will have, in particular, wide applications to the convergence and compactness theory for the Sobolev homeomorphisms as well as for the Orlicz–Sobolev homeomorphisms, cf. e.g. [17], [24] and [27], that will be published elsewhere.

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