Siberian Mathematical Journal, Vol. 52, *No.* 3, *pp.* 524–536, 2011

EQUICONTINUITY OF MEAN QUASICONFORMAL MAPPINGS

V. I. Ryazanov and E. A. Sevost′yanov

UDC 517.5

Abstract: We establish the equicontinuity and normality of the families $\mathfrak{R}^\Phi$ of ring $Q(x)$-homeomorphisms with integral-type restrictions $\int \Phi(Q(x))dm(x) < \infty$ on a domain $D \subset \mathbb{R}^n$ with $n \geq 2$. The resulting conditions on Φ are not only sufficient but also necessary for the equicontinuity and normality of these families of mappings. We give some applications of these results to the Sobolev classes $W^{1,n}_{\mathrm{loc}}$.

Keywords: equicontinuity, normal family, mean quasiconformal mapping, Sobolev class

1. Introduction

Throughout this article m stands for the Lebesgue measure in $\mathbb{R}^n$ with $n \geq 2$.

In the theory of mappings known as quasiconformal in the mean, conditions of the form

$$\int\limits_D \Phi(Q(x))\, dm(x) < \infty \tag{1.1}$$

are the standard for various characteristics Q of mappings (see [1–15] for instance). The study of classes with integral restrictions of this form is related to recent developments in the theory of degenerate Beltrami equations (see the monographs [16, 17] for instance, as well the surveys [18, 19]) and mappings with finite distortion (see Chapter VI of [20] and Section 8.4 of [17]).

This article is a natural continuation of [21]. Here we study some questions related to the equicontinuity and normality of the ring $Q(x)$-homeomorphisms satisfying (1.1) and give some applications to Sobolev classes, which include in particular the quasiconformal mappings whose geometric definition rests on the concept of modulus as well.

Recall that the *modulus* of a family of curves Γ is defined as

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm} \Gamma} \int\limits_D \rho^n(x)\, dm(x)$$

where Borel functions $\rho : \mathbb{R}^n \to [0, \infty]$ are admissible for Γ in D, which we express as $\rho \in \operatorname{adm} \Gamma$, provided that

$$\int\limits_\gamma \rho(x)\, |dx| \geq 1 \quad \forall\, \gamma \in \Gamma.$$

One of the several equivalent geometric definitions of a K-quasiconformal mapping f with $K \in [1, \infty)$ on a domain D in $\mathbb{R}^n$ with $n \geq 2$ reduces to the inequality

$$M(f\Gamma) \leq K\, M(\Gamma) \tag{1.2}$$

for an arbitrary family Γ of curves γ in D (see [22, Chapter II, Definition 13.1 and Theorem 34.3]). In other words, (1.2) means that the distortion of the outer measure M over the space of all curves in D is bounded under quasiconformal mappings.

Similarly, given a domain D in $\mathbb{R}^n$ with $n \geq 2$ and a Lebesgue measurable function $Q : D \to [1, \infty]$, refer to a homeomorphism $f : D \to \overline{\mathbb{R}^n}$, with $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, as a $Q(x)$-*homeomorphism* whenever

$$M(f\Gamma) \leq \int\limits_D Q(x) \cdot \rho^n(x)\, dm(x) \tag{1.3}$$

Donetsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 52, No. 3, pp. 665–679, May–June, 2011. Original article submitted April 13, 2010.

for every family Γ of curves γ in D and every $\rho \in \operatorname{adm}\Gamma$ (see [23, 24] for instance as well as [17, Section 4.1]).

In the case $Q(x) \le K$ almost everywhere we again arrive at (1.2). In the general case the latter means that we can estimate the modulus of the family $f\Gamma$ by the modulus of Γ with some weight $Q(x)$: $M(f\Gamma) \le M_Q(\Gamma)$ (see [25] for instance). The monograph of Miklyukov [26] discusses other classes of the mappings satisfying similar inequalities in terms of capacities. Originally an inequality of type (1.3) was established by Lehto and Virtanen for the quasiconformal mappings on the plane [27, Chapter V, Section 6.3, p. 221] and by Strugov for the spatial mappings quasiconformal in the mean [12]. In [28] an inequality of the form (1.3) is established for quasiconformal spatial mappings with $Q(x)$ equal to $K_I(x, f)$.

Recall that the *inner dilatation* of a mapping $f : D \to \mathbb{R}^n$, $n \ge 2$, at a point $x \in D$ where f is differentiable is

$$K_I(x, f) = \frac{|J(x, f)|}{l(f'(x))^n}$$

if $J(x, f) \ne 0$, $K_I(x, f) = 1$ if $f'(x) = 0$, and $K_I(x, f) = \infty$ at the remaining points, where $J(x, f)$ is the Jacobian of f at x, and

$$l(f'(x)) = \inf_{h \in \mathbb{R}^n \setminus \{0\}} \frac{|f'(x)h|}{|h|}.$$

The following concept generalizes and localizes the concept of a Q-homeomorphism. It is motivated by the ring definition of quasiconformal mappings in the sense of Gehring (see [29] for instance), introduced originally by Ryazanov, Srebro, and Yakubov on the plane, and later extended by Ryazanov and Sevost′yanov to the spatial case (see [21; 17, Chapters VII and XI] for instance). Given $E, F \subset \overline{\mathbb{R}^n}$, denote by $\Gamma(E, F, D)$ the family of all curves $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ connecting E and F in D; thus, $\gamma(a) \in E$, $\gamma(b) \in F$, and $\gamma(t) \in D$ for $t \in (a, b)$.

Given $x_0 \in D$ and a Lebesgue measurable function $Q : D \to [0, \infty]$, refer to a homeomorphism $f : D \to \overline{\mathbb{R}^n}$ as a *ring Q-homeomorphism at $x_0 \in D$* whenever f satisfies

$$M(f(\Gamma(S_1, S_2, R))) \le \int\limits_R Q(x)\eta^n(|x - x_0|)\, dm(x) \tag{1.4}$$

for every ring $R = R(r_1, r_2, x_0) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$, $S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n : |x - x_0| = r_i\}$, where $0 < r_1 < r_2 < r_0 := \operatorname{dist}(x_0, \partial D)$, and every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ with

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \ge 1.$$

Furthermore, f is called a *ring Q-homeomorphism in D* if f is a ring Q-homeomorphism at every point $x_0 \in D$. Observe that, in particular, the homeomorphisms $f : D \to \overline{\mathbb{R}^n}$ of class $W^{1,n}_{\mathrm{loc}}$ for $K_I(x, f) \in L^1_{\mathrm{loc}}$ are ring Q-homeomorphisms as well as Q-homeomorphisms with $Q(x) := K_I(x, f)$ (see Theorems 8.1 and 8.6 in [17] for instance, as well as Theorem 6.10 and Corollary 4.9 in [30]).

The concept of a ring Q-homeomorphism extends naturally to the case $x_0 = \infty$. Namely, for $\infty \in D \subseteq \overline{\mathbb{R}^n}$ a homeomorphism $f : D \to \overline{\mathbb{R}^n}$ is called a *ring Q-homeomorphism at* ∞ whenever the mapping $\widetilde{f} = f\big(\frac{x}{|x|^2}\big)$ is a ring Q'-homeomorphism at zero for $Q'(x) = Q\big(\frac{x}{|x|^2}\big)$. In other words, $f : \mathbb{R}^n \to \overline{\mathbb{R}^n}$ is a ring Q-homeomorphism at ∞ if and only if

$$M(f(\Gamma(S(R_1), S(R_2), R))) \le \int\limits_R Q(y)\eta^n(|y|)\, dm(y)$$

for every ring $R = R(R_1, R_2, 0) = \{y \in \mathbb{R}^n : R_1 < |y| < R_2\}$ in D with $0 < R_1 < R_2 < \infty$ and $S(R_i) = \{x \in \mathbb{R}^n : |x| = R_i\}$, and every measurable function $\eta : (R_1, R_2) \to [0, \infty]$ with $\int_{R_1}^{R_2} \eta(r)\, dr \ge 1$.

2. Preliminaries

Consider two metric spaces (X, d) and (X', d') with distances d and d'. A family $\mathfrak{F}$ of continuous mappings $f : X \to X'$ is called *normal* if from every sequence $f_m \in \mathfrak{F}$ we can select a subsequence f_{m_k} converging locally uniformly in X to a continuous mapping $f : X \to X'$. This concept is rather close to the following: A family $\mathfrak{F}$ of mappings $f : X \to X'$ is called *equicontinuous* at $x_0 \in X$ whenever given $\varepsilon > 0$ there is $\delta > 0$ such that $d'(f(x), f(x_0)) < \varepsilon$ for all x with $d(x, x_0) < \delta$ and all $f \in \mathfrak{F}$. Refer to $\mathfrak{F}$ as *equicontinuous* whenever $\mathfrak{F}$ is equicontinuous at every point of X.

The following version of the Arzelà–Ascoli theorem is useful below (see Section 20.4 in [22] for instance).

Proposition 2.1. *Given a separable metric space (X, d) and a compact metric space (X', d'), some family $\mathfrak{F}$ of mappings $f : X \to X'$ is normal if and only if $\mathfrak{F}$ is equicontinuous.*

In particular, Proposition 2.1 holds in the case $X = \mathbb{R}^n$ with the usual distance and X' is the one-point compactification $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ with the spherical metric.

Recall that the spherical (chordal) metric $h(x, y)$ equals $|\pi(x) - \pi(y)|$, where π is the stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n\left(\frac{1}{2}e_{n+1}, \frac{1}{2}\right)$ in $\mathbb{R}^{n+1}$; explicitly,

$$h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}, \quad h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}, \quad x \neq \infty \neq y.$$

Refer as the *spherical diameter* of a set E in $\overline{\mathbb{R}^n}$ to

$$h(E) = \sup_{x_1, x_2 \in E} h(x_1, x_2).$$

Denote by $\mathfrak{R}_{Q,\Delta}(D)$ the class of all ring Q-homeomorphisms f in a domain $D \subseteq \mathbb{R}^n$ with $n \geq 2$ satisfying $h(\overline{\mathbb{R}^n} \backslash f(D)) \geq \Delta > 0$. The following estimate for the distortion of spherical distances under ring Q-homeomorphisms appears in [21] (also see [17, Theorem 7.3]).

Proposition 2.2. *Given $\Delta > 0$ and a measurable function $Q : D \to [0, \infty]$, we have*

$$h(f(x), f(x_0)) \leq \frac{\alpha_n}{\Delta} \exp\left\{ - \int\limits_{|x - x_0|}^{\varepsilon(x_0)} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)} \right\} \tag{2.1}$$

for all $f \in \mathfrak{R}_{Q,\Delta}(D)$ and $x \in B(x_0, \varepsilon(x_0))$ with $\varepsilon(x_0) < \operatorname{dist}(x_0, \partial D)$, where $\alpha_n > 0$ depends only on n and $q_{x_0}(r)$ is the mean integral value of $Q(z)$ on the sphere $|z - x_0| = r$.

The *inverse function* Φ^{-1} is well defined for every nondecreasing function $\Phi : [0, \infty] \to [0, \infty]$:

$$\Phi^{-1}(\tau) = \inf_{\Phi(t) \geq \tau} t. \tag{2.2}$$

As usual, inf in (2.2) is equal to ∞ if the set of $t \in [0, \infty]$ with $\Phi(t) \geq \tau$ is empty. Observe that Φ^{-1} is also nondecreasing.

Remark 2.1. It is obvious from the definition that

$$\Phi^{-1}(\Phi(t)) \leq t \quad \forall\, t \in [0, \infty], \tag{2.3}$$

with equality holding except on the intervals where $\Phi(t)$ is constant.

Since for every positive p the mapping $t \mapsto t^p$ is an orientation-preserving homeomorphism of $[0, \infty]$ onto itself, we can recast Theorem 2.1 of [31] into the following form, more convenient for subsequent applications. Here, in (2.5) and (2.6), we extend the definitions of the integrals ∞ as $\Phi_p(t) = \infty$, respectively $H_p(t) = \infty$ for all $t \geq T \in [0, \infty)$. The integral in (2.6) is understood in the sense of Lebesgue–Stieltjes; while the integrals in (2.5) and (2.7)–(2.10), in the usual sense of Lebesgue.

Proposition 2.3. *Given a nondecreasing function* $\Phi : [0,\infty] \to [0,\infty]$, *put*

$$H_p(t) = \log \Phi_p(t), \quad \Phi_p(t) = \Phi(t^p), \quad p \in (0,\infty). \tag{2.4}$$

Then

$$\int_\delta^\infty H_p'(t)\frac{dt}{t} = \infty \tag{2.5}$$

yields

$$\int_\delta^\infty \frac{dH_p(t)}{t} = \infty, \tag{2.6}$$

and (2.6) *is equivalent to*

$$\int_\delta^\infty H_p(t)\frac{dt}{t^2} = \infty \tag{2.7}$$

for some $\delta > 0$. *Equality* (2.7) *is equivalent to each of the equalities:*

$$\int_0^\Delta H_p\left(\frac{1}{t}\right) dt = \infty \tag{2.8}$$

for some $\delta > 0$,

$$\int_{\delta_*}^\infty \frac{d\eta}{H_p^{-1}(\eta)} = \infty \tag{2.9}$$

for some $\delta_* > H(+0)$,

$$\int_{\delta_*}^\infty \frac{d\tau}{\tau \Phi_p^{-1}(\tau)} = \infty \tag{2.10}$$

for some $\delta_* > \Phi(+0)$.

Moreover, (2.5) is equivalent to (2.6), and consequently (2.5)–(2.10) are equivalent to each other under the additional assumption that Φ is absolutely continuous. In particular, all conditions (2.5)–(2.10) are equivalent to each other when Φ is a nondecreasing convex function.

It is easy to see that conditions (2.5)–(2.10) are weaker for large p (see (2.7) for instance). We should give one more explanation. We presume that the right-hand sides of (2.5)–(2.10) are $+\infty$. For $\Phi_p(t) = 0$ for $t \in [0,t_*]$, while $H_p(t) = -\infty$ for $t \in [0,t_*]$, and we put $H_p'(t) := 0$ for $t \in [0,t_*]$. Observe that (2.6) and (2.7) exclude the case that t_* belongs to the interval of integration in the relations mentioned above. Otherwise, the left-hand sides of (2.6) and (2.7) are simultaneously either equal to $-\infty$ or undefined. Consequently, we may assume in (2.5)–(2.8) that $\delta > t_0$, and accordingly $\Delta < 1/t_0$, where $t_0 := \sup_{\Phi_p(t)=0} t$, and $t_0 = 0$ if $\Phi_p(0) > 0$.

3. The Main Lemma and Its Corollaries

Recall that a function $\Phi : [0,\infty] \to [0,\infty]$ is called *convex* whenever

$$\Phi(\lambda t_1 + (1-\lambda)t_2) \le \lambda\Phi(t_1) + (1-\lambda)\Phi(t_2)$$

for all t_1, $t_2 \in [0,\infty]$ and $\lambda \in [0,1]$.

Henceforth $\mathbb{R}^n(\varepsilon)$, with $\varepsilon \in (0,1)$, stands for the spherical ring

$$\mathbb{R}^n(\varepsilon) = \{x \in \mathbb{R}^n : \varepsilon < |x| < 1\} \tag{3.1}$$

in $\mathbb{R}^n$ with $n \ge 2$. The following statement generalizes and strengthens Lemma 3.1 of [31].

Lemma 3.1. *Take a measurable function $Q : \mathbb{B}^n \to [0,\infty]$ and a nondecreasing convex function $\Phi : [0,\infty] \to (0,\infty]$. Suppose that the mean integral value $M(\varepsilon)$ of $\Phi \circ Q$ on the ring $\mathbb{R}^n(\varepsilon)$ is finite. Then*

$$\int\limits_{\varepsilon}^{1} \frac{dr}{rq^{\frac{1}{p}}(r)} \geq \frac{1}{n} \int\limits_{eM(\varepsilon)}^{\frac{M(\varepsilon)}{\varepsilon^n}} \frac{d\tau}{\tau[\Phi^{-1}(\tau)]^{\frac{1}{p}}} \quad \forall\, p \in (0,\infty),\ \varepsilon \in (0,1), \tag{3.2}$$

where $q(r)$ is the mean integral value of $Q(x)$ on the sphere $|x| = r$.

Remark 3.1. Observe that for every $p \in (0,\infty)$ the relation in (3.2) is equivalent to

$$\int\limits_{\varepsilon}^{1} \frac{dr}{rq^{\frac{1}{p}}(r)} \geq \frac{1}{n} \int\limits_{eM(\varepsilon)}^{\frac{M(\varepsilon)}{\varepsilon^n}} \frac{d\tau}{\tau\Phi_p^{-1}(\tau)}, \quad \Phi_p(t) := \Phi(t^p). \tag{3.3}$$

Proof of Lemma 3.1. Put $t_* = \sup_{\Phi_p(t)=\tau_0} t$ and $\tau_0 = \Phi(0) > 0$. Letting $H_p(t) := \log \Phi_p(t)$, we see that $H_p^{-1}(\eta) = \Phi_p^{-1}(e^\eta)$ and $\Phi_p^{-1}(\tau) = H_p^{-1}(\log\tau)$ (see Lemma 2.1 in [31]). Consequently,

$$q^{\frac{1}{p}}(r) = H_p^{-1}\left(\log\frac{h(r)}{r^n}\right) = H_p^{-1}\left(n\log\frac{1}{r} + \log h(r)\right) \quad \forall\, r \in R_*, \tag{3.4}$$

where $h(r) := r^n\Phi(q(r)) = r^n\Phi_p(q^{\frac{1}{p}}(r))$ and $R_* = \{r \in (\varepsilon,1) : q^{\frac{1}{p}}(r) > t_*\}$. Then

$$q^{\frac{1}{p}}(e^{-s}) = H_p^{-1}(ns + \log h(e^{-s})) \quad \forall\, s \in S_* \tag{3.5}$$

where $S_* = \{s \in (0,\log\frac{1}{\varepsilon}) : q^{\frac{1}{p}}(e^{-s}) > t_*\}$.

Since Φ is convex, the Jensen inequality yields

$$\int\limits_{0}^{\log\frac{1}{\varepsilon}} h(e^{-s})ds = \int\limits_{\varepsilon}^{1} h(r)\frac{dr}{r} = \int\limits_{\varepsilon}^{1} \Phi(q(r))r^{n-1}dr$$

$$\leq \int\limits_{\varepsilon}^{1} \Bigg(\;\; -\!\!\!\!\!\!\int\limits_{S(r)} \Phi(Q(x))d\mathscr{A}\Bigg) r^{n-1}dr \leq \frac{\Omega_n}{\omega_{n-1}}M(\varepsilon) = \frac{1}{n}M(\varepsilon), \tag{3.6}$$

where we use the mean value of $\Phi \circ Q$ on the sphere $S(r) = \{x \in \mathbb{R}^n : |x| = r\}$ with respect to the area measure. As usual, Ω_n and ω_{n-1} here are the volume of the unit ball and the area of the unit sphere in $\mathbb{R}^n$. Arguing by contradiction, we can easily see that

$$|T| = \int\limits_{T} ds \leq \frac{1}{n}, \tag{3.7}$$

where $T = \big\{s \in (0,\log\frac{1}{\varepsilon}) : h(e^{-s}) > M(\varepsilon)\big\}$. At the next step we verify that

$$q^{\frac{1}{p}}(e^{-s}) \leq H_p^{-1}(ns + \log M(\varepsilon)) \quad \forall\, s \in (0,\log(1/\varepsilon)) \setminus T_*, \tag{3.8}$$

where $T_* := T \cap S_*$. Observe that

$$\left(0,\log\frac{1}{\varepsilon}\right) \setminus T_* = \left[\left(0,\log\frac{1}{\varepsilon}\right) \setminus S_*\right] \cup \left[\left(0,\log\frac{1}{\varepsilon}\right) \setminus T\right] = \left[\left(0,\log\frac{1}{\varepsilon}\right) \setminus S_*\right] \cup [S_* \setminus T].$$

By (3.5), we have (3.8) for $s \in S_* \setminus T$ since H_p^{-1} is a nondecreasing function. Observe also that

$$e^{ns}M(\varepsilon) > \Phi(0) = \tau_0 \quad \forall\, s \in (0, \log(1/\varepsilon)), \tag{3.9}$$

as well as

$$t_* < \Phi_p^{-1}(e^{ns}M(\varepsilon)) = H_p^{-1}(ns + \log M(\varepsilon)) \quad \forall\, s \in (0, \log(1/\varepsilon)). \tag{3.10}$$

Consequently, (3.8) also holds for $s \in (0, \log \frac{1}{\varepsilon}) \setminus S_*$.

Since H_p^{-1} is nondecreasing, (3.7) and (3.8) yield

$$\begin{aligned} \int\limits_{\varepsilon}^{1} \frac{dr}{rq^{\frac{1}{p}}(r)} &= \int\limits_{0}^{\log\frac{1}{\varepsilon}} \frac{ds}{q^{\frac{1}{p}}(e^{-s})} \geq \int\limits_{(0,\log\frac{1}{\varepsilon})\setminus T_*} \frac{ds}{H_p^{-1}(ns+\Delta)} \\ &\geq \int\limits_{|T_*|}^{\log\frac{1}{\varepsilon}} \frac{ds}{H_p^{-1}(ns+\Delta)} \geq \int\limits_{\frac{1}{n}}^{\log\frac{1}{\varepsilon}} \frac{ds}{H_p^{-1}(ns+\Delta)} = \frac{1}{n} \int\limits_{1+\Delta}^{n\log\frac{1}{\varepsilon}+\Delta} \frac{d\eta}{H_p^{-1}(\eta)}, \end{aligned} \tag{3.11}$$

where $\Delta = \log M(\varepsilon)$. Observe that $1 + \Delta = \log eM(\varepsilon)$. Therefore,

$$\int\limits_{\varepsilon}^{1} \frac{dr}{rq^{\frac{1}{p}}(r)} \geq \frac{1}{n} \int\limits_{\log eM(\varepsilon)}^{\log\frac{M(\varepsilon)}{\varepsilon^n}} \frac{d\eta}{H_p^{-1}(\eta)}, \tag{3.12}$$

and upon changing the variable to $\eta = \log \tau$ we obtain (3.3), and so (3.2) as well. $\square$

Corollary 3.1. *Take a nondecreasing convex function $\Phi : [0, \infty] \to (0, \infty]$ and a Lebesgue measurable function $Q : \mathbb{B}^n \to [0, \infty]$. Put $Q_*(x) = 1$ for $Q(x) < 1$ and $Q_*(x) = Q(x)$ for $Q(x) \geq 1$. Assume that the mean value $M_*(\varepsilon)$ of the function $\Phi \circ Q_*$ on the ring $\mathbb{R}^n(\varepsilon)$, $\varepsilon \in (0, 1)$, is finite. Then*

$$\int\limits_{\varepsilon}^{1} \frac{dr}{rq^{\frac{\lambda}{p}}(r)} \geq \frac{1}{n} \int\limits_{eM_*(\varepsilon)}^{\frac{M_*(\varepsilon)}{\varepsilon^n}} \frac{d\tau}{\tau[\Phi^{-1}(\tau)]^{\frac{1}{p}}} \quad \forall\, \lambda \in (0, 1),\ p \in (0, \infty), \tag{3.13}$$

where $q(r)$ is the mean integral value of $Q(x)$ on the sphere $|x| = r$.

Indeed, denote by $q_*(r)$ the mean integral value of $Q_*(x)$ on the sphere $|x| = r$. Then $q(r) \leq q_*(r)$, and in addition $q_*(r) \geq 1$ for all $r \in (0, 1)$. Therefore, $q^{\frac{\lambda}{p}}(r) \leq q_*^{\frac{\lambda}{p}}(r) \leq q_*^{\frac{1}{p}}(r)$ for all $\lambda \in (0, 1)$, and Lemma 3.1 applied to $Q_*(x)$ yields (3.13).

Theorem 3.1. *Take a measurable function $Q : \mathbb{B}^n \to [0, \infty]$ with*

$$\int\limits_{\mathbb{B}^n} \Phi(Q(x))dm(x) < \infty, \tag{3.14}$$

where $\Phi : [0, \infty] \to [0, \infty]$ is a nondecreasing convex function satisfying

$$\int\limits_{\delta_0}^{\infty} \frac{d\tau}{\tau[\Phi^{-1}(\tau)]^{\frac{1}{p}}} = \infty, \quad p \in (0, \infty), \tag{3.15}$$

for some $\delta_0 > \tau_0 := \Phi(0)$. Then

$$\int\limits_0^1 \frac{dr}{rq^{\frac{1}{p}}(r)} = \infty, \tag{3.16}$$

where $q(r)$ is the mean integral value of $Q(x)$ on the sphere $|x| = r$.

PROOF. If $\Phi(0) \neq 0$ then Theorem 3.1 is a direct corollary of Lemma 3.1. In the case $\Phi(0) = 0$ fix some $\delta \in (0, \delta_0)$ and put $\Phi_*(t) = \Phi(t)$ if $\Phi(t) > \delta$ and $\Phi_*(t) = \delta$ if $\Phi(t) \leq \delta$. Then (3.14) implies that $\int_{\mathbb{B}^n} \Phi_*(Q(x))dm(x) < \infty$ since $|\Phi_*(t) - \Phi(t)| \leq \delta$, while the measure of $\mathbb{B}^n$ is finite. In addition, $\Phi_*(\tau) = \Phi^{-1}(\tau)$ for $\tau \geq \delta$, and then (3.15) yields

$$\int\limits_{\delta_0}^{\infty} \frac{d\tau}{\tau[\Phi_*^{-1}(\tau)]^{\frac{1}{p}}} = \infty.$$

Therefore, (3.16) holds by Lemma 3.1. $\Box$

REMARK 3.2. Since $[\Phi^{-1}(\tau)]^{\frac{1}{p}} = \Phi_p^{-1}(\tau)$, where $\Phi_p(t) = \Phi(t^p)$, it follows from (3.15) that

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau\Phi_p^{-1}(\tau)} = \infty \quad \forall\, \delta \in [0, \infty). \tag{3.17}$$

On the other hand, the relation of the form (3.17), fulfilled for some $\delta \in [0, \infty)$, in general fails to imply (3.15). Indeed, (3.15) obviously implies (3.17) for $\delta \in [0, \delta_0)$, while for $\delta \in (\delta_0, \infty)$ we have

$$0 \leq \int\limits_{\delta_0}^{\delta} \frac{d\tau}{\tau\Phi_p^{-1}(\tau)} \leq \frac{1}{\Phi_p^{-1}(\delta_0)} \log\frac{\delta}{\delta_0} < \infty \tag{3.18}$$

since Φ_p^{-1} is a nondecreasing function, and $\Phi_p^{-1}(\delta_0) > 0$. In addition, by the definition of the inverse function, $\Phi_p^{-1}(\tau) \equiv 0$ for all $\tau \in [0, \tau_0]$ with $\tau_0 = \Phi_p(0)$. Consequently, (3.17) for $\delta \in [0, \tau_0)$ in general fails to imply (3.15). If $\tau_0 > 0$ then

$$\int\limits_{\delta}^{\tau_0} \frac{d\tau}{\tau\Phi_p^{-1}(\tau)} = \infty \quad \forall\, \delta \in [0, \tau_0). \tag{3.19}$$

But (3.19) carries no information exactly about the function $Q(x)$, and so (3.16) cannot follow from (3.17) for $\delta < \Phi(0)$.

By analogy to Corollary 3.2 of [31] we have

Corollary 3.2. *If $\Phi : [0, \infty] \to [0, \infty]$ is a nondecreasing convex function, while Q satisfies* (3.14), *then each of the conditions* (2.5)–(2.10) *for $p \in (0, \infty)$ implies* (3.16).

If, in addition, $\Phi(1) < \infty$ or $q(r) \geq 1$ on a subset of the interval $(0, 1)$ of positive measure, each of the conditions (2.5)–(2.10) *for $p \in (0, \infty)$ implies that*

$$\int\limits_0^1 \frac{dr}{rq^{\frac{\lambda}{p}}(r)} = \infty \quad \forall\, \lambda \in (0, 1), \tag{3.20}$$

as well as

$$\int\limits_0^1 \frac{dr}{r^{\alpha}q^{\frac{\beta}{p}}(r)} = \infty \quad \forall\, \alpha \geq 1,\ \beta \in (0, \alpha]. \tag{3.21}$$

4. Sufficient Conditions for Equicontinuity

Henceforth D is a fixed domain in the compactification $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ with $n \geq 2$. Given a function $\Phi : [0, \infty] \to [0, \infty]$ and two numbers $M > 0$ and $\Delta > 0$, denote by $\mathfrak{R}^{\Phi}_{M,\Delta}$ the family of all ring $Q(x)$-homeomorphisms in D with $h(\overline{\mathbb{R}^n} \setminus f(D)) \geq \Delta$ and

$$\int\limits_D \Phi(Q(x)) \frac{dm(x)}{(1+|x|^2)^n} \leq M. \tag{4.1}$$

Sometimes we may use the notation $\mathfrak{R}^{\Phi}_{M,\Delta}(D)$ indicating the domain D explicitly.

Theorem 4.1. *Take a nondecreasing convex function $\Phi : [0, \infty] \to [0, \infty]$. If*

$$\int\limits_{\delta_0}^{\infty} \frac{d\tau}{\tau[\Phi^{-1}(\tau)]^{\frac{1}{n-1}}} = \infty \tag{4.2}$$

for some $\delta_0 > \tau_0 := \Phi(0)$ then the class $\mathfrak{R}^{\Phi}_{M,\Delta}$ is equicontinuous, and consequently it constitutes a normal family of mappings for all $M \in (0, \infty)$ and $\Delta \in (0, 1)$.

Remark 4.1. The condition

$$\int\limits_D \Phi(Q(x)) dm(x) \leq M \tag{4.3}$$

implies (4.1). Consequently, (4.1) is more general than (4.3), while the corresponding class of ring Q-homeomorphisms satisfying (4.3) amounts to a subclass of the family $\mathfrak{R}^{\Phi}_{M,\Delta}$. On the other hand, if D is bounded then (4.1) implies that

$$\int\limits_D \Phi(Q(x))\, dm(x) \leq M_*, \tag{4.4}$$

where $M_* = M \cdot (1 + \delta_*^2)^n$ and $\delta_* = \sup_{x \in D} |x|$.

Proof of Theorem 4.1. By Proposition 2.1, it suffices to verify that the family $\mathfrak{R}^{\Phi}_{M,\Delta}$ is equicontinuous at every point $x_0 \in D$. If $x_0 \neq \infty$ then Proposition 2.2 yields

$$h(f(x), f(x_0)) \leq \frac{\alpha_n}{\Delta} \exp\left\{ - \int\limits_{|x-x_0|}^{\rho} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)} \right\} \tag{4.5}$$

for all $x \in B(x_0, \rho)$ and every positive $\rho = \rho(x_0) < \operatorname{dist}(x_0, \partial D)$, where $q_{x_0}(r)$ stands for the mean value of $Q(x)$ on the sphere $|z - x_0| = r$, and the constant α_n depends only on n. Insert $y = (x - x_0)/\rho$ into the integral on the right-hand side of (4.5), then use Lemma 3.1 to estimate it as

$$\int\limits_{|x-x_0|}^{\rho} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)} = \int\limits_{\varepsilon}^{1} \frac{dr}{r q^{\frac{1}{n-1}}(r)} \geq \frac{1}{n} \int\limits_{eM(\varepsilon)}^{\frac{M(\varepsilon)}{\varepsilon^n}} \frac{d\tau}{\tau[\Phi^{-1}(\tau)]^{\frac{1}{n-1}}},$$

where $\varepsilon = |x - x_0|/\rho$, $q(r) = q_{x_0}(\rho r)$, and

$$M(\varepsilon) = ⨍_R \Phi(Q(z)) dm(z) = \frac{1}{\Omega_n \rho^n (1 - \varepsilon^n)} \int\limits_R \Phi(Q(z)) dm(z),$$

where $R = \{z \in \mathbb{R}^n : |x - x_0| < |z - x_0| < \rho\}$ is the ring centered at x_0, and Ω_n is the volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. Since

$$|z| \le |z - x_0| + |x_0| \le \rho(x_0) + |x_0|,$$

we deduce that

$$M(\varepsilon) \le \frac{\beta_n(x_0)}{\Omega_n(1-\varepsilon^n)} \int\limits_R \Phi(Q(z)) \frac{dm(z)}{(1+|z|^2)^n},$$

where

$$\beta_n(x_0) = (1 + (\rho(x_0) + |x_0|)^2)^n / \rho^n(x_0).$$

Consequently, for $\varepsilon \le 1/\sqrt[n]{2}$, and in particular for $\varepsilon \le 1/2$, we have

$$\Phi(0) \le M(\varepsilon) \le \frac{2\beta_n(x_0)}{\Omega_n} M.$$

Therefore, for all x satisfying $|x - x_0| < \rho(x_0)/2$,

$$h(f(x), f(x_0)) \le \frac{\alpha_n}{\Delta} \exp\left\{ -\frac{1}{n} \int\limits_{\lambda_n \beta_n(x_0) M}^{\frac{\Phi(0)\rho^n(x_0)}{|x-x_0|^n}} \frac{d\tau}{\tau[\Phi^{-1}(r)]^{\frac{1}{n-1}}} \right\}, \tag{4.6}$$

where the constant $\lambda_n = 2e/\Omega_n$ depends only on n. Consequently, the family $\mathfrak{R}^{\Phi}_{M,\Delta}$ is equicontinuous at x_0. Finally, the case $x_0 = \infty$ reduces to the case $x_0 = 0$ by inversion in the sphere $|x| = 1$. $\square$

Corollary 4.1. *Each of conditions* (2.5)–(2.10) *for* $p \in (0, n-1]$ *implies the equicontinuity and normality of the class* $\mathfrak{R}^{\Phi}_{M,\Delta}$ *for all* $M \in (0, \infty)$ *and* $\Delta \in (0, 1)$.

Given a function $\Phi : [0, \infty] \to [0, \infty]$ and numbers $M > 0$, $\Delta > 0$, denote by $S^{\Phi}_{M,\Delta}$ the family of all homeomorphisms f of the domain D of the Sobolev class $W^{1,n}_{\mathrm{loc}}$ possessing a locally integrable inner dilatation $K_I(x, f)$ as well as satisfying $h(\overline{\mathbb{R}^n} \setminus f(D)) \ge \Delta$ and (4.1) with $Q(x) := K_I(x, f)$. Observe that if Φ is a convex function which is nondecreasing and nonconstant on $[0, \infty)$ then (4.1) automatically implies that $K_I(x, f) \in L^1_{\mathrm{loc}}$. Observe also that $S^{\Phi}_{M,\Delta} \subset \mathfrak{R}^{\Phi}_{M,\Delta}$ (see Theorem 6.10 and Corollary 4.9 in [30] for instance). Therefore, this yields a corollary of Theorem 4.1.

Corollary 4.2. *Each of the conditions* (2.5)–(2.10) *for* $p \in (0, n-1]$ *implies the equicontinuity and normality of the family* $S^{\Phi}_{M,\Delta}$ *for all* $M \in (0, \infty)$ *and* $\Delta \in (0, 1)$.

Remark 4.2. For $p = n - 1$ the conditions of type (2.5)–(2.10) are the weakest of those asserting the equicontinuity and normality of the classes $S^{\Phi}_{M,\Delta}$ and $\mathfrak{R}^{\Phi}_{M,\Delta}$ (see Theorem 5.1 below). The most interesting of these conditions is (2.7), which we can rearrange as

$$\int\limits_{\delta}^{\infty} \log \Phi(t) \frac{dt}{t^{n'}} = \infty, \tag{4.7}$$

where $\frac{1}{n'} + \frac{1}{n} = 1$; thus, $n' = 2$ for $n = 2$, n' is strictly increasing with respect to n, and $n' = n/(n-1) \to 1$ as $n \to \infty$. Observe also that we can rearrange (4.2), as well as (5.1) below, as

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}_{n-1}(\tau)} = \infty, \quad \Phi_{n-1}(t) := \Phi(t^{n-1}). \tag{4.8}$$

5. Necessary Conditions for Equicontinuity

Theorem 5.1. *Assume that the classes of mappings $S^{\Phi}_{M,\Delta} \subset \mathfrak{R}^{\Phi}_{M,\Delta}$ are equicontinuous (normal) for a nondecreasing convex function $\Phi : [0,\infty] \to [0,\infty]$ and all $M \in (0,\infty)$ and $\Delta \in (0,1)$. Then*

$$\int\limits_{\delta_*}^{\infty} \frac{d\tau}{\tau[\Phi^{-1}(\tau)]^{\frac{1}{n-1}}} = \infty \tag{5.1}$$

for all $\delta_* \in (\tau_0, \infty)$, where $\tau_0 := \Phi(0)$.

It is clear that $\Phi(t)$ in Theorem 5.1 cannot be constant since otherwise no restrictions on K_I in the theorem arise, with the exception of the condition $\Phi(t) \equiv \infty$ when the class $S^{\Phi}_{M,\Delta}$ is empty. Moreover, by the convexity criterion (see [32, I.4.3, Proposition 5] for instance), the slope $[\Phi(t) - \Phi(0)]/t$ is a nondecreasing function. Therefore, the proof of Theorem 5.1 reduces to the following claim.

Lemma 5.1. *Take a nondecreasing function $\Phi : [0,\infty] \to [0,\infty]$ satisfying*

$$\Phi(t) \geq C \cdot t^{\frac{1}{n-1}} \quad \forall\, t \in [T, \infty] \tag{5.2}$$

for some $C > 0$ and $T \in (0,\infty)$. If the classes $S^{\Phi}_{M,\Delta} \subset \mathfrak{R}^{\Phi}_{M,\Delta}$ are equicontinuous (normal) for all $M \in (0,\infty)$ and $\Delta \in (0,1)$ then (5.1) holds for all $\delta_* \in (\tau_0, \infty)$, where $\tau_0 := \Phi(+0)$.

Remark 5.1. It is well known that the critical exponent $n-1$ plays a key role in many problems about spatial mappings. Rearrange (5.2) as

$$\Phi_{n-1}(t) \geq C \cdot t \quad \forall\, t \in [T, \infty], \tag{5.3}$$

where $\Phi_{n-1}(t) = \Phi(t^{n-1})$, $C > 0$, and $T \in (0,\infty)$, which emphasizes once more the importance of the function Φ_{n-1} in this context. In fact, in Theorem 5.1 it suffices to impose a weaker convexity condition on Φ_{n-1} instead of Φ.

Proof of Lemma 5.1. Assume that (5.1) fails:

$$\int\limits_{\delta_0}^{\infty} \frac{d\tau}{\tau \Phi^{-1}_{n-1}(\tau)} < \infty \tag{5.4}$$

for some $\delta_0 \in (\tau_0, \infty)$, where $\Phi_{n-1}(t) := \Phi(t^{n-1})$. Then also

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}_{n-1}(\tau)} < \infty \quad \forall\, \delta \in (\tau_0, \infty) \tag{5.5}$$

since $\Phi^{-1}(\tau) > 0$ for all $\tau > \tau_0$, and the function $\Phi^{-1}(\tau)$ is nondecreasing. Observe that by (5.2)

$$\Phi_{n-1}(t) \geq Ct \quad \forall\, t \geq T \tag{5.6}$$

for some $C > 0$ and $T \in (1,\infty)$. Moreover, applying the linear transformation $\alpha\Phi + \beta$, where $\alpha = 1/C$ and $\beta = T$ (see (2.7) for instance), we may assume that

$$\Phi_{n-1}(t) \geq t \quad \forall\, t \in [0, \infty). \tag{5.7}$$

Certainly, we can also assume that $\Phi(t) = t$ for all $t \in [0,1)$ since the values of Φ on the half-open interval $[0,1)$ carry no information about $K_I(x,f) \geq 1$ in (4.1). It is clear that (5.5) implies that $\Phi(t) < \infty$ for all $t < \infty$ (see (2.7), as well as (2.10)).

Observe now that $\Psi(t) := t\Phi_{n-1}(t)$ is a strictly increasing function, $\Psi(1) = \Phi(1)$, and $\Psi(t) \to \infty$ as $t \to \infty$. Therefore, the functional equation

$$\Psi(K(r)) = \left(\frac{\gamma}{r}\right)^2 \quad \forall\, r \in (0, 1], \tag{5.8}$$

where $\gamma = \Phi^{1/2}(1) \geq 1$, is solvable for $K(1) = 1$ and a strictly increasing continuous function $K(r)$ satisfying $K(r) < \infty$ for $r \in (0, 1]$ and $K(r) \to \infty$ as $r \to 0$. The logarithm of (5.8) yields

$$\log K(r) + \log \Phi_{n-1}(K(r)) = 2 \log \frac{\gamma}{r},$$

and by (5.7) we deduce that

$$\log K(r) \leq \log \frac{\gamma}{r};$$

thus,

$$K(r) \leq \gamma/r. \tag{5.9}$$

Then (5.8) yields $\Phi_{n-1}(K(r)) \geq \gamma/r$, and (2.3) yields

$$K(r) \geq \Phi_{n-1}^{-1}(\gamma/r). \tag{5.10}$$

It suffices to consider the case $D = \mathbb{B}^n$. Define the mappings

$$f(x) = \frac{x}{|x|}\rho(|x|), \quad f_m(x) = \frac{x}{|x|}\rho_m(|x|), \quad m = 1, 2, \ldots,$$

on the punctured unit ball $\mathbb{B}^n \setminus \{0\}$, where

$$\rho(t) = \exp\{I(0) - I(t)\}, \quad \rho_m(t) = \exp\{I(0) - I_m(t)\}, \quad I(t) = \int\limits_t^1 \frac{dr}{rK(r)}, \quad I_m(t) = \int\limits_t^1 \frac{dr}{rK_m(r)},$$

and

$$K_m(r) = \begin{cases} K(r), & \text{if } r \geq \frac{1}{m}, \\ K(\frac{1}{m}), & \text{if } r \in \left(0, \frac{1}{m}\right). \end{cases}$$

From (5.10) we obtain

$$I(0) - I(t) = \int\limits_0^t \frac{dr}{rK(r)} \leq \int\limits_0^t \frac{dr}{r\Phi_{n-1}^{-1}\left(\frac{\gamma}{r}\right)} = \int\limits_{\frac{\gamma}{t}}^{\infty} \frac{d\tau}{\tau\Phi_{n-1}^{-1}(\tau)} \quad \forall\, t \in (0, 1],$$

where $\gamma/t \geq \gamma \geq 1 > \Phi(0) = 0$. Therefore, (5.5) yields

$$I(0) - I(t) \leq I(0) = \int\limits_0^1 \frac{dr}{rK(r)} < \infty \quad \forall\, t \in (0, 1]. \tag{5.11}$$

In addition, $f_m, f \in C^1(\mathbb{B}^n \setminus \{0\})$ since $K_m(r)$ and $K(r)$ are continuous and, therefore, locally quasiconformal in $\mathbb{B}^n \setminus \{0\}$. Moreover, f_m is K_m-quasiconformal in $\mathbb{B}^n$, where $K_m = K(1/m)$. Now the tangential and radial dilatations of f on the sphere $|x| = \rho$ with $\rho \in (0, 1)$ are easy to calculate:

$$\delta_\tau(x) = \frac{|f(x)|}{|x|} = \frac{\exp\left\{\int\limits_0^\rho \frac{dr}{rK(r)}\right\}}{\rho}, \quad \delta_r(x) = \frac{\partial|f(x)|}{\partial|x|} = \frac{\exp\left\{\int\limits_0^\rho \frac{dr}{rK(r)}\right\}}{\rho K(\rho)},$$

and we see that $\delta_\tau(x) \geq \delta_r(x)$ since $K(r) \geq 1$. Consequently, the spherical symmetry yields

$$K_I(x,f) = \frac{\delta_\tau^{n-1}(x)\cdot\delta_r(x)}{\delta_r^n(x)} = K^{n-1}(|x|)$$

at all points $x \in \mathbb{B}^n \setminus \{0\}$ (see [33, Section I.2.1] for instance). Observe that

$$f_m(x) \equiv f(x) \quad \text{for all } x \text{ such that } \frac{1}{m} < |x| < 1,\ m = 1,2,\dots. \tag{5.12}$$

Therefore, we similarly calculate $K_I(x, f_m) = K_I(x, f) = K^{n-1}(|x|)$ for $\frac{1}{m} < |x| < 1$ and $K_I(x, f_m) = K^{n-1}\left(\frac{1}{m}\right)$ for $0 < |x| < \frac{1}{m}$. Thus, f_m is quasiconformal in $\mathbb{B}^n$, and so $f_m \in W^{1,n}_{\text{loc}}$. By (5.8) we have

$$\int_{\mathbb{B}^n} \Phi(K_I(x,f_m))\,dm(x) \leq \int_{\mathbb{B}^n} \Phi_{n-1}(K(|x|))\,dm(x)$$

$$= \omega_{n-1}\int_0^1 \frac{\Psi(K(r))}{rK(r)}\cdot r^n dr \leq \gamma^2\omega_{n-1}\int_0^1 \frac{dr}{rK(r)} \leq M := \gamma^2\omega_{n-1}I(0) < \infty.$$

Observe that f_m maps the unit ball $\mathbb{B}^n$ onto the ball of radius $R = e^{I(0)} < \infty$ centered at the origin. Therefore, $f_m \in S^\Phi_{M,\Delta}$ for some $\Delta > 0$, where M is indicated above.

On the other hand, it is easy to see that

$$\lim_{x\to 0}|f(x)| = \lim_{t\to 0}\rho(t) = e^0 = 1; \tag{5.13}$$

thus, f maps the punctured ball $\mathbb{B}^n \setminus \{0\}$ onto the ring $1 < |y| < R = e^{I(0)}$. Then by (5.12) and (5.13) we deduce that $|f_m(x)| = |f(x)| \geq 1$ for all x such that $|x| \geq 1/m$, $m = 1,2,\dots$, thus, the family $\{f_m\}_{m=1}^\infty$ is not equicontinuous at zero.

The resulting contradiction refutes the assumption in (5.4). $\square$

Remark 5.2. Theorem 5.1 shows that condition (4.2) of Theorem 4.1 is not only sufficient, but also necessary for the equicontinuity (normality) of the classes of mappings with integral restrictions of the form (4.1) or (4.4) for nondecreasing convex functions Φ. By Proposition 2.3, this also applies to each of conditions (2.5)–(2.10) with $p = n-1$.

Finally, note that already in [11] it was established that the requirement that Φ is nondecreasing and convex is necessary for the compactness (completeness) of the classes of mappings with integral-type restrictions (4.3).

Corollary 5.1. *If the classes $S^\Phi_{M,\Delta} \subset \mathfrak{R}^\Phi_{M,\Delta}$ are equicontinuous (normal) for all $M \in (0,\infty)$, $\Delta \in (0,1)$, and a nondecreasing convex function Φ then*

$$\int_\delta^\infty \log\Phi(t)\frac{dt}{t^{n'}} = \infty \tag{5.14}$$

for all $\delta > t_0$, where $t_0 := \sup_{\Phi(t)=0} t$, $t_0 = 0$ if $\Phi(0) > 0$, and $\frac{1}{n'} + \frac{1}{n} = 1$, i.e., $n' = n/(n-1)$.

By Remark 4.2 and Proposition 2.3, (5.14) is also a sufficient condition for the equicontinuity (normality) of the classes $S^\Phi_{M,\Delta}$ and $\mathfrak{R}^\Phi_{M,\Delta}$ for all $M \in (0,\infty)$ and $\Delta \in (0,1)$.

References

1. *Ahlfors L. V.*, "On quasiconformal mappings," J. Anal. Math., **3**, 1–58 (1951/1954).
2. *Biluta P. A.*, "Extremal problems for mean quasiconformal mappings," Sibirsk. Mat. Zh., **6**, 717–726 (1965).
3. *Golberg A.*, "Homeomorphisms with finite mean dilatations," Contemp. Math., **382**, 177–186 (2005).
4. *Kruglikov V. I.*, "Capacity of condensers and spatial mappings quasiconformal in the mean," Math. USSR-Sb., **58**, No. 1, 185–205 (1987).
5. *Krushkal′ S. L.*, "On mean quasiconformal mappings," Dokl. Akad. Nauk SSSR, **157**, No. 3, 517–519 (1964).
6. *Krushkal′ S. L. and R. Kühnau*, Quasiconformal Mappings: New Methods and Applications [in Russian], Nauka, Novosibirsk (1984).
7. *Kud′yavin V. S.*, "The behavior of a class of mappings, quasiconformal in the mean, at an isolated singular point," Soviet Math. Dokl., **30**, 209–210 (1984).
8. *Kühnau R.*, "Über Extremalprobleme bei im Mittel quasiconformen Abbildungen," in: Complex Analysis-5, Romanian-Finnish Seminar, Bucharest 1981, Springer-Verlag, Berlin, 1983, pp. 113–124 (Lecture Notes Math.; V. 1013).
9. *Perovich M.*, "Isolated singularity of the mean quasiconformal mappings," in: Romanian-Finnish Seminar on Complex Analysis, Proc., Bucharest 1976, Springer-Verlag, 1979, pp. 212–214 (Lecture Notes Math.; V. 743).
10. *Pesin I. N.*, "Mappings that are quasiconformal in the mean," Soviet Math. Dokl., **10**, 939–941 (1969).
11. *Ryazanov V. I.*, "On mean quasiconformal mappings," Siberian Math. J., **37**, No. 2, 325–334 (1996).
12. *Strugov Yu. F.*, "On the compactness of families of mean quasiconformal mappings," Soviet Math. Dokl., **19**, 1443–1446 (1978).
13. *Ukhlov A. and Vodopyanov S. K.*, "Mappings associated with weighted Sobolev spaces," Complex Anal. Dynam. Syst. III, Contemp. Math., **455**, 369–382 (2008).
14. *Zorich V. A.*, "Admissible order of growth of the quasiconformality characteristic in Lavrent′ev's theorem.," Soviet Math. Dokl., **9**, 866–869 (1968).
15. *Zorich V. A.*, "An isolated singularity of mappings with bounded distortion," Math. USSR-Sb., **10**, No. 4, 581–583 (1970).
16. *Astala K., Iwaniec T., and Martin G. J.*, Elliptic Differential Equations and Quasiconformal Mappings in the Plane, Princeton Univ. Press, Princeton (2009) (Princeton Math. Ser.; V. 48).
17. *Martio O., Ryazanov V., Srebro U., and Yakubov E.*, Moduli in Modern Mapping Theory, Springer Science + Business Media, LLC, New York (2009).
18. *Gutlyanskii V., Ryazanov V., Srebro U., and Yakubov E.*, "On recent advances in the Beltrami equations," J. Math. Sci. (2011) (to be published).
19. *Srebro U. and Yakubov E.*, "The Beltrami equation," in: Handbook in Complex Analysis: Geometric Function Theory. Vol. 2, Elsevier, Basel, 2005, pp. 555–597.
20. *Iwaniec T. and Martin G.*, Geometric Function Theory and Nonlinear Analysis, Oxford University Press, Oxford (2001).
21. *Ryazanov V. I. and Sevost′yanov E. A.*, "Equicontinuous classes of ring Q-homeomorphisms," Siberian Math. J., **48**, No. 6, 1361–1376 (2007).
22. *Väisälä J.*, Lectures on n-Dimensional Quasiconformal Mappings, Springer-Verlag, Berlin etc. (1971) (Lecture Notes Math.; V. 229).
23. *Martio O., Ryazanov V., Srebro U., and Yakubov E.*, "Q-Homeomorphisms," Contemp. Math., **364**, 193–203 (2004).
24. *Martio O., Ryazanov V., Srebro U., and Yakubov E.*, "On Q-homeomorphisms," Ann. Acad. Sci. Fenn., **30**, 49–69 (2005).
25. *Andreian Cazacu C.*, "On the length-area dilatation," Complex Var. Theory Appl., **50**, No. 7–11, 765–776 (2005).
26. *Miklyukov V. M.*, The Conformal Mapping of an Irregular Surface and Its Applications [in Russian], Izdat. Volgogradsk. Univ., Volgograd (2005).
27. *Lehto O. and Virtanen K.*, Quasiconformal Mappings in the Plane, Springer-Verlag, New York etc. (1973).
28. *Bishop C. J., Gutlyanskiĭ V. Ya., Martio O., and Vuorinen M.*, "On conformal dilatation in space," Internat. J. Math. Math. Sci., **22**, 1397–1420 (2003).
29. *Gehring F. W.*, "Rings and quasiconformal mappings in space," Trans. Amer. Math. Soc., **103**, No. 3, 353–393 (1962).
30. *Martio O. and Ryazanov V., Srebro U., and Yakubov E.*, "Mappings with finite length distortion," J. Anal. Math., **93**, 215–236 (2004).
31. *Ryazanov V., Srebro U., and Yakubov E.*, "On integral conditions in the mapping theory," J. Math. Sci., **173**, No. 4, 397–407 (2010).
32. *Bourbaki N.*, Functions of a Real Variable. Elementary Theory, Springer-Verlag, Berlin (2004).
33. *Reshetnyak Yu. G.*, Space Mappings with Bounded Distortion, Amer. Math. Soc., Providence (1989) (Transl. Math. Monographs; 73).

V. I. Ryazanov; E. A. Sevost′yanov
Institute of Applied Mathematics and Mechanics, Donetsk, Ukraine
E-mail address: `vlryazanov1@rambler.ru`; `brusin2006@rambler.ru`