

ON SPATIAL MAPPINGS WITH INTEGRAL RESTRICTIONS ON THE CHARACTERISTIC

E. A. SEVOST'YANOV

ABSTRACT. For a given domain $D \subset \mathbb{R}^n$, some families \mathfrak{F} of mappings $f : D \rightarrow \overline{\mathbb{R}^n}$, $n \geq 2$ are studied; such families are more general than the mappings with bounded distortion. It is proved that a family is equicontinuous if $\int_{\delta_0}^{\infty} \frac{d\tau}{\tau[\Phi^{-1}(\tau)]^{\frac{1}{n-1}}} = \infty$, where the integral depends on each mapping $f \in \mathfrak{F}$, Φ is a special function, and $\delta_0 > 0$ is fixed. Under similar restrictions, removability results are obtained for isolated singularities of f . Also, analogs of the well-known Sokhotsky–Weierstrass and Liouville theorems are proved.

§1. INTRODUCTION

In this paper, we use the following notation: $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$, $\mathbb{B}^n := B(0, 1)$, $S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$, $\mathbb{S}^{n-1} := S(0, 1)$; Ω_n denotes the volume of the unit ball \mathbb{B}^n of \mathbb{R}^n , ω_{n-1} is the area of the sphere \mathbb{S}^{n-1} in \mathbb{R}^n , D is a domain in \mathbb{R}^n , $n \geq 2$, m is Lebesgue measure on \mathbb{R}^n , $\text{dist}(A, B)$ is the Euclidean distance between sets $A, B \subset \mathbb{R}^n$, $|A|$ denotes the linear measure of a set $A \subset \mathbb{R}$ whenever there is no confusion. The notation $f : D \rightarrow \mathbb{R}^n$ assumes that the mapping f is continuous on the domain D . As usual, we write $f \in W_{\text{loc}}^{1,n}(D)$ if the coordinate functions $f = (f_1, \dots, f_n)$ have the first order generalized partial derivatives the n th power of which are locally integrable on D . A mapping $f : D \rightarrow \mathbb{R}^n$ is said to be *discrete* if, for every point $y \in \mathbb{R}^n$, the preimage $f^{-1}(y)$ consists of isolated points; the mapping is *open* if the image of any open set $U \subset D$ is an open set in \mathbb{R}^n .

Conditions of the following form are important in the theory of spacial mappings:

$$(1) \quad \int_D \Phi(Q(x)) dm(x) < \infty,$$

where $\Phi : [0, \infty] \rightarrow [0, \infty]$ and $Q : D \rightarrow [0, \infty]$ are fixed measurable functions. Note that properties of the form (1) arise in various problems; see, e.g., [1, 2, 6, 8, 13, 14, 18, 20] and [24]. In the present paper, we investigate how the restriction (1) can affect the properties of spacial mappings.

First, we give several definitions. Here and in what follows, a *curve* γ is a continuous mapping of a segment $[a, b]$ (or an open interval (a, b)) into \mathbb{R}^n , that is, $\gamma : [a, b] \rightarrow \mathbb{R}^n$. We use the symbol Γ to denote a fixed family of curves γ ; by definition, $f(\Gamma) = \{f \circ \gamma | \gamma \in \Gamma\}$. The following definitions can be found, for example, in [22, Chapter 1, §§1–6]. A Borel function $\rho : \mathbb{R}^n \rightarrow [0, \infty]$ is said to be *admissible* for a family Γ of curves γ in \mathbb{R}^n if the inequality $\int_{\gamma} \rho(x) |dx| \geq 1$ is true for the curvilinear integral of the first kind and for all curves $\gamma \in \Gamma$. If this is the case, then we write $\rho \in \text{adm } \Gamma$. The *modulus* of a family

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Γ of curves is defined by

$$(2) \quad M(\Gamma) = \inf_{\rho \in \text{adm } \Gamma} \int_D \rho^n(x) dm(x).$$

In a sense, the properties of the modulus are similar to those of Lebesgue measure m on \mathbb{R}^n . Namely, the modulus of the empty family is equal to zero: $M(\emptyset) = 0$; the modulus is monotone with respect to families of curves: $\Gamma_1 \subset \Gamma_2 \Rightarrow M(\Gamma_1) \leq M(\Gamma_2)$. Also, the modulus is semiadditive: $M\left(\bigcup_{i=1}^{\infty} \Gamma_i\right) \leq \sum_{i=1}^{\infty} M(\Gamma_i)$; see [22, Theorem 6.2]. Recall that $f : D \rightarrow \mathbb{R}^n$ is called a *mapping with bounded distortion* if the following conditions are satisfied: 1) $f \in W_{\text{loc}}^{1,n}$; 2) the Jacobian $J(x, f)$ of the mapping f at $x \in D$ has one and the same sign almost everywhere in D ; 3) $\|f'(x)\|^n \leq K \cdot |J(x, f)|$ for almost all $x \in D$ and for some constant $K < \infty$, where, as usual, $\|f'(x)\| := \sup_{h \in \mathbb{R}^n : |h|=1} |f'(x)h|$ (see, e.g., [16, §3, Chapter I] or [17, Definition 2.1, §2, Chapter I]). For us, it is important that each mapping with bounded distortion satisfies the inequality of E. A. Poletsky (see [15, §4, Theorem 1]). Namely, if $f : D \rightarrow \mathbb{R}^n$ is a mapping with bounded distortion, then

$$(3) \quad M(f(\Gamma)) \leq K' \cdot M(\Gamma)$$

for an arbitrary family Γ of curves γ in the domain D , where M is the conformal modulus introduced above for the family of curves (i.e., an outer measure defined on the family of curves in \mathbb{R}^n) and $K' < \infty$ is a constant (see also [17, Chapter II, §8, Theorem 8.1]). A homeomorphism $f : D \rightarrow \mathbb{R}^n$ or, depending on the context, a homeomorphism $f : D \rightarrow \overline{\mathbb{R}^n}$, $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, is called a *K' -quasiconformal mapping* on D if f satisfies (3) for an arbitrary family Γ of curves γ (see, e.g., [22, Chapter 2, §13; Theorem 34.3, Chapter IV]).

In the present paper, we investigate mappings f that are more general than those with bounded distortion. More precisely, given a measurable function $Q : D \rightarrow [0, \infty]$ that maps a domain D in \mathbb{R}^n into the real numbers, instead of (3) we assume that

$$(4) \quad M(f(\Gamma(S_1, S_2, A))) \leq \int_A Q(x) \cdot \eta^n(|x - x_0|) dm(x),$$

where

$$(5) \quad A = A(r_1, r_2, x_0) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$$

denotes the spherical annulus with center x_0 and with radii r_1, r_2 ; $S_i = S(x_0, r_i)$ denotes the sphere with center x_0 and with radius r_i , $i = 1, 2$; $\Gamma(S_1, S_2, A)$ denotes the family of all curves joining S_1 and S_2 inside the domain A . The real-valued function $\eta : (r_1, r_2) \rightarrow [0, \infty]$ in inequality (4) is assumed to be measurable and to satisfy the condition

$$(6) \quad \int_{r_1}^{r_2} \eta(r) dr \geq 1.$$

See, e.g., [3, 10, 11, 12] and [20] for further details about inequalities of type (4). Note that, in the case where $Q(x) \leq K$ a.e., an expression like $K \cdot M(\Gamma(S_1, S_2, A))$ arises on the right-hand side of inequality (4), see inequalities (4) and (3). It follows that an arbitrary mapping with bounded distortion satisfies (4) for a constant Q . In general, if the function $Q(x)$ is merely measurable, then the right-hand side of inequality (4) is, in a sense, the modulus M of the family $\Gamma(S_1, S_2, A)$ of curves with weight $Q(x)$. We are primarily interested in the case where the function Q in (4) is unbounded. Suppose that an open discrete mapping f satisfies inequality (4) and condition (1) is fulfilled for the function $Q(x)$, where Φ is a function such that

$$(7) \quad \int_{\delta_0}^{\infty} \frac{d\tau}{\tau [\Phi^{-1}(\tau)]^{\frac{1}{n-1}}} = \infty$$

for some $\delta_0 > 0$. In the present paper, we show that the mapping f extends by continuity to an isolated point of the boundary of D . Also we prove the equicontinuity of the families under consideration, assuming that f satisfies (4), where the function $Q(x)$ satisfies (1), and the function Φ occurring in (1) satisfies condition (7). Also, we shall show that the restriction on the function Φ mentioned above, that is, property (7), is not only sufficient, but, in an appropriate sense, also necessary. We assume that f is an open and discrete mapping, Q is merely a measurable function, and Φ is a monotone nondecreasing and convex function.

We say that $f : D \rightarrow \overline{\mathbb{R}^n}$ is an *annular Q -mapping at a point $x_0 \in D$* if f satisfies (4) for any annulus $A = A(r_1, r_2, x_0)$, $0 < r_1 < r_2 < r_0 = \text{dist}(x_0, \partial D)$, and for every measurable function $\eta : (r_1, r_2) \rightarrow [0, \infty]$ with property (6). If f in (4) is assumed to be a homeomorphism, then f is called an *annular Q -homeomorphism at the point $x_0 \in D$* . Finally, f is called an *annular Q -mapping* (an *annular Q -homeomorphism*, respectively) if (4) is true at each point $x_0 \in D$. Similarly, in the studies of questions related to the boundary behavior of mappings, one may introduce the notion of an annular Q -mapping $f : D \setminus \{x_0\} \rightarrow \overline{\mathbb{R}^n}$ that is defined *a priori* in a neighborhood of an isolated point x_0 of the boundary $D \setminus \{x_0\}$.

Let E be a compact set in $\overline{\mathbb{R}^n}$ with a positive conformal capacity: $\text{cap } E > 0$. Assume that we are given a function $\Phi : [0, \infty] \rightarrow [0, \infty]$, a Lebesgue measurable function $Q : D \rightarrow [0, \infty]$, and a number $M > 0$. Denote by $\mathfrak{R}_{M,E}^{\Phi,Q}$ the family of all open discrete annular Q -mappings $f : D \rightarrow \overline{\mathbb{R}^n} \setminus E$ such that

$$(8) \quad \int_D \Phi(Q(x)) \frac{dm(x)}{(1+|x|^2)^n} \leq M.$$

To mention explicitly the domain D , we sometimes use the notation $\mathfrak{R}_{M,E}^{\Phi,Q}(D)$. The following assertions are among the main results of the present paper.

Theorem 1. *Let $\Phi : [0, \infty] \rightarrow [0, \infty]$ be a monotone nondecreasing convex function. If (7) is true for some $\delta_0 > \tau_0 := \Phi(0)$, then the class $\mathfrak{R}_{M,E}^{\Phi,Q}$ is equicontinuous; hence, it forms a normal family of mappings for all $M \in (0, \infty)$.*

For a Lebesgue measurable function $Q : D \rightarrow [0, \infty]$ and a compact set $E \subset \overline{\mathbb{R}^n}$ with $\text{cap } E > 0$, denote by $K_{M,E}^{\Phi,Q}(D \setminus \{x_0\})$ the family of all open and discrete Q -mappings $f : D \setminus \{x_0\} \rightarrow \overline{\mathbb{R}^n} \setminus E$ annular at the point x_0 and satisfying (8).

Theorem 2. *Assume that $Q(x) \geq 1$ almost everywhere, $\Phi : [0, \infty] \rightarrow [0, \infty]$ is a monotone nondecreasing convex function, and $x_0 \in D$. If (7) is true for some $\delta_0 > \tau_0 := \Phi(0)$, then an arbitrary mapping $f : D \setminus \{x_0\} \rightarrow \overline{\mathbb{R}^n}$, $f \in K_{M,E}^{\Phi,Q}(D \setminus \{x_0\})$, extends by continuity to the point $x_0 \in D$ up to an open discrete mapping $f : D \rightarrow \overline{\mathbb{R}^n}$.*

Theorem 3. *Let $\Phi : [0, \infty] \rightarrow [0, \infty]$ be a monotone nondecreasing convex function. Assume that, for all sets E of positive capacity, all numbers $M > 0$, and all measurable functions $Q : D \rightarrow [0, \infty]$, the class $\mathfrak{R}_{M,E}^{\Phi,Q}$ is equicontinuous (normal). Then, for all $\delta_* \in (\tau_0, \infty)$, $\tau_0 := \Phi(0)$, we have*

$$(9) \quad \int_{\delta_*}^{\infty} \frac{d\tau}{\tau [\Phi^{-1}(\tau)]^{\frac{1}{n-1}}} = \infty.$$

Theorem 4. *Let $\Phi : [0, \infty] \rightarrow [0, \infty]$ be a monotone nondecreasing convex function. Assume that, for all sets E of positive capacity, all numbers $M > 0$, and all measurable functions $Q : D \rightarrow [0, \infty]$, an arbitrary mapping $f : D \setminus \{x_0\} \rightarrow \overline{\mathbb{R}^n}$, $x_0 \in D$, $f \in K_{M,E}^{\Phi,Q}(D \setminus \{x_0\})$, extends by continuity to the point x_0 . Then condition (9) is fulfilled for all $\delta_* \in (\tau_0, \infty)$, $\tau_0 := \Phi(0)$.*

Remark 1. Observe that the condition $\int_D \Phi(Q(x)) dm(x) \leq M$ implies property (8). Hence, property (8) is more general, and the corresponding class of the annular Q -mappings is a subclass of the family $\mathfrak{R}_{M,E}^{\Phi,Q}$. On the other hand, if the domain D is bounded, then (8) implies the condition $\int_D \Phi(Q(x)) dm(x) \leq M_*$, where $M_* = M \cdot (1 + \delta_*^2)^n$, $\delta_* = \sup_{x \in D} |x|$.

In the next section, we accurately introduce the main definitions and notation used above.

§2. PRELIMINARIES

By definition, a *condenser* in \mathbb{R}^n , $n \geq 2$, is a pair $E = (A, C)$, where A is an open set in \mathbb{R}^n and C is a compact subset of A . A mapping $f : D \rightarrow \mathbb{R}^n$ is said to be *absolutely continuous on lines*, $f \in ACL$, if in any n -dimensional parallelepiped P with edges parallel to the coordinate axes, $\overline{P} \subset D$, all coordinate functions $f = (f_1, \dots, f_n)$ are absolutely continuous on almost all lines that are parallel to the coordinate axes. By the *conformal capacity* (or merely *capacity*) of a condenser E we mean the following quantity:

$$\text{cap } E = \text{cap}(A, C) = \inf_{u \in W_0(E)} \int_A |\nabla u|^n dm(x),$$

where $W_0(E) = W_0(A, C)$ is the family of all nonnegative continuous functions $u : A \rightarrow \mathbb{R}$ compactly supported in A and such that $u(x) \geq 1$ for $x \in C$ and $u \in ACL$. We say that a compact set C in \mathbb{R}^n , $n \geq 2$, is of *zero capacity* and we write $\text{cap } C = 0$ if $\text{cap}(A, C) = 0$ for at least one bounded open set A that contains C . Otherwise, $\text{cap } C > 0$.

It is known that the sets of capacity zero are discontinuous; see, e.g., Corollary 2.5 in Chapter 3 of [17]. In particular, the condition $\text{cap } C = 0$ implies that $\text{Int } C = \emptyset$. We say that a set A in \mathbb{R}^n is of capacity zero if an arbitrary compact subset of A is of capacity zero. The notions of a condenser and of a set of capacity zero in $\overline{\mathbb{R}^n}$ are defined similarly (see, e.g., [17]).

Let (X, d) and (X', d') be metric spaces with distances d and d' , respectively. We say that a family \mathfrak{F} of continuous mappings $f : X \rightarrow X'$ is *normal* if the following is true: for any sequence of mappings $f_m \in \mathfrak{F}$, there is a subsequence f_{m_k} that converges locally uniformly on X to a continuous function $f : X \rightarrow X'$. The above notion is closely related with the following one: a family \mathfrak{F} of mappings $f : X \rightarrow X'$ is *equicontinuous at a point* $x_0 \in X$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d'(f(x), f(x_0)) < \varepsilon$ for all x such that $d(x, x_0) < \delta$ and for all $f \in \mathfrak{F}$. We say that \mathfrak{F} is *equicontinuous* if \mathfrak{F} is equicontinuous at each point of X . If (X, d) is a separable metric space and (X', d') is a compact metric space, then a family \mathfrak{F} of mappings $f : X \rightarrow X'$ is normal if and only if \mathfrak{F} is equicontinuous. This is a version of the well-known Arzelà–Ascoli theorem (see, e.g., [22, §20.4]).

Recall that an isolated point x_0 of the boundary ∂D of a domain D is said to be *removable* for a mapping f if there exists a finite limit $\lim_{x \rightarrow x_0} f(x)$. An isolated point x_0 of the boundary ∂D is called an *essential singular point* of a mapping $f : D \rightarrow \mathbb{R}^n$ if the mapping has neither finite, nor infinite limit as $x \rightarrow x_0$.

Suppose that a mapping $f : D \rightarrow \mathbb{R}^n$, $n \geq 2$, has the first partial derivatives for almost all $x \in D$. Then the *inner dilatation* of f at a point $x \in D$ is the quantity

$$K_I(x, f) = \frac{|J(x, f)|}{l(f'(x))^n}$$

if $J(x, f) \neq 0$, $K_I(x, f) = 1$ if $f'(x) = 0$, and $K_I(x, f) = \infty$ for other points. Here $l(f'(x)) = \inf_{h \in \mathbb{R}^n : |h|=1} |f'(x)h|$. We shall need the following proposition.

Proposition 1. *Let $f : D \rightarrow \overline{\mathbb{R}^n}$, $f \in W_{\text{loc}}^{1,n}$, be a homeomorphism. If $K_I(x, f) \in L_{\text{loc}}^1$, then f is an annular Q -homeomorphism at each point $x_0 \in D$ with $Q = K_I(x, f)$; see, e.g., [11, Theorems 8.1 and 8.6]. Moreover, f is an annular Q -mapping at every isolated point of the boundary of D ; see op. cit.*

To compute inner dilatations, sometimes it is useful to apply the quantities introduced below. Assume that a mapping $f : D \rightarrow \mathbb{R}^n$ is differentiable at a point $x_0 \in D$ and the Jacobi matrix $f'(x_0)$ is nonsingular: $J(x_0, f) = \det f'(x_0) \neq 0$. Then there exist vector systems e_1, \dots, e_n and $\tilde{e}_1, \dots, \tilde{e}_n$ as well as positive numbers $\lambda_1(x_0), \dots, \lambda_n(x_0)$ with $\lambda_1(x_0) \leq \dots \leq \lambda_n(x_0)$ such that $f'(x_0)e_i = \lambda_i(x_0)\tilde{e}_i$; see Theorem 2.1 in Chapter 1 of [16]. Moreover, $\lambda_1^2(x_0), \dots, \lambda_n^2(x_0)$ are the eigenvalues of the symmetric mapping $(f'(x_0))^*f'(x_0)$, see Theorem 2.2 in Chapter 1 of [16], and

$$(10) \quad \begin{aligned} |J(x_0, f)| &= \lambda_1(x_0) \dots \lambda_n(x_0), \quad \|f'(x_0)\| = \lambda_n(x_0), \quad l(f'(x_0)) = \lambda_1(x_0), \\ K_I(x_0, f) &= \frac{\lambda_1(x_0) \dots \lambda_n(x_0)}{\lambda_1^n(x_0)}; \end{aligned}$$

see property (2.5) and additional comments in §2.1 in Chapter 1 of [16]. The numbers $\lambda_1(x_0), \dots, \lambda_n(x_0)$ mentioned above are called the *principal values*, and the vectors e_1, \dots, e_n and $\tilde{e}_1, \dots, \tilde{e}_n$ are called the *principal vectors* of the mapping $f'(x_0)$. More details on this issue can be found in the corresponding comment after the proof of Theorem 2.2 in Chapter 1 of [16]. Clearly, the principal vectors and principal values depend on the point x_0 and on the mapping f . However, to simplify the notation here and in what follows, we omit (x_0) if there is no confusion.

Recall that the *spherical (chordal) metric* $h(x, y)$ is equal to $|\pi(x) - \pi(y)|$, where π is the stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in \mathbb{R}^{n+1} , that is,

$$h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}, \quad h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}, \quad x \neq \infty \neq y.$$

Let $Q : D \rightarrow [0, \infty]$ be a Lebesgue measurable function. Then $q_{x_0}(r)$ denotes the integral mean of $Q(x)$ over the sphere $|x - x_0| = r$:

$$(11) \quad q_{x_0}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int_{|x-x_0|=r} Q(x) dS,$$

where dS is the area element of the surface S . Below we agree that $a/\infty = 0$ for $a \neq \infty$, $a/0 = \infty$ for $a > 0$, and $0 \cdot \infty = 0$.

The following results were formulated and proved in [25].

Proposition 2. *Let $Q : \mathbb{B}^n \rightarrow [1, \infty]$ be a Lebesgue measurable function, and let $f : \mathbb{B}^n \setminus \{0\} \rightarrow \overline{\mathbb{R}^n}$, $n \geq 2$, be an open, discrete, annular Q -mapping at the point $x_0 = 0$ such that $\text{cap}(\overline{\mathbb{R}^n} \setminus f(\mathbb{B}^n \setminus \{0\})) > 0$. Assume that there exists $\varepsilon_0 > 0$, $\varepsilon_0 < 1$, such that*

$$(12) \quad \int_0^{\varepsilon_0} \frac{dt}{t q_{x_0}^{\frac{1}{n-1}}(t)} = \infty.$$

Then f has a continuous extension onto \mathbb{B}^n , that is, $f : \mathbb{B}^n \rightarrow \overline{\mathbb{R}^n}$. Continuity is understood in the sense of the space $\overline{\mathbb{R}^n}$ with respect to the chordal metric h . Moreover, the extended mapping $f : \mathbb{B}^n \rightarrow \overline{\mathbb{R}^n}$ is open and discrete; see Theorem 6 in [25].

In what follows, equicontinuity, normality, and other notions are considered with respect to the spherical (chordal) metric h .

Proposition 3. *Assume that $x_0 \in D$, $E \subset \overline{\mathbb{R}^n}$ is a compact set of positive capacity, and \mathfrak{F}_Q is a family of open, discrete, annular Q -mappings $f : D \rightarrow \overline{\mathbb{R}^n} \setminus E$ at the point*

x_0 . Assume that there exists a number $\varepsilon_0 < \text{dist}(x_0, \partial D)$ such that (12) is true. Then the family \mathfrak{F}_Q is equicontinuous and, hence, normal at the point x_0 .

Recall that a function $\Phi : [0, \infty] \rightarrow [0, \infty]$ is said to be *convex* if

$$\Phi(\lambda t_1 + (1 - \lambda)t_2) \leq \lambda \Phi(t_1) + (1 - \lambda) \Phi(t_2)$$

for all $t_1, t_2 \in [0, \infty]$ and $\lambda \in [0, 1]$. The *inverse function* Φ^{-1} is well defined for any monotone nondecreasing function $\Phi : [0, \infty] \rightarrow [0, \infty]$ as follows:

$$(13) \quad \Phi^{-1}(\tau) = \inf_{\Phi(t) \geq \tau} t.$$

As usual, the infimum in (13) is equal to ∞ if there is no $t \in [0, \infty]$ for which $\Phi(t) \geq \tau$. Observe that the function Φ^{-1} is also monotone nondecreasing.

Remark 2. Obviously, the above definition shows that

$$(14) \quad \Phi^{-1}(\Phi(t)) \leq t, \quad t \in [0, \infty].$$

Moreover, equality occurs in (14) outside of the intervals where $\Phi(t)$ is constant.

For the proposition formulated below, see, e.g., [19, Theorem 2.1]. In (15) and (16), the integrals are assumed to be equal to infinity if $\Phi_p(t) = \infty$ or if there exists $T \in \mathbb{R}$ such that $H_p(t) = \infty$ for all $t \geq T \in [0, \infty)$. The integral in (16) is that of Lebesgue–Stieltjes, the integrals in (15) and in (17)–(20) are those of Lebesgue.

Proposition 4. *Let $\Phi : [0, \infty] \rightarrow [0, \infty]$ be a monotone nondecreasing function. Put $H_p(t) = \log \Phi_p(t)$, $\Phi_p(t) = \Phi(t^p)$, $p \in (0, \infty)$. Then the relation*

$$(15) \quad \int_{\delta}^{\infty} H_p'(t) \frac{dt}{t} = \infty$$

implies

$$(16) \quad \int_{\delta}^{\infty} \frac{dH_p(t)}{t} = \infty.$$

Identity (16) is equivalent to the following property:

$$(17) \quad \int_{\delta}^{\infty} H_p(t) \frac{dt}{t^2} = \infty$$

for some $\delta > 0$. Also, (17) is equivalent to each of the following relations:

$$(18) \quad \int_0^{\Delta} H_p\left(\frac{1}{t}\right) dt = \infty$$

for some $\Delta > 0$;

$$(19) \quad \int_{\delta_*}^{\infty} \frac{d\eta}{H_p^{-1}(\eta)} = \infty$$

for some $\delta_ > H(+0)$;*

$$(20) \quad \int_{\delta_*}^{\infty} \frac{d\tau}{\tau \Phi_p^{-1}(\tau)} = \infty$$

for some $\delta_ > \Phi(+0)$.*

Moreover, (15) is equivalent to (16); hence, properties (15)–(20) are equivalent to each other if Φ is additionally assumed to be absolutely continuous. In particular, all conditions (15)–(20) are equivalent to each other whenever Φ is convex and monotone nondecreasing.

It is easily seen that conditions (15)–(20) become weaker if p grows; see, e.g., (17). Yet one more clarification is necessary. The right-hand sides of conditions (15)–(20) should be understood as $+\infty$. If $\Phi_p(t) = 0$ for $t \in [0, t_*]$, then $H_p(t) = -\infty$ for $t \in [0, t_*]$; if this is the case, we put $H_p'(t) := 0$ for $t \in [0, t_*]$. Note that conditions (16) and (17) exclude the case where t_* belongs to the interval of integration in the above properties. Indeed, otherwise the left-hand sides of (16) and (17) are either simultaneously equal to $-\infty$ or not defined. Hence, in (15)–(17) we may assume that $\delta > t_0$ and $\Delta < 1/t_0$ where $t_0 := \sup_{\Phi_p(t)=0} t$ and $t_0 = 0$ if $\Phi_p(0) > 0$.

§3. MAIN LEMMA

The following assertion generalizes and refines Lemma 3.1 in [19].

Lemma 1. *Let $Q : \mathbb{B}^n \rightarrow [0, \infty]$ be a measurable function, and let $\Phi : [0, \infty] \rightarrow (0, \infty]$ be a monotone nondecreasing convex function. Then*

$$(21) \quad \int_{\varepsilon}^1 \frac{dr}{r q_0^{\frac{1}{p}}(r)} \geq \frac{1}{n} \int_{eM(\varepsilon)}^{\frac{M(\varepsilon)}{\varepsilon^n}} \frac{d\tau}{\tau [\Phi^{-1}(\tau)]^{\frac{1}{p}}}, \quad p \in (0, \infty), \quad \varepsilon \in (0, 1),$$

where $q_0(r)$ is defined by (11) for $x_0 = 0$, and

$$(22) \quad M(\varepsilon) = \frac{1}{\Omega_n(1-\varepsilon^n)} \int_{A(\varepsilon, 1, 0)} \Phi(Q(x)) \, dm(x)$$

is the integral mean of the function $\Phi \circ Q$ on the annulus $A(\varepsilon, 1, 0)$ defined by (5) with $x_0 = 0$, $r_1 = \varepsilon$ and $r_2 = 1$.

Remark 3. Observe that, for every $p \in (0, \infty)$, property (21) is equivalent to the following inequality:

$$(23) \quad \int_{\varepsilon}^1 \frac{dr}{r q_0^{\frac{1}{p}}(r)} \geq \frac{1}{n} \int_{eM(\varepsilon)}^{\frac{M(\varepsilon)}{\varepsilon^n}} \frac{d\tau}{\tau \Phi_p^{-1}(\tau)}, \quad \Phi_p(t) := \Phi(t^p).$$

Proof of Lemma 1.

Step 1. Denote

$$(24) \quad t_* = \sup_{\Phi_p(t)=\tau_0} t, \quad \tau_0 = \Phi(0) > 0.$$

Putting $H_p(t) := \log \Phi_p(t)$, we see that

$$(25) \quad H_p^{-1}(\eta) = \Phi_p^{-1}(e^\eta), \quad \Phi_p^{-1}(\tau) = H_p^{-1}(\log \tau).$$

Hence, using Remark 2 and setting

$$h(r) := r^n \Phi(q_0(r)) = r^n \Phi_p(q_0^{\frac{1}{p}}(r)), \quad R_* = \{r \in (\varepsilon, 1) : q_0^{\frac{1}{p}}(r) > t_*\},$$

we obtain

$$q_0^{\frac{1}{p}}(r) = H_p^{-1}\left(\log \frac{h(r)}{r^n}\right) = H_p^{-1}\left(n \log \frac{1}{r} + \log h(r)\right), \quad r \in R_*.$$

Then we also have

$$(26) \quad q_0^{\frac{1}{p}}(e^{-s}) = H_p^{-1}(ns + \log h(e^{-s})), \quad s \in S_*,$$

where

$$(27) \quad S_* = \left\{s \in \left(0, \log \frac{1}{\varepsilon}\right) : q_0^{\frac{1}{p}}(e^{-s}) > t_*\right\}.$$

Step 2. Applying Jensen's inequality and the convexity of the function Φ , we have

$$(28) \quad \begin{aligned} \int_0^{\log \frac{1}{\varepsilon}} h(e^{-s}) ds &= \int_{\varepsilon}^1 h(r) \frac{dr}{r} = \int_{\varepsilon}^1 \Phi(q_0(r)) r^{n-1} dr \\ &\leq \int_{\varepsilon}^1 \left(\frac{1}{\omega_{n-1} r^{n-1}} \int_{S(0,r)} \Phi(Q(x)) dS \right) r^{n-1} dr \leq \frac{\Omega_n}{\omega_{n-1}} \cdot M(\varepsilon) = \frac{1}{n} \cdot M(\varepsilon), \end{aligned}$$

where $M(\varepsilon)$ is defined by (22) and, as above, dS is the area element of the surface of integration. Hence, (28) implies that the linear measure $|T|$ of the set $T = \{s \in (0, \log \frac{1}{\varepsilon}) : h(e^{-s}) > M(\varepsilon)\}$ satisfies

$$(29) \quad |T| = \int_T ds \leq \frac{1}{n}.$$

Step 3. We show that

$$(30) \quad q_0^{\frac{1}{p}}(e^{-s}) \leq H_p^{-1}(ns + \log M(\varepsilon)), \quad s \in \left(0, \log \frac{1}{\varepsilon}\right) \setminus T_*,$$

where $T_* := T \cap S_*$ and the set S_* is defined by (27). Observe that

$$\begin{aligned} \left(0, \log \frac{1}{\varepsilon}\right) \setminus T_* &= \left[\left(0, \log \frac{1}{\varepsilon}\right) \setminus S_*\right] \cup \left[\left(0, \log \frac{1}{\varepsilon}\right) \setminus T\right] \\ &= \left[\left(0, \log \frac{1}{\varepsilon}\right) \setminus S_*\right] \cup [S_* \setminus T]. \end{aligned}$$

Remark that inequality (30) is true for $s \in S_* \setminus T$, because (26) is true and the function H_p^{-1} is monotone nondecreasing. Also, by (24),

$$(31) \quad e^{ns} M(\varepsilon) > \Phi(0) + \varepsilon_1 = \tau_0 + \varepsilon_1, \quad s \in \left(0, \log \frac{1}{\varepsilon}\right),$$

for some sufficiently small $\varepsilon_1 > 0$. Thus, since the function $H_p^{-1}(\eta)$ is strictly monotone increasing for $\eta > \tau_0$, (25) and (31) guarantee that

$$(32) \quad t_* < \Phi_p^{-1}(e^{ns} M(\varepsilon)) = H_p^{-1}(ns + \log M(\varepsilon)), \quad s \in \left(0, \log \frac{1}{\varepsilon}\right).$$

Relation (32) shows that (30) is also valid for $s \in (0, \log \frac{1}{\varepsilon}) \setminus S_*$, and hence, for all $s \in (0, \log \frac{1}{\varepsilon}) \setminus T_*$.

Step 4. Since the function H_p^{-1} is monotone nondecreasing, (29) and (30) guarantee that

$$(33) \quad \begin{aligned} \int_{\varepsilon}^1 \frac{dr}{r q_0^{\frac{1}{p}}(r)} &= \int_0^{\log \frac{1}{\varepsilon}} \frac{ds}{q_0^{\frac{1}{p}}(e^{-s})} \geq \int_{(0, \log \frac{1}{\varepsilon}) \setminus T_*} \frac{ds}{H_p^{-1}(ns + \log M(\varepsilon))} \\ &\geq \int_{|T_*|}^{\log \frac{1}{\varepsilon}} \frac{ds}{H_p^{-1}(ns + \log M(\varepsilon))} \geq \int_{\frac{1}{n}}^{\log \frac{1}{\varepsilon}} \frac{ds}{H_p^{-1}(ns + \log M(\varepsilon))} \\ &= \frac{1}{n} \int_{1 + \log M(\varepsilon)}^{n \log \frac{1}{\varepsilon} + \log M(\varepsilon)} \frac{d\eta}{H_p^{-1}(\eta)}. \end{aligned}$$

Note that $1 + \log M(\varepsilon) = \log eM(\varepsilon)$. Therefore, by (33), $\int_{\varepsilon}^1 \frac{dr}{r q_0^{\frac{1}{p}}(r)} \geq \frac{1}{n} \int_{\log eM(\varepsilon)}^{\log \frac{M(\varepsilon)}{\varepsilon^n}} \frac{d\eta}{H_p^{-1}(\eta)}$.

Introducing the new variable $\eta = \log \tau$ and applying (25), we obtain inequality (23). Hence, (21) is true. \square

Theorem 5. Let $Q : \mathbb{B}^n \rightarrow [0, \infty]$ be a Lebesgue measurable function such that

$$(34) \quad \int_{\mathbb{B}^n} \Phi(Q(x)) dm(x) < \infty,$$

where $\Phi : [0, \infty] \rightarrow [0, \infty]$ is a monotone nondecreasing convex function such that

$$(35) \quad \int_{\delta_0}^{\infty} \frac{d\tau}{\tau [\Phi^{-1}(\tau)]^{\frac{1}{p}}} = \infty, \quad p \in (0, \infty),$$

for some $\delta_0 > \tau_0 := \Phi(0)$. Then

$$(36) \quad \int_0^1 \frac{dr}{r q_0^{\frac{1}{p}}(r)} = \infty.$$

Remark 4. Observe that, in Theorem 5, the assumptions about Φ are more general, namely, Φ is allowed to take the value 0. Indeed, the above case reduces to the function $\Phi : [0, \infty] \rightarrow (0, \infty]$ by consideration of the following auxiliary function:

$$\Phi_*(t) = \begin{cases} \Phi(t) & \text{if } \Phi(t) > \delta, \\ \delta & \text{if } \Phi(t) \leq \delta, \end{cases}$$

where δ is an arbitrary number such that $\delta \in (0, \delta_0)$. Observe that properties (34) and (35) remain valid for $\Phi_*(t)$.

Remark 5. We have $[\Phi^{-1}(\tau)]^{\frac{1}{p}} = \Phi_p^{-1}(\tau)$, where $\Phi_p(t) = \Phi(t^p)$; hence, (35) implies that

$$(37) \quad \int_{\delta}^{\infty} \frac{d\tau}{\tau \Phi_p^{-1}(\tau)} = \infty, \quad \delta \in [0, \infty).$$

On the other hand, property (37) with some $\delta \in [0, \infty)$ does not imply (35), in general.

Indeed, property (35) with some $\delta_0 > \tau_0$ clearly implies (37) for $\delta \in [0, \delta_0)$. Also, for $\delta \in (\delta_0, \infty)$ we have

$$(38) \quad 0 \leq \int_{\delta_0}^{\delta} \frac{d\tau}{\tau \Phi_p^{-1}(\tau)} \leq \frac{1}{\Phi_p^{-1}(\delta_0)} \log \frac{\delta}{\delta_0} < \infty,$$

because the function Φ_p^{-1} is monotone nondecreasing and $\Phi_p^{-1}(\delta_0) > 0$. Therefore, (37) is true for all $\delta \in [0, \infty)$.

On the other hand, by the definition of the inverse function, $\Phi_p^{-1}(\tau) \equiv 0$ for all $\tau \in [0, \tau_0]$ with $\tau_0 = \Phi_p(0)$. Thus, in general, condition (37) with $\delta \in [0, \tau_0)$ does not imply (35). If $\tau_0 > 0$, then

$$(39) \quad \int_{\delta}^{\tau_0} \frac{d\tau}{\tau \Phi_p^{-1}(\tau)} = \infty, \quad \delta \in [0, \tau_0).$$

However, property (39) contains no proper information about the function $Q(x)$. Hence, condition (37) with $\delta < \Phi(0)$ definitely does not imply (36).

By property (37), the proof of Theorem 5 reduces to the claim of Lemma 1.

§4. CONDITIONS SUFFICIENT FOR EQUICONTINUITY

Proof of Theorem 1. Without loss of generality, we may assume that $\Phi(0) > 0$. By Proposition 3, it suffices to show that the assumptions of Theorem 1 imply property (12) for every $x_0 \in D$ and for some $\varepsilon_0 < \text{dist}(x_0, \partial D)$. Introducing the variable $t = r/\varepsilon_0$, we obtain

$$(40) \quad \int_{\varepsilon}^{\varepsilon_0} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)} = \int_{\varepsilon/\varepsilon_0}^1 \frac{dt}{t q_{x_0}^{\frac{1}{n-1}}(t\varepsilon_0)} = \int_{\varepsilon/\varepsilon_0}^1 \frac{dt}{t \tilde{q}_0^{\frac{1}{n-1}}(t)},$$

where $\tilde{q}_0(t)$ is the integral mean value of the function $\tilde{Q}(x) := Q(\varepsilon_0 x + x_0)$ on the sphere $|x| = t$, see (11) for the definition. Now, applying Lemma 1 with $p = n - 1$, we obtain

$$(41) \quad \int_{\varepsilon/\varepsilon_0}^1 \frac{dt}{t \tilde{q}_0^{\frac{1}{n-1}}(t)} \geq \frac{1}{n} \int_{\varepsilon M_*(\varepsilon/\varepsilon_0)}^{\frac{M_*(\varepsilon/\varepsilon_0) \varepsilon_0^n}{\varepsilon^n}} \frac{d\tau}{\tau [\Phi^{-1}(\tau)]^{\frac{1}{n-1}}},$$

$$(42) \quad \begin{aligned} M_*(\varepsilon/\varepsilon_0) &= \frac{1}{\Omega_n (1 - (\varepsilon/\varepsilon_0)^n)} \int_{A(\varepsilon/\varepsilon_0, 1, 0)} \Phi(Q(\varepsilon_0 x + x_0)) dm(x) \\ &= \frac{1}{\Omega_n (\varepsilon_0^n - \varepsilon^n)} \int_{A(\varepsilon, \varepsilon_0, x_0)} \Phi(Q(x)) dm(x). \end{aligned}$$

Observe that if $x \in A(\varepsilon, \varepsilon_0, x_0)$, then $|x| \leq |x - x_0| + |x_0| \leq \varepsilon_0 + |x_0|$; thus, (42) implies that

$$M_*(\varepsilon/\varepsilon_0) \leq \frac{\beta_n(x_0)}{\Omega_n (\varepsilon_0^n - \varepsilon^n)} \int_{A(\varepsilon, \varepsilon_0, x_0)} \Phi(Q(x)) \frac{dm(x)}{(1 + |x|^2)^n},$$

where $\beta_n(x_0) = (1 + (\varepsilon + |x_0|)^2)^n$. Hence, for $\varepsilon \leq 1/\sqrt[n]{\varepsilon_0^n - 1/2}$ we have

$$M_*(\varepsilon/\varepsilon_0) \leq \frac{2\beta_n(x_0)}{\Omega_n} M,$$

where M is the constant on the right-hand side of (8). Also, observe that

$$M_*(\varepsilon/\varepsilon_0) > \Phi(0) > 0.$$

Therefore, by (40) and (41),

$$\int_{\varepsilon}^{\varepsilon_0} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)} \geq \frac{1}{n} \int_{\frac{2\varepsilon\beta_n(x_0)M}{\Omega_n}}^{\frac{\Phi(0)\varepsilon_0^n}{\varepsilon^n}} \frac{d\tau}{\tau [\Phi^{-1}(\tau)]^{\frac{1}{n-1}}}.$$

However, the right-hand side of the above inequality tends to infinity by (7); hence, $\int_0^{\varepsilon_0} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)} = \infty$. Now, the required conclusion follows from Proposition 3. The theorem is proved. \square

Corollary 1. *Each of conditions (15)–(20) with $p \in (0, n - 1]$ implies the equicontinuity and normality of the class $\mathfrak{R}_{M,E}^{\Phi,Q}$ for all $M \in (0, \infty)$.*

§5. CONDITIONS SUFFICIENT FOR THE REMOVABILITY OF SINGULARITIES

The proof of Theorem 2. is quite similar to that of Theorem 1; namely, first we verify that the integral $\int_0^{\varepsilon_0} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)}$ diverges. Second, we apply Proposition 2. \square

For a function $f : D \rightarrow \mathbb{R}^n$, a set $E \subset D$, and a point $y \in \mathbb{R}^n$, we define the *multiplicity function* $N(y, f, E)$ as the number of preimages of y in the set E , that is,

$$N(y, f, E) = \text{card} \{x \in E : f(x) = y\}.$$

Corollary 2 (a Sokhotsky–Weierstrass type theorem). *Let $\Phi : [0, \infty) \rightarrow [0, \infty)$ be a monotone nondecreasing convex function. Assume that property (7) is fulfilled for some $\delta_0 > \tau_0 := \Phi(0)$, and that an open discrete Q -mapping $f : D \setminus \{x_0\} \rightarrow \overline{\mathbb{R}^n}$ has an essential singular point $x_0 \in D$. Suppose Q satisfies condition (8) in the domain D and $Q(x) \geq 1$ almost everywhere. Then there exists an F_σ -set $C \subset \overline{\mathbb{R}^n}$ of zero capacity and such that*

$$N(y, f, U \setminus \{x_0\}) = \infty$$

for any neighborhood U of the point x_0 and for all $y \in \overline{\mathbb{R}^n} \setminus C$.

Proof. Let U be an arbitrary neighborhood of the point x_0 . Without loss of generality, we may assume that $x_0 = 0$ and $U = \mathbb{B}^n$. Consider the sets $V_k = B(0, 1/k) \setminus \{0\}$, $k = 1, 2, \dots$. Put

$$(43) \quad C = \bigcup_{k=1}^{\infty} \overline{\mathbb{R}^n} \setminus f(V_k).$$

By Theorem 2, the union of each set $B_k := \overline{\mathbb{R}^n} \setminus f(V_k)$ and the set on the right-hand side of (43) has zero capacity. Hence, C also has zero capacity; see, e.g., [7]. Fix $y \in \overline{\mathbb{R}^n} \setminus C$. Then

$$(44) \quad y \in \bigcap_{k=1}^{\infty} f(V_k).$$

By (44), there exists a sequence $\{x_i\}_{i=1}^{\infty}$ such that $x_i \rightarrow 0$ as $i \rightarrow \infty$ and $f(x_i) = y$, $i = 1, 2, \dots$. Corollary 2 is proved. \square

Corollary 3 (a Liouville type theorem). *Let $\Phi : [0, \infty] \rightarrow [0, \infty]$ be a monotone nondecreasing convex function. Assume that condition (7) is fulfilled for some $\delta_0 > \tau_0 := \Phi(0)$ and that $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}^n}$ is an open discrete Q -mapping at the point $x_0 = \infty$, that is, the mapping $\tilde{f} := f \circ \varphi$ with $\varphi(x) = \frac{x}{|x|^2}$ is an annular $Q'(x)$ -mapping at zero, where $Q'(x) = Q(\frac{x}{|x|^2})$. Suppose Q satisfies condition (8) in the domain $D = \mathbb{R}^n$ and $Q(x) \geq 1$ almost everywhere. Then $\text{cap}(\overline{\mathbb{R}^n} \setminus f(\mathbb{R}^n)) = 0$. In particular, f cannot map the space \mathbb{R}^n into a bounded domain.*

Proof. Suppose the contrary, that is,

$$\text{cap}(\overline{\mathbb{R}^n} \setminus f(\mathbb{R}^n)) > 0.$$

Then, for the auxiliary mapping $\tilde{f} = f(\frac{x}{|x|^2})$, $\tilde{f} : \mathbb{R}^n \setminus \{0\} \rightarrow \overline{\mathbb{R}^n}$, we have $\tilde{f}(\mathbb{R}^n \setminus \{0\}) = f(\mathbb{R}^n \setminus \{0\})$, so that

$$\text{cap}(\overline{\mathbb{R}^n} \setminus \tilde{f}(\mathbb{R}^n \setminus \{0\})) = \text{cap}(\overline{\mathbb{R}^n} \setminus f(\mathbb{R}^n \setminus \{0\})) \geq \text{cap}(\overline{\mathbb{R}^n} \setminus f(\mathbb{R}^n)) > 0.$$

Moreover, note that $\varphi(x) = \frac{x}{|x|^2}$ is a similarity map from the sphere $S(0, r)$ onto the sphere $S(0, 1/r)$, whence $|J(x, \psi)| = (1/|x|)^{2n}$. Using the above observations and changing the variable in the integral in (8), we have

$$\begin{aligned} \int_{\mathbb{R}^n} Q'(x) \frac{dm(x)}{(1+|x|^2)^n} &= \int_{\mathbb{R}^n} Q\left(\frac{x}{|x|^2}\right) \frac{dm(x)}{(1+|x|^2)^n} = \int_{\mathbb{R}^n} Q(y) \cdot \frac{1}{|y|^{2n}} \cdot \frac{dm(y)}{\left(1+\frac{1}{|y|^2}\right)^n} \\ &= \int_{\mathbb{R}^n} Q(y) \frac{dm(y)}{(1+|y|^2)^n} \leq M. \end{aligned}$$

So, by Theorem 2, the mapping \tilde{f} extends by continuity to an open discrete mapping $\tilde{f} : \mathbb{R}^n \rightarrow \overline{\mathbb{R}^n}$. Equivalently, f also extends by continuity to an open discrete mapping $\bar{f} : \overline{\mathbb{R}^n} \rightarrow \overline{\mathbb{R}^n}$. In this case, the set $\bar{f}(\overline{\mathbb{R}^n})$ is simultaneously open and closed in $\overline{\mathbb{R}^n}$, whence $\bar{f}(\overline{\mathbb{R}^n}) = \overline{\mathbb{R}^n}$. However, this contradicts the following assumption made above: $\text{cap}(\overline{\mathbb{R}^n} \setminus \bar{f}(\overline{\mathbb{R}^n})) > 0$. \square

§6. ON THE BRANCHING POINTS OF MAPPINGS WITH INTEGRAL RESTRICTIONS ON THE CHARACTERISTIC

Recall that $y_0 \in D$ is a *branching point* of a mapping $f : D \rightarrow \mathbb{R}^n$ if the restriction $f|_U$ is not a homeomorphism for any neighborhood U of y_0 . Given f , the set of all branching points of f is denoted by B_f . We say that a point $z_0 \in \overline{\mathbb{R}^n}$ is an *asymptotic limit* of a

mapping $f : D \rightarrow \overline{\mathbb{R}^n}$ at a point $b \in \partial D$ if there exists a curve $\alpha : [0, 1) \rightarrow D$, $\alpha(t) \rightarrow b$ as $t \rightarrow 1$, such that $f(\alpha(t)) \rightarrow z_0$ as $t \rightarrow 1$; see §2 in Chapter 7 of [17]. Informally, a mapping f defined on a domain D has $z_0 \in \overline{\mathbb{R}^n}$ as an asymptotic limit at a point b on the boundary of D if there exists a curve that approaches b in D and is such that, along this curve, the mapping f approaches z_0 . The present section contains several consequences of the following assertion obtained by the author earlier; see Theorem 1 in [26].

Proposition 5. *Let $x_0 \in D$. Assume that an open discrete mapping $f : D \setminus \{x_0\} \rightarrow \overline{\mathbb{R}^n}$, $n \geq 2$, satisfies (4) at the point x_0 for any measurable function η with property (6) and for some measurable function $Q(x)$, $Q : D \rightarrow [1, \infty]$. Suppose that x_0 is an essential singular point of f and (12) is true for some $\delta(x_0) > 0$, $\delta(x_0) < \text{dist}(x_0, \partial D)$. Then:*

- I. *If $n \geq 3$ and a point $z_0 \in \overline{\mathbb{R}^n}$ is an asymptotic limit of f at x_0 , then $z_0 \in \overline{f(B_f \cap U)}$ for any neighborhood $U \subset D$ of x_0 .*
- II. *Each point of the set $\overline{\mathbb{R}^n} \setminus f(D \setminus \{x_0\})$ is an asymptotic limit of f at x_0 .*
- III. *If $n \geq 3$, then $(\overline{\mathbb{R}^n} \setminus f(D \setminus \{x_0\})) \subset \overline{f(B_f)}$.*
- IV. *If $n \geq 3$ and $\infty \notin f(D \setminus \{x_0\})$, then*
 - a) *the set $f(B_f)$ is unbounded in \mathbb{R}^n ;*
 - b) *$x_0 \in \overline{B_f}$.*

Theorem 6. *Let $\Phi : [0, \infty] \rightarrow [0, \infty]$ be a monotone nondecreasing convex function. Assume that (7) is true for some $\delta_0 > \tau_0 := \Phi(0)$, and that an open discrete Q -mapping $f : D \setminus \{x_0\} \rightarrow \overline{\mathbb{R}^n}$ has an essential singular point $x_0 \in D$. Suppose Q satisfies condition (8) in the domain D and $Q(x) \geq 1$ almost everywhere. Then the claims I–IV in Proposition 5 are true.*

Proof. As in the proof of Theorem 1, first we prove that the integral $\int_0^{\varepsilon_0} \frac{dr}{r^{q_{x_0} - 1}(r)}$ in (12) diverges. Second, we apply Proposition 5 to obtain the required conclusion. \square

Corollary 4. *Let $\Phi : [0, \infty] \rightarrow [0, \infty]$ be a monotone nondecreasing convex function. Suppose that (7) is true for some $\delta_0 > \tau_0 := \Phi(0)$, and that a Q -mapping $f : D \setminus \{x_0\} \rightarrow \overline{\mathbb{R}^n}$ is a local homeomorphism in a domain $D \setminus \{x_0\}$. Assume that the function Q satisfies condition (8) in D and $Q(x) \geq 1$ almost everywhere. Then f extends to a local homeomorphism $\overline{f} : D \rightarrow \overline{\mathbb{R}^n}$.*

Proof of Corollary 4. easily follows from Theorem 6. Indeed, the condition $B_f = \emptyset$ implies that f extends continuously to the point x_0 . It remains to show that the extended mapping $\overline{f} : D \rightarrow \overline{\mathbb{R}^n}$ is a homeomorphism on a certain ball $B(x_0, \varepsilon_1)$. Assume that this is not the case. Then $x_0 \in \overline{B_f}$. On the other hand, for an arbitrary open discrete mapping $g : G \rightarrow \overline{\mathbb{R}^n}$ of a domain $G \subset \mathbb{R}^n$, $n \geq 2$, the following inequality for the Hausdorff dimension is known: $\dim_H g(B_g) \geq n - 2$; see, e.g., Proposition 5.3 in Chapter 3 of [17]. In the case under consideration, this means that the domain D contains certain points of the set $\overline{B_f}$ besides the point x_0 , a contradiction. Indeed, by assumption, f is a local homeomorphism in $D \setminus \{x_0\}$. \square

§7. CONDITIONS NECESSARY FOR EQUICONTINUITY AND FOR THE REMOVABILITY OF SINGULARITIES

Before proving Theorem 3, we make the following remark.

Remark 6. Without loss of generality, we may assume that, in the definition of the class $\mathfrak{R}_{M,E}^{\Phi,Q}$ used in Theorem 3, the measurable function $Q(x)$ related to Φ by (8) is greater than or equal to 1.

Also, observe that the function $\Phi(t)$ used in Theorem 3 is not constant. Indeed, there are no restrictions on Q in the theorem under consideration. The only exception is the condition $\Phi(t) \equiv \infty$, which implies that the class $\mathfrak{R}_{M,E}^\Phi$ is empty. Moreover, by a well-known convexity criterion (see, e.g., Proposition 5 in § I.4.3 in [4]) the slope $[\Phi(t) - \Phi(0)]/t$ is a monotone nondecreasing function. Thus, *the proof of Theorem 3 reduces to the following statement.*

Lemma 2. *Assume that $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a monotone nondecreasing function, and that*

$$(45) \quad \Phi(t) \geq C \cdot t^{\frac{1}{n-1}}, \quad t \in [T, \infty),$$

for some $C > 0$ and $T \in (0, \infty)$. If the class $\mathfrak{R}_{M,E}^{\Phi,Q}$ is equicontinuous (normal) for all $M \in (0, \infty)$, all sets E of positive capacity, and all Lebesgue measurable functions Q , then (9) is true for all $\delta_* \in (\tau_0, \infty)$, where $\tau_0 := \Phi(+0)$.

Proof. It suffices to consider the case where $D = \mathbb{B}^n$. Suppose the contrary: let property (9) fail, that is,

$$(46) \quad \int_{\delta_0}^{\infty} \frac{d\tau}{\tau \Phi_{n-1}^{-1}(\tau)} < \infty$$

for some $\delta_0 \in (\tau_0, \infty)$, where $\Phi_{n-1}(t) := \Phi(t^{n-1})$. Then we also have

$$(47) \quad \int_{\delta}^{\infty} \frac{d\tau}{\tau \Phi_{n-1}^{-1}(\tau)} < \infty, \quad \delta \in (\tau_0, \infty),$$

because $\Phi^{-1}(\tau) > 0$ for all $\tau > \tau_0$, and $\Phi^{-1}(\tau)$ is a monotone nondecreasing function. Note that, by (45), the inequality $\Phi_{n-1}(t) \geq C \cdot t$, $t \geq T$, is also valid for some $C > 0$ and $T \in (1, \infty)$. Moreover, applying the linear transformation $\alpha\Phi + \beta$, where $\alpha = 1/C$ and $\beta = T$ (see, e.g., (17)), we may assume that

$$(48) \quad \Phi_{n-1}(t) \geq t, \quad t \in [0, \infty).$$

Clearly, we may also assume that $\Phi(t) = t$ for all $t \in [0, 1)$ because the values of the function Φ on the half-open interval $[0, 1)$ provide no information about the set $Q(x) \geq 1$ in (8). Also, property (47) implies the condition $\Phi(t) < \infty$ for all $t < \infty$; see the criterion (17), and also (20). Now, observe that the function $\Psi(t) := t\Phi_{n-1}(t)$ is strictly monotone increasing, $\Psi(1) = \Phi(1)$, and $\Psi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Thus, the functional equation

$$(49) \quad \Psi(K(r)) = \left(\frac{\gamma}{r}\right)^2, \quad r \in (0, 1],$$

with $\gamma = \Phi^{1/2}(1) \geq 1$, is solvable if $K(1) = 1$, and $K(r)$ is a strictly monotone decreasing continuous function such that $K(r) < \infty$, $r \in (0, 1]$, and $K(r) \rightarrow \infty$ as $r \rightarrow 0$. Taking the logarithm in (49), we have $\log K(r) + \log \Phi_{n-1}(K(r)) = 2 \log \frac{\gamma}{r}$. So, by (48), we obtain $\log K(r) \leq \log \frac{\gamma}{r}$, that is,

$$(50) \quad K(r) \leq \frac{\gamma}{r}.$$

Hence, by (49), $\Phi_{n-1}(K(r)) \geq \frac{\gamma}{r}$ and, by (14),

$$(51) \quad K(r) \geq \Phi_{n-1}^{-1}\left(\frac{\gamma}{r}\right).$$

We define the following mappings in the punctured unit ball $\mathbb{B}^n \setminus \{0\}$:

$$f(x) = \frac{x}{|x|} \rho(|x|), \quad f_m(x) = \frac{x}{|x|} \rho_m(|x|), \quad m = 1, 2, \dots,$$

where $\rho(t) = \exp\{I(0) - I(t)\}$, $\rho_m(t) = \exp\{I(0) - I_m(t)\}$, $t \in [0, 1]$, and put

$$I(t) = \int_t^1 \frac{dr}{rK(r)}, \quad I_m(t) = \int_t^1 \frac{dr}{rK_m(r)},$$

and

$$K_m(r) = \begin{cases} K(r) & \text{for } r \geq 1/m, \\ K(\frac{1}{m}) & \text{for } r \in (0, 1/m). \end{cases}$$

Using (51), we obtain

$$I(0) - I(t) = \int_0^t \frac{dr}{rK(r)} \leq \int_0^t \frac{dr}{r\Phi_{n-1}^{-1}(\frac{\gamma}{r})} = \int_{\frac{\gamma}{t}}^{\infty} \frac{d\tau}{\tau\Phi_{n-1}^{-1}(\tau)}, \quad t \in (0, 1],$$

where $\gamma/t \geq \gamma \geq 1 > \Phi(0) = 0$. Thus, by (47),

$$(52) \quad I(0) - I(t) \leq I(0) = \int_0^1 \frac{dr}{rK(r)} < \infty, \quad t \in (0, 1].$$

Also, we have $f_m, f \in C^1(\mathbb{B}^n \setminus \{0\})$ because $K_m(r)$ and $K(r)$ are continuous. We show that the f_m are K_m -quasiconformal in \mathbb{B}^n , where $K_m = K^{n-1}(1/m)$, $f_m(0) = 0$.

Fix $\rho \in (0, 1)$ and a point $x \in \mathbb{B}^n \setminus \{0\}$ such that $|x| = \rho$. Observe that the principal distortions $\lambda_{i_1}(x), \dots, \lambda_{i_{n-1}}(x)$ of the mapping f at the point x in $n-1$ directions i_1, \dots, i_{n-1} tangential to the sphere $|x| = \rho$ are all equal and are given by the following formula:

$$(53) \quad \lambda_{i_k}(x) := \delta_\tau(x) = \frac{|f(x)|}{|x|} = \frac{\exp\left\{\int_0^\rho \frac{dr}{rK(r)}\right\}}{\rho},$$

$k = 1, \dots, n-1$. The value $\lambda_{i_n}(x)$ that corresponds to the perpendicular (radial) direction is computed as follows:

$$(54) \quad \lambda_{i_n}(x) := \delta_r(x) = \frac{\partial|f(x)|}{\partial|x|} = \frac{\exp\left\{\int_0^\rho \frac{dr}{rK(r)}\right\}}{\rho K(\rho)}.$$

For more details about the definition and computation of distortions, see, e.g., §5 in Chapter 1 of [16], see also Proposition 6.3 in Chapter 4 of [11]. Properties (53) and (54) imply that $\delta_r(x) \leq \delta_\tau(x)$, because $K(r) \geq 1$. Hence, applying (10), we see that

$$K_I(x, f) = \frac{\delta_\tau^{n-1}(x) \cdot \delta_r(x)}{\delta_r^n(x)} = K^{n-1}(|x|)$$

at all points $x \in \mathbb{B}^n \setminus \{0\}$; see also [16, § I.2.1]. Note that

$$(55) \quad f_m(x) \equiv f(x) \quad \text{for all } x \text{ with } \frac{1}{m} < |x| < 1, \quad m = 1, 2, \dots$$

Thus, $K_I(x, f_m) = K_I(x, f) = K^{n-1}(|x|)$ for $\frac{1}{m} < |x| < 1$, and $K_I(x, f_m) = K^{n-1}(1/m)$ for $0 < |x| < \frac{1}{m}$ can be computed similarly. Therefore, by Proposition 1, each f_m is an annular Q -mapping in \mathbb{B}^n with $Q(x) = K_I(x, f)$, because $K(r)$ is monotone nonincreasing. By (49),

$$(56) \quad \begin{aligned} \int_{\mathbb{B}^n} \Phi(K_I(x, f_m)) dm(x) &\leq \int_{\mathbb{B}^n} \Phi_{n-1}(K(|x|)) dm(x) = \omega_{n-1} \int_0^1 \frac{\Psi(K(r))}{rK(r)} \cdot r^n dr \\ &\leq \gamma^2 \omega_{n-1} \int_0^1 \frac{dr}{rK(r)} \leq M := \gamma^2 \omega_{n-1} I(0) < \infty. \end{aligned}$$

Note that f_m maps the unit ball \mathbb{B}^n onto the ball with center at the origin and of radius $R = e^{I(0)} < \infty$. Therefore, $f_m \in \mathfrak{R}_{M,E}^{\Phi,Q}$, where M is defined above; so, we may put $E = \overline{\mathbb{R}^n} \setminus B(0, e^{I(0)})$. On the other hand, it is easily seen that

$$(57) \quad \lim_{x \rightarrow 0} |f(x)| = \lim_{t \rightarrow 0} \rho(t) = e^0 = 1,$$

that is, f maps the punctured ball $\mathbb{B}^n \setminus \{0\}$ onto the annulus $1 < |y| < R = e^{I(0)}$. Hence, by (55) and (57),

$$|f_m(x)| = |f(x)| \geq 1 \quad \text{for all } x \text{ with } |x| \geq 1/m, \quad m = 1, 2, \dots,$$

that is, the family $\{f_m\}_{m=1}^{\infty}$ is not equicontinuous at zero. The contradiction obtained shows that (46) cannot be true. \square

Remark 7. Theorem 3 shows that (7) is not merely sufficient, but also necessary for the continuity (normality) of the mapping classes satisfying the integral conditions (8) with a convex monotone nondecreasing function Φ . By Proposition 4, this remark extends to conditions (15)–(20) for $p = n - 1$.

Proof of Theorem 4. Applying the construction described in the proof of Theorem 3 and Lemma 2 and retaining the above notation, consider the mapping

$$f(x) = \frac{x}{|x|} \rho(|x|)$$

on the domain $D = \mathbb{B}^n \setminus \{0\}$. As was mentioned above, f maps the punctured ball $\mathbb{B}^n \setminus \{0\}$ onto the annulus $1 < |y| < R = e^{I(0)}$. This means that f has an essential singularity at zero. Also, as was proved above, f is an annular Q -mapping at zero for $Q(x) = K_I(x, f) = K^{n-1}(|x|)$; moreover, Q satisfies (56), that is, $f \in K_{M,E}^{\Phi,Q}(\mathbb{B}^n \setminus \{0\})$ for some $M > 0$ and a set E such that $\text{cap } E > 0$. \square

§8. APPLICATIONS TO THE SOBOLEV CLASSES

Assume that the first order partial derivatives are defined for a mapping $f : D \rightarrow \mathbb{R}^n$, $n \geq 2$, at almost all points $x \in D$. Then the *outer dilatation* of f at a point x is defined as

$$K_O(x, f) = \frac{\|f'(x)\|^n}{|J(x, f)|},$$

provided that $J(x, f) \neq 0$. If $J(x, f) = 0$, then $K_O(x, f) = 1$ when $f'(x) = 0$, otherwise $K_O(x, f) = \infty$. It is possible to verify that, for almost all $x \in D$, we have $K_I(x, f) \leq K_O^{n-1}(x, f)$, $K_O(x, f) \leq K_I^{n-1}(x, f)$; see, e.g., §3 in Chapter 1 of [16]. The following theorem was obtained by the author earlier (see [27, Theorem 1]):

Theorem 7. *Let $x_0 \in D$, and let $f : D \rightarrow \mathbb{R}^n$ be an open discrete mapping of class $W_{\text{loc}}^{1,n}(D)$ such that $Q := K_O^{n-1}(x, f) \in L_{\text{loc}}^1(D)$ and $m(B_f) = 0$. Then f satisfies condition (4) at the point x_0 for every nonnegative measurable function η with property (6) and for $Q := K_O^{n-1}(x, f)$. The case where x_0 is an essential singular point in D is admissible.*

Therefore, the results mentioned above apply to the Sobolev classes. Also, such results have direct reformulations in terms of the Sobolev classes.

Postscriptum. The present work continues the studies initiated by the prominent mathematician G. D. Suvorov who described “*the ideal (and the goal!) of the function theory as achieving the situation when we have a large number of function classes and, for each class, we have an elaborate catalog of properties (metric and topological)*”.

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INSTITUTE OF APPLIED MATHEMATICS AND MECHANICS, NAS OF UKRAINE, 83114, DONETSK, ROZY LUXEMBOURG STR., 74, UKRAINE

E-mail address: `brusin2006@rambler.ru`, `esevostyanov2009@mail.ru`

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