

# On Generalized Fibonacci Quaternions and Fibonacci-Narayana Quaternions

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**Abstract.** In this paper, we investigate some properties of generalized Fibonacci quaternions and Fibonacci-Narayana quaternions in a generalized quaternion algebra.

**Keywords:** Fibonacci quaternions, generalized Fibonacci quaternions, Fibonacci-Narayana quaternions.

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## 0. Introduction

The Fibonacci numbers was introduced by *Leonardo of Pisa (1170-1240)* in his book *Liber abbaci*, book published in 1202 AD (see [Ko; 01], p. 1, 3). These numbers was used as a model for investigate the growth of rabbit populations (see [Dr, Gi, Gr, Wa; 03]). The Latin name of Leonardo was *Leonardus Pisanus*, also called *Leonardus filius Bonaccii*, shortly *Fibonacci*. This name is attached to the following sequence of numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \dots,$$

with the  $n$ th term given by the formula:

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 2,$$

where  $f_0 = 0, f_1 = 1$ .

Fibonacci numbers was known in India before Leonardo's time and used by the Indian authorities on metrical sciences (see [Pa; 85], p. 230). These numbers have many properties which was studied by many authors (see [Ho; 61], [Cu; 76], [Pa; 85], [Ko; 01]).

Narayana was an outstanding Indian mathematician of the XIV century. From him came to us the manuscript "Bidzhahanity" (incomplete), written in the middle of the XIV century. For Narayana was interesting summation of arithmetic series and magic squares. In the middle of the XIV century he proved a more general summation. Using the following sums

$$\begin{aligned} 1 + 2 + 3 + \dots + n &= S_n^{(1)}, \\ S_1^{(1)} + S_2^{(1)} + \dots + S_n^{(1)} &= S_n^{(2)}, \\ S_1^{(2)} + S_2^{(2)} + \dots + S_n^{(2)} &= S_n^{(3)}, \dots, \end{aligned}$$

Narayana calculated that

$$S_n^{(m)} = \frac{n(n+1)(n+2)\dots(n+m)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot (m+1)}. \quad (*)$$

Narayana applied its rules to the problem of a herd of cows and heifers (see [Yu; 61], [Si; 36], [Si; 85], [Al, Jo; 96]).

**Narayana problem** ([Al, Jo; 96]). *A cow annually brings heifers. Every heifer, beginning from the fourth year of his life also brings heifer. How many cows and calves will be after 20 years?*

Narayana's calculation is in the following:

- 1) a cow within 20 years brings 20 heifers of the first generation;
- 2) the first heifer of the first generation brings 17 heifers second generation, the second heifer of the first generation brings 16 heifers second generation etc. The total in the second generation will be  $17 + 16 + \dots + 1 = S_{17}^{(1)}$  cows and calves;
- 3) the first heifer of the seventeen heifers of the second generation brings 14 heifers of the third generation, the second heifer of the seventeen heifers of the second generation brings 13 heifers of third generation, etc. The total heifers of the first generation bring  $13 + 12 + \dots + 1 = S_{13}^{(1)}$  heads. Now, all heifers of the second generation brings  $S_{14}^{(1)} + S_{13}^{(1)} + \dots + S_1^{(1)} = S_{14}^{(2)}$  heads in the third generation.

Similarly, Narayana calculated total number in the herd after 20 years:

$$n = 1 + 20 + S_{17}^{(1)} + S_{14}^{(2)} + \dots + S_2^{(6)}.$$

Using formula (\*), he obtained:

$$n = 1 + 20 + \frac{17 \cdot 18}{1 \cdot 2} + \frac{14 \cdot 15 \cdot 16}{1 \cdot 2 \cdot 3} + \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = 2745.$$

This problem can be solved in the same way that Fibonacci solved its problem about rabbits (see [Ka; 04], [Ko; 01], [Si; 36], [Si; 85]).

In the beginning of the first year was 1 cow and 1 heifer which born. That is had 2 heads. In the beginning of the second year and in the beginning of the third year the number of heads increased by one. Therefore the number of heads are 3 and 4, respectively. From the fourth year, the number of heads in the herd is defined by recurrence formulae:

$$x_4 = x_3 + x_1, x_5 = x_4 + x_2, \dots, x_n = x_{n-1} + x_{n-3},$$

since the number of cows for any year is equal with the number of cows of the previous year plus the number of heifers which was born (= number of heads that were three years ago) (see [Al, Jo; 96 ]).

We have the sequence

$$2, 3, 4, 6, 9, \dots, u_{n+1} = u_n + u_{n-2}.$$

Computing, we obtain that  $u_{20} = 2745$  (see [Ka; 04], [Ko; 01], [Si; 36], [Si; 85], [Al, Jo; 96 ]).

Now, we can consider the sequence

$$1, 1, 1, 2, 3, 4, 6, 9, \dots, u_{n+1} = u_n + u_{n-2},$$

with  $n \geq 2, u_0 = 0, u_1 = 1, u_2 = 1$ . These numbers are called the *Fibonacci-Narayana numbers* (see [Di, St; 03]).

In the same paper [Di, St; 03], authors proved some basic properties of Fibonacci-Narayana numbers, namely:

- 1)  $u_1 + u_2 + \dots + u_n = u_{n+3} - 1$ .
- 2)  $u_1 + u_4 + u_7 + \dots + u_{3n-2} = u_{3n-1}$ .
- 3)  $u_2 + u_5 + u_8 + \dots + u_{3n-1} = u_{3n}$ .
- 4)  $u_3 + u_6 + u_9 + \dots + u_{3n} = u_{3n+1} - 1$ .

- 5)  $u_{n+m} = u_{n-1}u_{m+2} + u_{n-2}u_m + u_{n-3}u_{m+1}$ .  
6)  $u_{2n} = u_{n+1}^2 + u_{n-1}^2 - u_{n-2}^2$ .  
7) If in the sequences  $\{u_n\}$ ,  $n = 7k + 4$ ,  $n = 7k + 6$ ,  $n = 7k$ , when  $k = 0, 1, 2, \dots$ , then  $u_n$  is even.  
8) If in the sequences  $\{u_n\}$   $n = 8k$ ,  $n = 8k - 1$ ,  $n = 8k - 3$ , when  $k = 0, 1, 2, \dots$ , then  $3 \mid u_n$ .

Another property of Fibonacci-Narayana numbers was proved in [Sh; 06]. For all natural  $n \geq 2$ , we have

$$u_n = \sum_{m=0}^{\lfloor n/3 \rfloor} \mathfrak{C}_{\lfloor n/3 \rfloor}^m u_{n-\lfloor n/3 \rfloor-2m},$$

where  $\lfloor a \rfloor$  is an integer part of  $a$  and  $\mathfrak{C}_n^k = \frac{n!}{k!(n-k)!}$ ,  $k! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot k$ ,  $k \in \mathbb{N}$ .

Let  $\mathbb{H}(\beta_1, \beta_2)$  be the generalized real quaternion algebra, the algebra of the elements of the form  $a = a_1 \cdot 1 + a_2e_2 + a_3e_3 + a_4e_4$ , where  $a_i \in \mathbb{R}$ ,  $i \in \{1, 2, 3, 4\}$ , and the basis elements  $1, e_2, e_3, e_4$  satisfy the following multiplication table:

$\cdot$	1	$e_2$	$e_3$	$e_4$
1	1	$e_2$	$e_3$	$e_4$
$e_2$	$e_2$	$-\beta_1$	$e_4$	$-\beta_1e_3$
$e_3$	$e_3$	$-e_4$	$-\beta_2$	$\beta_2e_2$
$e_4$	$e_4$	$\beta_1e_3$	$-\beta_2e_2$	$-\beta_1\beta_2$

We denote by  $\mathbf{t}(a)$  and  $\mathbf{n}(a)$  the trace and the norm of a real quaternion  $a$ . The norm of a generalized quaternion has the following expression  $\mathbf{n}(a) = a_1^2 + \beta_1a_2^2 + \beta_2a_3^2 + \beta_1\beta_2a_4^2$ . For  $\beta_1 = \beta_2 = 1$ , we obtain the real division algebra  $\mathbb{H}$ .

## 1. Preliminaries

In the present days, several mathematicians studied properties of the Fibonacci sequence. In [Ho; 61], the author generalized Fibonacci numbers and gave many properties of them:

$$h_n = h_{n-1} + h_{n-2}, \quad n \geq 2,$$

where  $h_0 = p, h_1 = q$ , with  $p, q$  being arbitrary integers. In the same paper [Ho; 61], relation (7), the following relation between Fibonacci numbers and generalized Fibonacci numbers was obtained:

$$h_{n+1} = pf_n + qf_{n+1}. \quad (1.1)$$

The same author, in [Ho; 63], defined and studied Fibonacci quaternions and generalized Fibonacci quaternions in the real division quaternion algebra and found a lot of properties of them. For the generalized real quaternion algebra, the Fibonacci quaternions and generalized Fibonacci quaternions are defined in the same way:

$$F_n = f_n \cdot 1 + f_{n+1}e_2 + f_{n+2}e_3 + f_{n+3}e_4,$$

for the  $n$ th Fibonacci quaternions, and

$$H_n = h_n \cdot 1 + h_{n+1}e_2 + h_{n+2}e_3 + h_{n+3}e_4,$$

for the  $n$ th generalized Fibonacci quaternions.

In the same paper, we find the norm formula for the  $n$ th Fibonacci quaternions:

$$\mathbf{n}(F_n) = F_n \overline{F}_n = 3f_{2n+3}, \quad (1.2)$$

where  $\overline{F}_n = f_n \cdot 1 - f_{n+1}e_2 - f_{n+2}e_3 - f_{n+3}e_4$  is the conjugate of the  $F_n$  in the algebra  $\mathbb{H}$ . After that, many authors studied Fibonacci and generalized Fibonacci quaternions in the real division quaternion algebra giving more and surprising new properties (for example, see [Sw; 73], [Sa-Mu; 82] and [Ha; 12]).

M. N. S. Swamy, in [Sw; 73], formula (17), obtained the norm formula for the  $n$ th generalized Fibonacci quaternions:

$$\begin{aligned} \mathbf{n}(H_n) &= H_n \overline{H}_n = \\ &= 3(2pq - p^2)f_{2n+2} + (p^2 + q^2)f_{2n+3}, \end{aligned}$$

where  $\overline{H}_n = h_n \cdot 1 - h_{n+1}e_2 - h_{n+2}e_3 - h_{n+3}e_4$  is the conjugate of the  $H_n$  in the algebra  $\mathbb{H}$ .

Similar to A. F. Horadam, we define the Fibonacci-Narayana quaternions as

$$U_n = u_n \cdot 1 + u_{n+1}e_2 + u_{n+2}e_3 + u_{n+3}e_4,$$

where  $u_n$  are the  $n$ th Fibonacci-Narayana number.

In this paper, we give some properties of generalized Fibonacci quaternions and Fibonacci-Narayana quaternions.

## 2. Generalized Fibonacci Quaternions

As in the case of Fibonacci numbers, numerous results between Fibonacci generalized numbers can be deduced. In the following, we will study some properties of the generalized Fibonacci quaternions in the generalized real quaternion algebra  $\mathbb{H}(\beta_1, \beta_2)$ . Let  $F_n = f_n \cdot 1 + f_{n+1}e_2 + f_{n+2}e_3 + f_{n+3}e_4$  be the  $n$ th Fibonacci quaternion and  $H_n = h_n \cdot 1 + h_{n+1}e_2 + h_{n+2}e_3 + h_{n+3}e_4$  be the  $n$ th generalized Fibonacci quaternion. A first question which can arise is what algebraic structure have these elements? The answer will be found in the below theorem, denoting first a  $n$ th generalized Fibonacci number and a  $n$ th generalized Fibonacci element with  $h_n^{p,q}$ , respectively  $H_n^{p,q}$ . In this way, we emphasis the starting integers  $p$  and  $q$ .

**Theorem 2.1.** *The set  $\mathcal{H}_n = \{H_n^{p,q} / p, q \in \mathbb{Z}\} \cup \{0\}$  is a  $\mathbb{Z}$ -module.*

**Proof.** Indeed,  $aH_n^{p,q} + bH_n^{p',q'} = H_n^{ap+bp',aq+bq'} \in \mathcal{H}_n$ , where  $a, b, p, q, p', q' \in \mathbb{Z}$ .  $\square$

**Theorem 2.2.** *i) For the Fibonacci quaternion elements, we have*

$$\sum_{m=1}^n (-1)^{m+1} F_m = (-1)^{n+1} F_{n-1} + 1 + e_3 + e_4. \quad (2.1)$$

*ii) For the generalized Fibonacci quaternion elements, the following relation is true*

$$\sum_{m=1}^n (-1)^{m+1} H_m^{p,q} = (-1)^{n+1} H_{n-1}^{p,q} - p+q+pe_2+qe_3+pe_4+qe_4 \quad (2.2)$$

**Proof.**

i) From [Cu; 76], we know that

$$\sum_{m=1}^n (-1)^{m+1} f_m = (-1)^{n+1} f_{n-1} + 1. \quad (2.3)$$

It results:

$$\begin{aligned} & \sum_{m=1}^n (-1)^{m+1} F_m = \\ & = \sum_{m=1}^n (-1)^{m+1} f_m + e_2 \sum_{m=1}^n (-1)^{m+1} f_{m+1} + \\ & + e_3 \sum_{m=1}^n (-1)^{m+1} f_{m+2} + e_4 \sum_{m=1}^n (-1)^{m+1} f_{m+3} = \\ & = (-1)^{n+1} f_{n-1} + 1 - e_2 [(-1)^{n+1} f_{n-1} + (-1)^{n+2} f_{n+1}] + \\ & + e_3 [(-1)^{n+1} f_{n-1} + 1 + (-1)^n f_{n+1} + (-1)^{n+1} f_{n+2}] - \\ & - e_4 [(-1)^{n+1} f_{n-1} - 1 + (-1)^{n+2} f_{n+1} + (-1)^{n+3} f_{n+2} + (-1)^{n+4} f_{n+3}] = \\ & = (-1)^{n+1} f_{n-1} + 1 + (-1)^{n+1} e_2 f_n + e_3 (-1)^{n+1} [f_{n+1} + (-1)^{n+1}] - \\ & - e_4 (-1)^{n+1} [-f_{n+2} - (-1)^{n+1}] = \\ & = (-1)^{n+1} (f_{n-1} + f_n e_2 + f_{n+1} e_3 + f_{n+2} e_4) + 1 + e_3 + e_4 = \\ & = (-1)^{n+1} F_{n-1} + 1 + e_3 + e_4. \end{aligned}$$

ii) Using relations (1.1) and (2.3), we have

$$\begin{aligned} & \sum_{m=1}^n (-1)^{m+1} H_m^{p,q} = \\ & = \sum_{m=1}^n (-1)^{m+1} h_m^{p,q} + e_2 \sum_{m=1}^n (-1)^{m+1} h_{m+1}^{p,q} + \\ & + e_3 \sum_{m=1}^n (-1)^{m+1} h_{m+2}^{p,q} + e_4 \sum_{m=1}^n (-1)^{m+1} h_{m+3}^{p,q} = \\ & = \sum_{m=1}^n (-1)^{m+1} p f_{m-1} + \sum_{m=1}^n (-1)^{m+1} q f_m + \\ & + e_2 \sum_{m=1}^n (-1)^{m+1} p f_m + e_2 \sum_{m=1}^n (-1)^{m+1} q f_{m+1} + \\ & + e_3 \sum_{m=1}^n (-1)^{m+1} p f_{m+1} + e_3 \sum_{m=1}^n (-1)^{m+1} q f_{m+2} + \\ & + e_4 \sum_{m=1}^n (-1)^{m+1} p f_{m+2} + e_4 \sum_{m=1}^n (-1)^{m+1} q f_{m+3} = \\ & = p (-1)^{n+1} f_{n-2} - p + q (-1)^{n+1} f_{n-1} + q + \\ & + e_2 p (-1)^{n+1} f_{n-1} + p e_2 + e_2 q [(-1)^{n+1} f_{n+1} - (-1)^{n+1} f_{n-1}] + \\ & + e_3 p [(-1)^{n+1} f_{n+1} - (-1)^{n+1} f_{n-1}] + \end{aligned}$$

$$\begin{aligned}
& +e_3q [(-1)^{n+1} f_{n-1} + 1 + (-1)^n f_{n+1} + (-1)^{n+1} f_{n+2}] + \\
& +e_4p [(-1)^{n+1} f_{n-1} + 1 + (-1)^n f_{n+1} + (-1)^{n+1} f_{n+2}] - \\
& -e_4q [(-1)^{n+1} f_{n-1} - 1 + (-1)^{n+2} f_{n+1} + (-1)^{n+3} f_{n+2} + (-1)^{n+4} f_{n+3}] = \\
& = p(-1)^{n+1} f_{n-2} - p + q(-1)^{n+1} f_{n-1} + q + \\
& +e_2p(-1)^{n+1} f_{n-1} + pe_2 + e_2q(-1)^{n+1} f_n + e_3p(-1)^{n+1} f_n + \\
& +e_3q(-1)^{n+1} [f_{n-1} + (-1)^{n+1} - f_{n+1} + f_{n+2}] + \\
& +e_4p(-1)^{n+1} [f_{n-1} + (-1)^{n+1} - f_{n+1} + f_{n+2}] - \\
& -e_4q(-1)^{n+1} [f_{n-1} - (-1)^{n+1} - f_{n+1} + f_{n+2} - f_{n+3}] = \\
& = p(-1)^{n+1} f_{n-2} - p + q(-1)^{n+1} f_{n-1} + q + \\
& +e_2p(-1)^{n+1} f_{n-1} + pe_2 + e_2q(-1)^{n+1} f_n + e_3p(-1)^{n+1} f_n + \\
& +e_3q(-1)^{n+1} [f_{n+1} + (-1)^{n+1}] + e_4p(-1)^{n+1} [(-1)^{n+1} + f_{n+1}] - \\
& -e_4q(-1)^{n+1} [-f_{n+2} - (-1)^{n+1}] = \\
& = (-1)^{n+1} H_{n-1}^{p,q} - p + q + pe_2 + qe_3 + pe_4 + qe_4. \square
\end{aligned}$$

From the above Theorem, we can remark that all identities valid for the Fibonacci quaternions can be easily adapted in an approximative similar expression for the generalized Fibonacci quaternions, if we use relation (1.1), a true relation in the both algebras  $\mathbb{H}(\beta_1, \beta_2)$  and  $\mathbb{H}$ .

**Proposition 2.3.** *If  $h_{n+1} = pf_n + qf_{n+1} = 0$ , then we have:*

$$H_{n+1}^2 = 3 \frac{q^2}{f_n^2} [f_{2n+1}^2 - f_{n+1}f_{n-2}f_{2n+2}], \quad (2.4)$$

where  $H_{n+1}^2 \in \mathbb{H}(\beta_1, \beta_2)$ .

**Proof.** Since  $h_{n+1} = 0$ , it results that  $\mathbf{t}(H_{n+1}) = h_{n+1} = 0$ , therefore  $\mathbf{n}(H_{n+1}) = H_{n+1}^2$ . From  $h_n = pf_n + qf_{n+1} = 0$ , we have  $p = -\frac{qf_{n+1}}{f_n}$  and we obtain:

$$p^2 + 2pq = \frac{q^2 f_{n+1}^2}{f_n^2} - 2q^2 \frac{f_{n+1}}{f_n} = -\frac{q^2 f_{n+1} f_{n-2}}{f_n^2}$$

and

$$p^2 + q^2 = \frac{q^2 f_{n+1}^2}{f_n^2} + q^2 = q^2 \frac{f_{n+1}^2 + f_n^2}{f_n^2} = q^2 \frac{f_{2n+1}}{f_n^2},$$

since  $f_{n+1}^2 + f_n^2 = f_{2n+1}$ .

It results

$$\begin{aligned}
\mathbf{n}(H_{n+1}) &= 3[(p^2 + 2pq)f_{2n+2} + (p^2 + q^2)f_{2n+1}] = \\
&= 3 \frac{q^2}{f_n^2} [-f_{n+1}f_{n-2}f_{2n+2} + f_{2n+1}^2]. \square
\end{aligned}$$

In the following, we will compute the norm of a Fibonacci quaternion and of a generalized Fibonacci quaternion in the algebra  $\mathbb{H}(\beta_1, \beta_2)$ .

Let  $F_n = f_n \cdot 1 + f_{n+1}e_2 + f_{n+2}e_3 + f_{n+3}e_4$  be the  $n$ th Fibonacci quaternion, then its norm is

$$\mathbf{n}(F_n) = f_n^2 + \beta_1 f_{n+1}^2 + \beta_2 f_{n+2}^2 + \beta_1 \beta_2 f_{n+3}^2.$$

Using recurrence of Fibonacci numbers and relations

$$f_n^2 + f_{n-1}^2 = f_{2n-1}, \quad n \in \mathbb{N}, \quad (2.5)$$

$$f_{2n} = f_n^2 + 2f_n f_{n-1}, \quad n \in \mathbb{N}, \quad (2.6)$$

from [Cu; 76], we have

$$\begin{aligned} \mathbf{n}(F_n) &= f_n^2 + \beta_1 f_{n+1}^2 + \beta_2 f_{n+2}^2 + \beta_1 \beta_2 f_{n+3}^2 = \\ &= f_n^2 + \beta_1 f_{n+1}^2 + \beta_2 (f_{n+2}^2 + \beta_1 f_{n+3}^2) = \\ &= f_{2n+1} + (\beta_1 - 1) f_{n+1}^2 + \beta_2 (f_{2n+5} + (\beta_1 - 1) f_{n+3}^2) = \\ &= f_{2n+1} + \beta_2 f_{2n+5} + (\beta_1 - 1) (f_{n+1}^2 + \beta_2 f_{n+3}^2) = \\ &= (1 + 2\beta_2) f_{2n+1} + 3\beta_2 f_{2n+2} + (\beta_1 - 1) (f_{n+1}^2 + \beta_2 f_{n+3}^2) = \\ &= h_{2n+2}^{1+2\beta_2, 3\beta_2} + (\beta_1 - 1) (f_{n+1}^2 + \beta_2 f_{n+3}^2) = \\ &= h_{2n+2}^{1+2\beta_2, 3\beta_2} + (\beta_1 - 1) (f_{2n+2} - 2f_n f_{n+1} + \beta_2 f_{2n+6} - 2\beta_2 f_{n+2} f_{n+3}) = \\ &= h_{2n+2}^{1+2\beta_2, 3\beta_2} + (\beta_1 - 1) [f_{2n+2} + \beta_2 f_{2n+6} - 2(f_n f_{n+1} + \beta_2 f_{n+2} f_{n+3})] = \\ &= h_{2n+2}^{1+2\beta_2, 3\beta_2} + (\beta_1 - 1) [f_{2n+2} + \beta_2 f_{2n+6} - 2(f_n f_{n+1} + \beta_2 f_{n+2}^2 + \beta_2 f_{n+1} f_{n+2})] = \\ &= h_{2n+2}^{1+2\beta_2, 3\beta_2} + (\beta_1 - 1) [f_{2n+2} + \beta_2 f_{2n+6} - 2(f_n f_{n+1} + \beta_2 f_{n+2}^2 + \beta_2 f_{n+1}^2 + \beta_2 f_n f_{n+1})] = \\ &= h_{2n+2}^{1+2\beta_2, 3\beta_2} + (\beta_1 - 1) [f_{2n+2} + \beta_2 f_{2n+6} - 2(1 + \beta_2) f_n f_{n+1} - 2\beta_2 f_{2n+3}] = \\ &= h_{2n+2}^{1+2\beta_2, 3\beta_2} + (\beta_1 - 1) [f_{2n+2} + \beta_2 f_{2n+4} + \beta_2 f_{2n+3} + \beta_2 f_{2n+4} - \\ &\quad - 2\beta_2 f_{2n+3} - 2(1 + \beta_2) f_n f_{n+1}] = \\ &= h_{2n+2}^{1+2\beta_2, 3\beta_2} + (\beta_1 - 1) [f_{2n+2} + 2\beta_2 f_{2n+4} - \beta_2 f_{2n+3} - 2(1 + \beta_2) f_n f_{n+1}] = \\ &= h_{2n+2}^{1+2\beta_2, 3\beta_2} + (\beta_1 - 1) [f_{2n+2} + 2\beta_2 f_{2n+2} + 2\beta_2 f_{2n+3} - \\ &\quad - \beta_2 f_{2n+3} - 2(1 + \beta_2) f_n f_{n+1}] = \\ &= h_{2n+2}^{1+2\beta_2, 3\beta_2} + (\beta_1 - 1) [(1 + 2\beta_2) f_{2n+2} + \beta_2 f_{2n+3} - 2(1 + \beta_2) f_n f_{n+1}] = \\ &= h_{2n+2}^{1+2\beta_2, 3\beta_2} + (\beta_1 - 1) [h_{2n+3}^{1+2\beta_2, \beta_2} - 2(1 + \beta_2) f_n f_{n+1}] = \\ &= h_{2n+2}^{1+2\beta_2, 3\beta_2} + (\beta_1 - 1) h_{2n+3}^{1+2\beta_2, \beta_2} - 2(\beta_1 - 1) (1 + \beta_2) f_n f_{n+1}. \end{aligned}$$

We just proved

**Theorem 2.4.** *The norm of the  $n$ th Fibonacci quaternion  $F_n$  in a generalized quaternion algebra is*

$$\mathbf{n}(F_n) = h_{2n+2}^{1+2\beta_2, 3\beta_2} + (\beta_1 - 1) h_{2n+3}^{1+2\beta_2, \beta_2} - 2(\beta_1 - 1) (1 + \beta_2) f_n f_{n+1}. \quad (2.7)$$

□

Using the formula (2.7) and the relation (1.1) when  $\beta_1 = \beta_2 = 1$ , we obtain the formula (1.2).

Using the above theorem and relations (2.5) and (2.6), we can compute the norm of a generalized Fibonacci quaternion in a generalized quaternion algebra. Let  $H_n = h_n \cdot 1 + h_{n+1}e_2 + h_{n+2}e_3 + h_{n+3}e_4$  be the  $n$ th generalized Fibonacci quaternion. The norm is

$$\begin{aligned}
\mathbf{n}(H_n^{p,q}) &= h_n^2 + \beta_1 h_{n+1}^2 + \beta_2 h_{n+2}^2 + \beta_1 \beta_2 h_{n+3}^2 = \\
&= (p f_{n-1} + q f_n)^2 + \beta_1 (p f_n + q f_{n+1})^2 + \beta_2 (p f_{n+1} + q f_{n+2})^2 + \\
&+ \beta_1 \beta_2 (p f_{n+2} + q f_{n+3})^2 = \\
&= p^2 (f_{n-1}^2 + \beta_1 f_n^2 + \beta_2 f_{n+1}^2 + \beta_1 \beta_2 f_{n+2}^2) + \\
&+ q^2 (f_n^2 + \beta_1 f_{n+1}^2 + \beta_2 f_{n+2}^2 + \beta_1 \beta_2 f_{n+3}^2) + \\
&+ 2pq (f_{n-1} f_n + \beta_1 f_n f_{n+1} + \beta_2 f_{n+1} f_{n+2} + \beta_1 \beta_2 f_{n+3} f_{n+2}) = \\
&= p^2 h_{2n}^{1+2\beta_2, 3\beta_2} + p^2 (\beta_1 - 1) h_{2n+1}^{1+2\beta_2, \beta_2} - 2p^2 (\beta_1 - 1) (1 + \beta_2) f_{n-1} f_n + \\
&+ q^2 h_{2n+2}^{1+2\beta_2, 3\beta_2} + q^2 (\beta_1 - 1) h_{2n+3}^{1+2\beta_2, \beta_2} - 2q^2 (\beta_1 - 1) (1 + \beta_2) f_n f_{n+1} + \\
&+ 2pq (1 - \beta_1) f_n f_{n-1} + 2pq \beta_1 f_{2n} + 2pq \beta_2 (1 - \beta_1) f_{n+1} f_{n+2} + 2pq \beta_1 \beta_2 f_{2n+4} = \\
&= p^2 h_{2n}^{1+2\beta_2, 3\beta_2} + p^2 (\beta_1 - 1) h_{2n+1}^{1+2\beta_2, \beta_2} + q^2 h_{2n+2}^{1+2\beta_2, 3\beta_2} + q^2 (\beta_1 - 1) h_{2n+3}^{1+2\beta_2, \beta_2} - \\
&- 2p (\beta_1 - 1) (p \beta_2 + p + q) f_{n-1} f_n - 2q^2 (\beta_1 - 1) (1 + \beta_2) f_n f_{n+1} + \\
&+ h_{2n+1}^{2pq\beta_1, 2pq\beta_1\beta_2} + 2pq \beta_1 \beta_2 (f_{2n} + f_{2n+3}) + 2pq \beta_2 (1 - \beta_1) f_{n+1} f_{n+2}.
\end{aligned}$$

From the above, we proved

**Theorem 2.5.** *The norm of the  $n$ th generalized Fibonacci quaternion  $H_n^{p,q}$  in a generalized quaternion algebra is*

$$\begin{aligned}
\mathbf{n}(H_n^{p,q}) &= p^2 h_{2n}^{1+2\beta_2, 3\beta_2} + p^2 (\beta_1 - 1) h_{2n+1}^{1+2\beta_2, \beta_2} + q^2 h_{2n+2}^{1+2\beta_2, 3\beta_2} + q^2 (\beta_1 - 1) h_{2n+3}^{1+2\beta_2, \beta_2} - \\
&- 2p (\beta_1 - 1) (p \beta_2 + p + q) f_{n-1} f_n - 2q^2 (\beta_1 - 1) (1 + \beta_2) f_n f_{n+1} + \\
&+ h_{2n+1}^{2pq\beta_1, 2pq\beta_1\beta_2} + 2pq \beta_1 \beta_2 (f_{2n} + f_{2n+3}) + 2pq \beta_2 (1 - \beta_1) f_{n+1} f_{n+2}. \quad (2.8)
\end{aligned}$$

□

It is known that the expression for the  $n$ th term of a Fibonacci element is

$$f_n = \frac{1}{\sqrt{5}} [\alpha^n - \beta^n] = \frac{\alpha^n}{\sqrt{5}} \left[ 1 - \frac{\beta^n}{\alpha^n} \right], \quad (2.9)$$

where  $\alpha = \frac{1+\sqrt{5}}{2}$  and  $\beta = \frac{1-\sqrt{5}}{2}$ .

From the above, we can compute the following

$$\lim_{n \rightarrow \infty} \mathbf{n}(F_n) = \lim_{n \rightarrow \infty} (f_n^2 + \beta_1 f_{n+1}^2 + \beta_2 f_{n+2}^2 + \beta_1 \beta_2 f_{n+3}^2) =$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \left( \frac{\alpha^{2n}}{5} + \beta_1 \frac{\alpha^{2n+2}}{5} + \beta_2 \frac{\alpha^{2n+4}}{5} + \beta_1 \beta_2 \frac{\alpha^{2n+6}}{5} \right) = \\
&= \operatorname{sgn} E(\beta_1, \beta_2) \cdot \infty
\end{aligned}$$

where

$$\begin{aligned}
E(\beta_1, \beta_2) &= \left( \frac{1}{5} + \frac{\beta_1}{5} \alpha^2 + \frac{\beta_2}{5} \alpha^4 + \frac{\beta_1 \beta_2}{5} \alpha^6 \right) = \\
&= \frac{1}{5} (1 + \beta_1 (\alpha + 1) + \beta_2 (3\alpha + 2) + \beta_1 \beta_2 (8\alpha + 5)) = \\
&= \frac{1}{5} [1 + \beta_1 + 2\beta_2 + 5\beta_1 \beta_2 + \alpha (\beta_1 + 3\beta_2 + 8\beta_1 \beta_2)], \text{ since } \alpha^2 = \alpha + 1.
\end{aligned}$$

If  $E(\beta_1, \beta_2) > 0$ , there exist a number  $n_1 \in \mathbb{N}$  such that for all  $n \geq n_1$  we have

$$h_{2n+2}^{1+2\beta_2, 3\beta_2} + (\beta_1 - 1)h_{2n+3}^{1+2\beta_2, \beta_2} - 2(\beta_1 - 1)(1 + \beta_2) f_n f_{n+1} > 0.$$

In the same way, if  $E(\beta_1, \beta_2) < 0$ , there exist a number  $n_2 \in \mathbb{N}$  such that for all  $n \geq n_2$  we have

$$h_{2n+2}^{1+2\beta_2, 3\beta_2} + (\beta_1 - 1)h_{2n+3}^{1+2\beta_2, \beta_2} - 2(\beta_1 - 1)(1 + \beta_2) f_n f_{n+1} < 0.$$

Therefore for all  $\beta_1, \beta_2 \in \mathbb{R}$  with  $E(\beta_1, \beta_2) \neq 0$ , in the algebra  $\mathbb{H}(\beta_1, \beta_2)$  there is a natural number  $n_0 = \max\{n_1, n_2\}$  such that  $\mathbf{n}(F_n) \neq 0$ , hence  $F_n$  is an invertible element for all  $n \geq n_0$ . Using the same arguments, we can compute

$$\begin{aligned}
\lim_{n \rightarrow \infty} (\mathbf{n}(H_n^{p,q})) &= \lim_{n \rightarrow \infty} (h_n^2 + \beta_1 h_{n+1}^2 + \beta_2 h_{n+2}^2 + \beta_1 \beta_2 h_{n+3}^2) = \\
&= \lim_{n \rightarrow \infty} [(pf_{n-1} + qf_n)^2 + \beta_1 (pf_n + qf_{n+1})^2 + \beta_2 (pf_{n+1} + qf_{n+2})^2 + \\
&\quad + \beta_1 \beta_2 (pf_{n+2} + qf_{n+3})^2] = \\
&= \operatorname{sgn} E'(\beta_1, \beta_2) \cdot \infty
\end{aligned}$$

where

$$\begin{aligned}
E'(\beta_1, \beta_2) &= \frac{1}{5} [(p + \alpha q)^2 + \beta_1 (p\alpha + \alpha^2 q)^2 + \beta_2 (p\alpha^2 + \alpha^3 q)^2 + \\
&\quad + \beta_1 \beta_2 (p\alpha^3 + \alpha^4 q)^2] = \\
&= \frac{1}{5} (p + \alpha q)^2 [1 + \beta_1 \alpha^2 + \beta_2 \alpha^4 + \beta_1 \beta_2 \alpha^6] = \\
&= \frac{1}{5} (p + \alpha q)^2 E(\beta_1, \beta_2).
\end{aligned}$$

Therefore for all  $\beta_1, \beta_2 \in \mathbb{R}$  with  $E'(\beta_1, \beta_2) \neq 0$  in the algebra  $\mathbb{H}(\beta_1, \beta_2)$  there exist a natural number  $n'_0$  such that  $\mathbf{n}(H_n^{p,q}) \neq 0$ , hence  $H_n^{p,q}$  is an invertible element for all  $n \geq n'_0$ .

Now, we proved

**Theorem 2.6.** *For all  $\beta_1, \beta_2 \in \mathbb{R}$  with  $E'(\beta_1, \beta_2) \neq 0$ , there exist a natural number  $n'$  such that for all  $n \geq n'$  Fibonacci elements  $F_n$  and generalized Fibonacci elements  $H_n^{p,q}$  are invertible elements in the algebra  $\mathbb{H}(\beta_1, \beta_2)$ .  $\square$*

**Remark 2.7.** Algebra  $\mathbb{H}(\beta_1, \beta_2)$  is not always a division algebra, and sometimes can be difficult to find an example of invertible element. Above Theorem provides us infinite sets of invertible elements in this algebra, namely Fibonacci elements and generalized Fibonacci elements.

### 3. Fibonacci-Narayana Quaternions

In this section, we will study some properties of Fibonacci-Narayana elements in the algebra  $\mathbb{H}(\beta_1, \beta_2)$ .

**Theorem 3.1.** *For the Fibonacci-Narayana quaternion  $U_n$ , we have*

$$\begin{aligned} a) \quad & \sum_{m=0}^n U_m = U_{n+3} - U_2, \\ b) \quad & \sum_{m=0}^n U_{3m} = U_{3n+1} - 1 - e_4. \end{aligned}$$

**Proof.** a)

$$\sum_{m=0}^n U_m = \sum_{m=0}^n u_m + e_2 \sum_{m=1}^{n+1} u_m + e_3 \sum_{m=2}^{n+2} u_m + e_4 \sum_{m=3}^{n+3} u_m =$$

Since  $u_0 = 0$ , we consider that the term  $\sum_{m=0}^n u_m$  is equal with  $\sum_{m=1}^n u_m$ . We can use property 1) from Introduction and we obtain

$$= u_{n+3} - 1 + e_2(u_{n+4} - 1) + e_3(u_{n+5} - 2) + e_4(u_{n+6} - 3) =$$

$$= U_{n+3} - (1 + e_2 + 2e_3 + 3e_4) = U_{n+3} - U_2.$$

b) Since  $u_0 = 0$ , the term  $\sum_{m=0}^n u_{3m}$  is equal with  $\sum_{m=1}^n u_{3m}$ , therefore

$$\sum_{m=0}^n U_{3m} = \sum_{m=0}^n u_{3m} + e_2 \sum_{m=0}^n u_{3m+1} + e_3 \sum_{m=0}^n u_{3m+2} + e_4 \sum_{m=0}^n u_{3m+3} =$$

using properties 4), 2), 3), and again 4), we have

$$= u_{3n+1} - 1 + u_{3n+2}e_2 + u_{3n+3}e_3 + (u_{3n+4} - 1)e_4 = U_{3n+1} - 1 - e_4.$$

□

Let  $\{u_n\}$  be a Fibonacci-Narayana sequence, and let  $U_n = u_n \cdot 1 + u_{n+1}e_2 + u_{n+2}e_3 + u_{n+3}e_4$  be the  $n$ th Fibonacci-Narayana quaternion.

The function  $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$  is called *the generating function* for the sequence  $\{a_0, a_1, a_2, \dots\}$ . In [Ha; 12], the author found a generating function for Fibonacci quaternions. In the following theorem, we established the generating function for Fibonacci-Narayana quaternions.

**Theorem 3.2.** *The generating function for the Fibonacci-Narayana quaternion  $U_n$  is*

$$G(t) = \frac{U_0 + (U_1 - U_0)t + (U_2 - U_1)t^2}{1 - t - t^3} = \frac{e_1 + e_2 + e_3 + (1 + e_3)t + (e_2 + e_3)t^2}{1 - t - t^3}. \quad (3.1)$$

**Proof.** Assuming that the generating function of the quaternion Fibonacci-Narayana sequence  $\{U_n\}$  has the form  $G(t) = \sum_{n=0}^{\infty} U_n t^n$ , we obtain

that

$$\begin{aligned} & \sum_{n=0}^{\infty} U_n t^n - t \sum_{n=0}^{\infty} U_n t^n - t^3 \sum_{n=0}^{\infty} U_n t^n = \\ & = U_0 + U_1 t + U_2 t^2 + U_3 t^3 + \dots - \\ & - U_0 t - U_1 t^2 - U_2 t^3 - U_3 t^4 - \dots - \\ & - U_0 t^3 - U_1 t^4 - U_2 t^5 - U_3 t^6 - \dots = \\ & = U_0 + (U_1 - U_0)t + (U_2 - U_1)t^2, \text{ since } U_n = U_{n-1} + U_{n-3}, n \geq 3 \text{ and the} \\ & \text{coefficients of } t^n \text{ for } n \geq 3 \text{ are equal with zero.} \end{aligned}$$

It results

$$U_0 + (U_1 - U_0)t + (U_2 - U_1)t^2 = \sum_{n=0}^{\infty} U_n t^n (1 - t - t^3),$$

or in equivalent form

$$\frac{U_0 + (U_1 - U_0)t + (U_2 - U_1)t^2}{1 - t - t^3} = \sum_{n=0}^{\infty} U_n t^n.$$

The theorem is proved.  $\square$

**Theorem 3.3.** (The Binet-Cauchy formula for Fibonacci-Narayana numbers) *Let  $u_n = u_{n-1} + u_{n-3}$ ,  $n \geq 3$  be the  $n$ th Fibonacci-Narayana number, then*

$$u_n = \frac{1}{(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)} [\alpha^{n+1}(\gamma - \beta) + \beta^{n+1}(\alpha - \gamma) + \gamma^{n+1}(\beta - \alpha)], \quad (3.2)$$

where  $\alpha, \beta, \gamma$  are the solutions of the equation  $t^3 - t^2 - 1 = 0$ .

**Proof.** Supposing that  $u_n = A\alpha^n + B\beta^n + C\gamma^n$ ,  $A, B, C \in \mathbb{C}$  and using the recurrence formula for the Fibonacci-Narayana numbers,  $u_n = u_{n-1} + u_{n-3}$ , it results that  $\alpha, \beta, \gamma$  are the solutions of the equation  $t^3 - t^2 - 1 = 0$ . Since  $u_0 = 0, u_1 = 1, u_2 = 1$ , we obtain the following system

$$\begin{cases} A + B + C = 0, \\ A\alpha + B\beta + C\gamma = 1, \\ A\alpha^2 + B\beta^2 + C\gamma^2 = 1. \end{cases} \quad (3.3)$$

The determinant of this system is a Vandermonde determinant and can be computed easily. It is  $\Delta = (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) \neq 0$ .

Using the Cramer's rule, the solutions of the system (3.3) are

$$A = \frac{\alpha(\gamma - \beta)}{(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)} = \frac{\alpha}{(\beta - \alpha)(\gamma - \alpha)},$$

$$B = \frac{\beta(\alpha - \gamma)}{(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)} = \frac{\beta}{(\alpha - \beta)(\gamma - \beta)},$$

$$C = \frac{\gamma(\beta - \alpha)}{(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)} = \frac{\gamma}{(\beta - \gamma)(\alpha - \gamma)},$$

therefore relation (3.2) is true.  $\square$

**Theorem 3.4.** (The Binet-Cauchy formula for the Fibonacci-Narayana quaternions) *Let  $U_n = u_n \cdot 1 + u_{n+1}e_2 + u_{n+2}e_3 + u_{n+3}e_4$  be the  $n$ th Fibonacci-Narayana quaternion, then*

$$U_n = D \frac{\alpha^{n+1}}{(\beta-\alpha)(\gamma-\alpha)} + E \frac{\beta^{n+1}}{(\alpha-\beta)(\gamma-\beta)} + F \frac{\gamma^{n+1}}{(\beta-\gamma)(\alpha-\gamma)}, \quad (3.4)$$

where  $\alpha, \beta, \gamma$  are the solutions of the equation  $t^3 - t^2 - 1 = 0$  and

$$D = 1 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3,$$

$$E = 1 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3,$$

$$F = 1 + \gamma e_1 + \gamma^2 e_2 + \gamma^3 e_3.$$

**Proof.** Using relation (3.2), we have that

$$\begin{aligned} U_n &= u_n \cdot 1 + u_{n+1}e_2 + u_{n+2}e_3 + u_{n+3}e_4 = \\ &= \frac{1}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)} [(\alpha^{n+1}(\gamma-\beta) + \beta^{n+1}(\alpha-\gamma) + \gamma^{n+1}(\beta-\alpha)) \cdot 1 + \\ &+ (\alpha^{n+2}(\gamma-\beta) + \beta^{n+2}(\alpha-\gamma) + \gamma^{n+2}(\beta-\alpha)) e_1 + \\ &+ (\alpha^{n+3}(\gamma-\beta) + \beta^{n+3}(\alpha-\gamma) + \gamma^{n+3}(\beta-\alpha)) e_2 + \\ &+ (\alpha^{n+4}(\gamma-\beta) + \beta^{n+4}(\alpha-\gamma) + \gamma^{n+4}(\beta-\alpha)) e_3] = \\ &= \frac{1}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)} [\alpha^{n+1}(\gamma-\beta)(1 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3) + \\ &+ \beta^{n+1}(\alpha-\gamma)(1 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3) + \\ &+ \gamma^{n+1}(\beta-\alpha)(1 + \gamma e_1 + \gamma^2 e_2 + \gamma^3 e_3)]. \end{aligned}$$

□

For negative  $n$ , the  $n$ th Fibonacci-Narayana number will be defined as  $u_n = u_{n+3} - u_{n+2}$ ,  $u_0 = 0, u_1 = 1, u_2 = 1$ . Accordingly defined the Fibonacci-Narayana quaternion  $U_n$  for negative  $n$ .

**Theorem 3.5.** *Let  $U_n = u_n \cdot 1 + u_{n+1}e_2 + u_{n+2}e_3 + u_{n+3}e_4$  be the  $n$ th Fibonacci-Narayana quaternion, therefore the following relations are true:*

$$1) \sum_{i=0}^n \mathfrak{C}_n^i U_{2n-2i-1} = U_{3n-1}.$$

$$2) \sum_{i=0}^n \mathfrak{C}_n^i U_{3n-2i-1} = U_{4n-1}.$$

**Proof.**

1) Using the Newton's formula, it results that

$$(t^2 + 1)^n = \mathfrak{C}_n^0 (t^2)^n + \mathfrak{C}_n^1 (t^2)^{n-1} + \mathfrak{C}_n^2 (t^2)^{n-2} + \dots + \mathfrak{C}_n^n =$$

$$\begin{aligned}
&= \mathbb{C}_n^0 t^{2n} + \mathbb{C}_n^1 t^{2n-2} + \mathbb{C}_n^2 t^{2n-4} + \dots + \mathbb{C}_n^n. \text{ From here, we have that} \\
&\sum_{i=0}^n \mathbb{C}_n^i U_{2n-2i-1} = \mathbb{C}_n^0 U_{2n-1} + \mathbb{C}_n^1 U_{2n-3} + \mathbb{C}_n^2 U_{2n-5} + \dots + \mathbb{C}_n^n U_{-1} = \\
&= \mathbb{C}_n^0 \left( D \frac{\alpha^{2n}}{(\beta-\alpha)(\gamma-\alpha)} + E \frac{\beta^{2n}}{(\alpha-\beta)(\gamma-\beta)} + F \frac{\gamma^{2n}}{(\beta-\gamma)(\alpha-\gamma)} \right) + \\
&+ \mathbb{C}_n^1 \left( D \frac{\alpha^{2n-2}}{(\beta-\alpha)(\gamma-\alpha)} + E \frac{\beta^{2n-2}}{(\alpha-\beta)(\gamma-\beta)} + F \frac{\gamma^{2n-2}}{(\beta-\gamma)(\alpha-\gamma)} \right) + \dots + \\
&+ \mathbb{C}_n^n \left( D \frac{1}{(\beta-\alpha)(\gamma-\alpha)} + E \frac{1}{(\alpha-\beta)(\gamma-\beta)} + F \frac{1}{(\beta-\gamma)(\alpha-\gamma)} \right) = \\
&= D \frac{1}{(\beta-\alpha)(\gamma-\alpha)} (\mathbb{C}_n^0 \alpha^{2n} + \mathbb{C}_n^1 \alpha^{2n-2} + \dots + \mathbb{C}_n^n 1) + \\
&+ E \frac{1}{(\alpha-\beta)(\gamma-\beta)} (\mathbb{C}_n^0 \beta^{2n} + \mathbb{C}_n^1 \beta^{2n-2} + \dots + \mathbb{C}_n^n 1) + \\
&+ F \frac{1}{(\beta-\gamma)(\alpha-\gamma)} (\mathbb{C}_n^0 \gamma^{2n} + \mathbb{C}_n^1 \gamma^{2n-2} + \dots + \mathbb{C}_n^n 1) = \\
&= D \frac{1}{(\beta-\alpha)(\gamma-\alpha)} (\alpha^2 + 1)^n + E \frac{1}{(\alpha-\beta)(\gamma-\beta)} (\beta^2 + 1)^n + F \frac{1}{(\beta-\gamma)(\alpha-\gamma)} (\gamma^2 + 1)^n = \\
&= D \frac{1}{(\beta-\alpha)(\gamma-\alpha)} \alpha^{3n} + E \frac{1}{(\alpha-\beta)(\gamma-\beta)} \beta^{3n} + F \frac{1}{(\beta-\gamma)(\alpha-\gamma)} \gamma^{3n} = U_{3n-1}.
\end{aligned}$$

We used that  $\alpha^3 = \alpha^2 + 1$ ,  $\beta^3 = \beta^2 + 1$ ,  $\gamma^3 = \gamma^2 + 1$ .

2) Since  $t^3 = t^2 + 1$ , starting from relation  $(t^3 + t)^n = t^{4n}$ , for  $t \in \{\alpha, \beta, \gamma\}$ , by straightforward calculations as in 2), we obtain the asked relation.  $\square$

**Conclusions.** In this paper we investigated some new properties of generalized Fibonacci quaternions and Fibonacci-Narayana quaternions. Since Fibonacci-Narayana quaternions was not intensive studied until now, we expect to find in the future more and surprising new properties. We study these elements for the beauty of the relations obtained, but the main reason is that the elements of this type, namely Fibonacci  $X$  elements, where  $X \in \{\text{quaternions}, \text{generalized quaternions}\}$ , can provide us many important information in the algebra  $\mathbb{H}(\beta_1, \beta_2)$ , as for example: sets of invertible elements in algebraic structures without division.

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## References

- [Al, Jo; 96 ] J.P. Allouche, T. Johnson, *Narayana's Cows and Delayed Morphisms*, [recherche.ircam.fr/equipes/repmus/jim96/actes/Allouche.ps](http://recherche.ircam.fr/equipes/repmus/jim96/actes/Allouche.ps)
- [Cu; 76] I. Cucurezeanu, *Exercices in Arithmetics and Numbers Theory*, Ed. Tehnică, Bucharest, 1976. [in Romanian]
- [Di, St; 03] T. V. Didkivska, M. V. St'opochkina, *Properties of Fibonacci-Narayana numbers*, In the World of Mathematics, **9**(1)(2003), 29–36. [in Ukrainian]
- [Dr, Gi, Gr, Wa; 03], A. Dress, R. Giegerich, S. Grūnewald, H. Wagner, *Fibonacci-Cayley Numbers and Repetition Patterns in Genomic DNA*, Ann. Comb. **7**(2003) 259–279.
- [Ha; 12] S. Halici, *On Fibonacci Quaternions*, Adv. in Appl. Clifford Algebras, **22**(2)(2012), 321-327.
- [Ho; 61] A. F. Horadam, *A Generalized Fibonacci Sequence*, Amer. Math. Monthly, **68**(1961), 455-459.
- [Ho; 63] A. F. Horadam, *Complex Fibonacci Numbers and Fibonacci Quaternions*, Amer. Math. Monthly, **70**(1963), 289-291.
- [Ka; 04] S. Kak, *The Golden Mean and the Physics of Aesthetics*, Archive of Physics: physics/0411195, 2004, <http://arxiv.org/ftp/physics/papers/0411/0411195.pdf>.
- [Ko; 01] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, A Wiley-Interscience publication, U.S.A, 2001.
- [Pa; 85] P. Singh, *The So-called Fibonacci Numbers in Ancient and Medieval India*, Historia Mathematica, **12**(1985), 229-244.
- [Sa-Mu; 82] P. V. Satyanarayana Murthy, *Fibonacci-Cayley Numbers*, The Fibonacci Quarterly, **20**(1)(1982), 59-64.
- [Si; 36] A.N. Singh, *On the use of series in Hindu mathematics*, Osiris 1, 1936, p. 606-628. (<http://www.anaphoria.com/hindu.PDF>)
- [Si; 85] P. Singh, *The so-called Fibonacci numbers in ancient and medieval India*, Historia Mathematica **12**(1985), 229-244.
- [Sh; 06] V. Shpakivskyi, *On the one property of Fibonacci-Narayana numbers*, Mathematics, no. **11**(359)(2006), 12–14. [in Ukrainian]

[Sw; 73] M. N. S. Swamy, *On generalized Fibonacci Quaternions*, The Fibonacci Quarterly, **11**(5)(1973), 547-549.

[Yu; 61] A. P. Yushkevich, *History of Mathematics in the Middle Ages*, Gos. izd. fiz.-mat. lit., Moskow, 1961.

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