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Solutions of some partial differential equations with variable coefficients by properties of monogenic functions

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Abstract. We study some partial differential equations, by using the properties of Gateaux differentiable functions on a commutative algebra. It is proved that components of differentiable functions satisfy some partial differential equations with coefficients related to properties of the bases of subspaces of the corresponding algebra.

Keywords. Monogenic function, commutative algebra, partial differential equations.

1. Introduction

The idea of studying the partial differential equation by using the properties of differentiable functions on algebras is not new. The first step in this direction was the connection established between complex differential functions and harmonic functions. Ketchum [1] extended this idea to the three-dimensional Laplace equation, by using the algebra of functions associated with the equation.

The so-called biharmonic bases in commutative algebras and monogenic functions on these algebras associated with the biharmonic equation are studied in [2, 3]. An interesting solution of the three-dimensional Laplace equation has been elaborated in [4], by defining a related commutative and associative algebra over the field of complex numbers. These ideas were generalized in [5] to a wide class of partial differential equations (PDEs) with constant coefficients. Here, we propose a further generalization to the case of PDEs with linearly dependent variable coefficients.

2. Differentiability on commutative algebras

Let $\mathbf{A}$ be an infinite-dimensional (or finite-dimensional) commutative unitary Banach algebra over a field K of characteristic 0. Assume that the set of vectors $\vec{e}_n$, $n = 1, 2, \dots$ is a basis of $\mathbf{A}$. Suppose $\mathbf{B}$ is an m-dimensional subspace of $\mathbf{A}$ with the basis $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m$, $m \in \mathbf{N}$. Now, any function $\vec{f} : \mathbf{B} \to \mathbf{A}$ is of the form

$$\vec{f}(\vec{x}) = \sum_{k=1}^{\infty} \vec{e}_k u_k(\vec{x}),$$

where $u_k(\vec{x}) = u_k(x_1, x_2, \dots, x_m)$ are K-valued functions of m variables $x_i \in K$. We will assume that all considered series are convergent in $\mathbf{A}$.

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Definition 2.1. *$\vec{f}(\vec{x})$ is called differentiable at a point $\vec{x}_0 \in \mathbf{B}$, if there exists a unique element $\vec{f}'(\vec{x}_0) \in \mathbf{A}$ such that, for any $\vec{h} \in \mathbf{B}$,*

$$\vec{h}\vec{f}'(\vec{x}_0) = \lim_{\varepsilon \to 0} \frac{\vec{f}(\vec{x}_0 + \varepsilon\vec{h}) - \vec{f}(\vec{x}_0)}{\varepsilon}, \tag{2.1}$$

where $\vec{h}\vec{f}'(\vec{x}_0)$ is the product of elements $\vec{h}$ and $\vec{f}'(\vec{x}_0)$ in the algebra $\mathbf{A}$.

It should be keep in mind that $\varepsilon \in K$ in accordance with the algebra $\mathbf{A}$. The classification of monogenic functions in a finite-dimensional commutative algebra is performed in [6]. The element $\vec{f}'(\vec{x}_0)$ is called as the Gateaux derivative of $\vec{f}$ at the point $\vec{x}_0$. For $\mathbf{A} = \mathbf{B} = \mathbf{C}$, this definition is also equivalent to the (complex) differentiability of the complex function $\vec{f}$, and $\vec{f}'(\vec{x}_0)$ becomes the usual complex derivative.

We say that $\vec{f} : \mathbf{B} \longrightarrow \mathbf{A}$ is differentiable (in $\mathbf{B}$) or monogenic if it is differentiable at any point of $\mathbf{B}$.

Theorem 2.1. *Suppose that, for some $1 \leq l \leq m$, there exists $\vec{e}_l^{\,-1}$. Then a function $\vec{f}(\vec{x}) = \sum_{k=1}^{\infty} \vec{e}_k u_k(\vec{x})$ is monogenic, iff there exists the function $\vec{f}' : \mathbf{B} \to \mathbf{A}$ such that, for all $k = 0, 1, \dots, m$, and $\forall\, \vec{x} \in \mathbf{B}$,*

$$\vec{e}_k \vec{f}'(\vec{x}) = \lim_{\varepsilon \to 0} \frac{\vec{f}(\vec{x} + \varepsilon\vec{e}_k) - \vec{f}(\vec{x})}{\varepsilon}, \tag{2.2}$$

where $\vec{f}'$ does not depend on $\vec{e}_k$.

Proof. Suppose that (2.2) is fulfilled. Since, by assumption, there exists $\vec{e}_l^{\,-1}$ $(1 \leq l \leq m)$, we have

$$\vec{e}_l \vec{f}' = \lim_{\varepsilon \to 0} \frac{\vec{f}(\vec{x} + \varepsilon\vec{e}_1) - \vec{f}(\vec{x})}{\varepsilon} = \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_l}$$

or, equivalently,

$$\begin{aligned}
\vec{e}_1 \vec{f}' &= \lim_{\varepsilon \to 0} \tfrac{\vec{f}(\vec{x}+\varepsilon\vec{e}_1) - \vec{f}(\vec{x})}{\varepsilon} = \textstyle\sum_{k=1}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_1} = \vec{e}_1 \vec{e}_l^{\,-1} \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_l}, \\
\vec{e}_2 \vec{f}' &= \lim_{\varepsilon \to 0} \tfrac{\vec{f}(\vec{x}+\varepsilon\vec{e}_2) - \vec{f}(\vec{x})}{\varepsilon} = \textstyle\sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_2} = \vec{e}_2 \vec{e}_l^{\,-1} \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_l}, \\
&\vdots \\
\vec{e}_m \vec{f}' &= \lim_{\varepsilon \to 0} \tfrac{\vec{f}(\vec{x}+\varepsilon\vec{e}_m) - \vec{f}(\vec{x})}{\varepsilon} = \textstyle\sum_{k=0}^{n} \vec{e}_k \frac{\partial u_k}{\partial x_m} = \vec{e}_m \vec{e}_l^{\,-1} \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_l}.
\end{aligned} \tag{2.3}$$

Now, let us consider $\vec{h} = \sum_{k=1}^{m} h_k \vec{e}_k$. It follows from (2.3) that

$$\begin{aligned}
h_1 \vec{e}_1 \vec{f}' &= h_1 \textstyle\sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_1}, \\
h_2 \vec{e}_2 \vec{f}' &= h_2 \textstyle\sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_2}, \\
&\vdots \\
h_m \vec{e}_m \vec{f}' &= h_m \textstyle\sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_m}.
\end{aligned}$$

This implies that

$$\vec{h}\vec{f}' = h_1 \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_1} + h_2 \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_2} + \cdots + h_m \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_m}$$
$$= \lim_{\varepsilon \to 0} \frac{\vec{f}(\vec{x}+\varepsilon\vec{h}) - \vec{f}(\vec{x})}{\varepsilon}.$$

Furthermore, it follows from (2.3) that

$$h_1 \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_1} + h_2 \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_2} + \cdots + h_m \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_m}$$
$$= h_1 \vec{e}_1 \vec{e}_l^{\,-1} \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_l} + h_2 \vec{e}_2 \vec{e}_l^{\,-1} \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_l} + \cdots$$
$$+ h_m \vec{e}_m \vec{e}_l^{\,-1} \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_l}.$$

Hence, for every $\vec{h} \in \mathbf{B}$,

$$\vec{h}\vec{e}_l^{\,-1} \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_l} = \lim_{\varepsilon \to 0} \frac{\vec{f}(\vec{x}+\varepsilon\vec{h}) - \vec{f}(\vec{x})}{\varepsilon}$$

or

$$\vec{f}' = \vec{e}_l^{\,-1} \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_l}. \tag{2.4}$$

□

The set of (2.4) implies the following Cauchy–Riemann type of conditions for a differentiable function $\vec{f}$:

$$\begin{aligned}
\vec{e}_l \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_1} &= \vec{e}_1 \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_l}, \\
\vec{e}_l \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_2} &= \vec{e}_2 \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_l}, \\
&\vdots \\
\vec{e}_l \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_m} &= \vec{e}_m \sum_{k=1}^{\infty} \vec{e}_k \frac{\partial u_k}{\partial x_l}
\end{aligned} \tag{2.5}$$

or, in the vector form,

$$\vec{e}_l \frac{\partial \vec{f}}{\partial x_k} = \vec{e}_k \frac{\partial \vec{f}}{\partial x_l}, \quad k = 1, 2, \ldots, m. \tag{2.6}$$

3. Differentiable functions providing solutions to partial differential equations

In this section, we extend the basic idea of relating analytic and harmonic functions into more general situations. For given integers m, $r \geq 1$, let

$$P(\xi_1, \xi_2, \ldots, \xi_m) := \sum_{i_1+i_2+\cdots+i_m=r} C_{i_1,i_2,\ldots,i_m}(x_1, x_2, \ldots, x_m)\xi_1^{i_1}\xi_2^{i_2}\ldots\xi_m^{i_m}, \tag{3.1}$$

where $C_{i_1,i_2,\ldots,i_m}(x_1,x_2,\ldots,x_m)$ are K-valued continuous functions of m variables x_i, $i=1,2,\ldots,m$. Consider the partial differential equation

$$P(\partial_0,\partial_1,\ldots,\partial_m)\,[u(x_1,x_2,\ldots,x_m)]=0, \tag{3.2}$$

where $\partial_k := \dfrac{\partial^k}{\partial x^k}$.

Theorem 3.1. *Let P be a polynomial as in (3.1). Let a function $\vec{f}:\boldsymbol{B}\longrightarrow\boldsymbol{A}$ and its derivatives $\vec{f}',\vec{f}'',\ldots,\vec{f}^r$ be differentiable, $\vec{f}(\vec{x})=\sum_{k=0}^{n}\vec{e_k}u_k(\vec{x})$. Assume that the functions $C_{i_1,i_2,\ldots,i_m}(x_1,x_2,\ldots,x_m)$ are linearly dependent in K^m, and the basis $\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_m$ of the subspace $\boldsymbol{B}$ of the algebra $\boldsymbol{A}$ is such that*

$$P(\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_m)=0. \tag{3.3}$$

Then the functions $u_k(\vec{x})$, $k=1,2,\ldots$ are solutions of (3.2).

Proof. It follows from the Cauchy–Riemann condition (2.6) that

$$\frac{\partial^{i_n}\vec{f}}{\partial x_k^{i_n}}=\vec{e_k}^{\,i_n}\vec{e_l}^{\,-i_n}\frac{\partial\vec{f}}{\partial x_l},\qquad k=1,2,\ldots,m. \tag{3.4}$$

This implies, for $i_1+i_2+\cdots+i_m\le r$, that

$$\frac{\partial^{i_1+i_2+\cdots+i_m}\vec{f}}{\partial x_1^{i_1}\partial x_2^{i_2}\cdots\partial x_m^{i_m}}=\vec{e}_l^{\,-(i_1+i_2+\cdots+i_m)}\vec{e}_1^{\,i_1}\vec{e}_2^{\,i_2}\cdots\vec{e}_m^{\,i_m}\frac{\partial^{i_1+i_2+\cdots+i_m}\vec{f}}{\partial x_1^{i_1+i_2+\cdots+i_m}}. \tag{3.5}$$

Therefore, we obtain

$$\sum_{i_1+i_2+\cdots+i_m=r}C_{i_1,i_2,\ldots,i_m}(x_1,x_2,\ldots,x_m)\frac{\partial^r}{\partial x_1^{i_1}\partial x_2^{i_2}\cdots\partial x_m^{i_m}}\vec{f}(x_1,x_2,\ldots,x_m)$$

$$=\vec{e_l}^{\,-r}\frac{\partial^r\vec{f}(x_1,x_2,\ldots,x_m)}{\partial x_l^r}\sum_{i_1+i_2+\cdots+i_m=r}C_{i_1,i_2,\ldots,i_m}(x_1,x_2,\ldots,x_m)(\vec{e}_1)^{\,i_1}(\vec{e}_2)^{\,i_2}\cdots(\vec{e}_m)^{\,i_m}=0.$$

Hence, every component $u_k(\vec{x})$, $k=1,2,\ldots,n$ of the function $\vec{f}$ is a solution of (3.2). □

Remark 3.1. We should note that if subspace **B** contains the unity, then there is an invertible element among its basis vectors.

4. Examples

4.1. The three-dimensional equation

Let us consider the PDE

$$\left(\frac{\partial^2}{\partial x^2}+x^2\frac{\partial^2}{\partial y^2}+(x^2+1)\frac{\partial^2}{\partial z^2}\right)u(x,y,z)=0. \tag{4.1}$$

This equation implies that the polynomial $P(\xi_1,\xi_2,\xi_3)=\xi_1^2+x^2\xi_2^2+(x^2+1)\xi_3^2$, and (3.3) has the view $e_1^2+x^2e_2^2+(x^2+1)e_3^2=0$. In this case, we can use the bicomplex algebra $BC=\{a_0+a_1\mathbf{i}+a_2\mathbf{j}+$

$a_3\mathbf{e}|\, a_k \in \mathbb{R}\}$. This algebra is commutative, and its basis vectors $1, \mathbf{i}, \mathbf{j}, \mathbf{e}$ satisfy $\mathbf{i}^2 = \mathbf{j}^2 = -\mathbf{e}^2 = 1$, and $\mathbf{ij} = \mathbf{ji} = \mathbf{e}$, $\mathbf{ie} = \mathbf{ei} = -\mathbf{j}$, $\mathbf{je} = \mathbf{ej} = -\mathbf{i}$.

It is easy to see that $x^2\mathbf{i}^2 + \mathbf{j}^2 + (x^2+1)\mathbf{e}^2 = 0$. So, we may consider BC as the algebra $\mathbf{A}$ and $\mathbf{B} = \{a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{e}|\, a_k \in \mathbb{R}\} \subset BC$ as a subspace of $\mathbf{A}$.

Consider a function $\vec{f}: \mathbf{B} \to \mathbf{A}$, namely,

$$\vec{f}(x,y,z) = u_0(x,y,z) + u_1(x,y,z)\,\mathbf{i} + u_2(x,y,z)\,\mathbf{j} + u_3(x,y,z)\,\mathbf{e}$$

where $x, y, z \in \mathbb{R}$ and $u_k : \mathbb{R}^3 \to \mathbb{R}$, $k = 0,1,2,3$.

According to Theorem 3.1, if the function $\vec{f}$ is monogenic, then the functions $u_k(x,y,z)$ are solutions of (4.1).

Thus, to obtain solutions of (4.1), it is enough to find a monogenic function $\vec{f}: \mathbf{B} \to \mathbf{A}$. As an example, consider the function $\vec{f}(z) = e^{x\mathbf{i}+y\mathbf{j}+z\mathbf{e}}$, where $z = x\mathbf{i} + y\mathbf{j} + z\mathbf{e}$. It is easily seen that f is monogenic and

$$\begin{aligned}
\vec{f}(z) &= e^{x\mathbf{i}+y\mathbf{j}+z\mathbf{e}} \\
&= (\cos(x) + \mathbf{i}\sin(x))(\cos(y) + \mathbf{j}\sin(y))(\cosh(z) + \mathbf{e}\sinh(z)) \\
&= \cos(x)\cos(y)\cosh(z) + \sin(x)\sin(y)\sinh(z) \\
&+ \mathbf{i}(\sin(x)\cos(y)\cosh(z) - \cos(x)\sin(y)\sinh(z)) \\
&+ \mathbf{j}(\cos(x)\sin(y)\cosh(z) - \sin(x)\cos(y)\cosh(z)) \\
&+ \mathbf{e}(\cos(x)\cos(y)\sinh(z) + \sin(x)\sin(y)\cosh(z)).
\end{aligned}$$

Therefore, we obtain four solutions of (4.1)

$$\begin{aligned}
u_0(x,y,z) &= \cos(x)\cos(y)\cosh(z) + \sin(x)\sin(y)\sinh(z); \\
u_1(x,y,z) &= \sin(x)\cos(y)\cosh(z) - \cos(x)\sin(y)\sinh(z); \\
u_2(x,y,z) &= \cos(x)\sin(y)\cosh(z) - \sin(x)\cos(y)\cosh(z); \\
u_3(x,y,z) &= \cos(x)\cos(y)\sinh(z) + \sin(x)\sin(y)\cosh(z).
\end{aligned}$$

4.2. The four-dimensional equation

Now, let us consider the PDE

$$\left(y^2\frac{\partial^2}{\partial x^2} + v\frac{\partial^2}{\partial y^2} - (y^2+v+1)\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial v^2}\right) u(x,y,z,v) = 0. \tag{4.2}$$

This equation implies that the polynomial $P(\xi_1,\xi_2,\xi_3,\xi_4) = y^2\xi_1^2 + v\xi_2^2 - (y^2+v+1)\xi_3^2 + \xi_4^2$, and (3.3) has the form $y^2e_1^2 + ve_2^2 - (y^2+v+1)e_3^2 + e_4^2 = 0$. Suppose that $\mathbf{A}$ is a commutative algebra of the form $\mathbf{A} = \{a_0 + \mathbf{e}a_1 + \mathbf{f}a_2 + \mathbf{g}a_3 | a_k \in \mathbb{R}\}$, where $\mathbf{e}^2 = \mathbf{f}^2 = \mathbf{g}^2 = 1$ and $\mathbf{efg} = 1$.

We will find a monogenic function such as

$$\vec{f}: \mathbf{A} \to \mathbf{A},$$

namely,

$$\vec{f}(x,y,z,v) = u_0(x,y,z,v) + u_1(x,y,z,v)\,\mathbf{e} + u_2(x,y,z,v)\,\mathbf{f}$$
$$+ u_3(x,y,z,v)\,\mathbf{g}$$

where $x, y, z, v \in \mathbb{R}$ and $u_k : \mathbb{R}^4 \to \mathbb{R}$, $k = 0,1,2,3$.

Let us define

$$\vec{f}(x, y, z, v) = (x + y\,\mathbf{e} + z\,\mathbf{f} + v\,\mathbf{g})^3.$$

It is easily verified that this function is monogenic. Hence, by calculating $u_i(x, y, z, v)$, $i = 0, 1, 2, 3$, we obtain four solutions of (4.2). After a simple computation, we obtain a solution of (4.2) in the following form:

$$u_0(x, y, z, v) = x^3 + 3xy^2 + 3xz^2 + 3xv^2 + 6yzv.$$

We may obtain solutions of (4.2), by using such monogenic functions as $\vec{f}(z) = e^{x+y\mathbf{e}+z\mathbf{f}+v\mathbf{g}}$, $\vec{f}(z) = \cos(x + y\mathbf{e} + z\mathbf{f} + v\mathbf{g})$, and so on.

4.3. The linearized Korteweg–de-Vries equation

A linearized version of the KdV equation is

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} = 0. \tag{4.3}$$

Then, in order to show the potentialities of our method, we consider the slightly different equation

$$\frac{\partial^3 w}{\partial z^2 \partial t} + \frac{\partial^3 w}{\partial x^3} = 0. \tag{4.4}$$

The corresponding polynomial can be defined as

$$P(\xi_1, \xi_2, \xi_3) = \xi_3^2 \xi_1 + \xi_2^3.$$

Let $\mathbf{A}_0$ be a three-dimensional commutative algebra over $\mathbb{R}$. We assume that the set $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$, is a basis of $\mathbf{A}_0$ with the Cayley table:

$$\mathbf{e}_i \mathbf{e}_j = \mathbf{e}_{i \oplus j},$$

where $\oplus j = i + j \ (mod\ 3)$.

The algebra $\mathbf{A}_0$ has the following matrix representation:

$$\mathbf{e}_k \to P_k = P_1^k,$$

where

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We define

$$\boldsymbol{\tau}_0^{(l)} = \mathbf{e}_l,\ l = 0, 1, 2, \quad \boldsymbol{\tau}_1^{(m)} = \mathbf{e}_m \mathbf{i} = \mathbf{i}\mathbf{e}_m,\ m = 0, 1, 2,$$

where $\mathbf{i}$ is the imaginary unit.

Let us consider the commutative algebra with the basis $\boldsymbol{\tau}_0^{(l)}$, $\boldsymbol{\tau}_1^{(m)}$, $l = 0, 1, 2$, $m = 0, 1, 2$, as $\mathbf{A}$ and the element $\boldsymbol{\tau}_0^{(0)} = \mathbf{e}_0$ is a unit of $\mathbf{A}$. We state $\mathbf{B} = \left\{ t\mathbf{i} + x\boldsymbol{\tau}_1^{(1)} + z\mathbf{e}_0 \right\}$, since it is easily seen that

$$P(\xi_1, \xi_2, \xi_3) = \xi_3^2 \xi_1 + \xi_2^3.$$

$$P\left(\mathbf{i}, \boldsymbol{\tau}_1^{(1)}, \mathbf{e}_0\right) = 0.$$

Consider a function $\overrightarrow{f} : \mathbf{B} \to \mathbf{A}$ such that

$$\overrightarrow{f}(z,t,x) = \sum_{l=0}^{2} u_l(t,x,z)\,\mathbf{e}_l + \sum_{m=0}^{2} u_{m+3}(t,x,z)\,\mathbf{e}_m\mathbf{i},$$

where $u_l : \mathbb{R}^3 \to \mathbb{R}$, $l = 0, 1, \cdots, 5$, are functions continuously differentiable three times.

We will find $\overrightarrow{f}$ as an exponential function of the following form:

$$\overrightarrow{f}(t,z,x) = e^{t\mathbf{i}+x\boldsymbol{\tau}_1^{(1)}+z\mathbf{e}_0} = (\cos t \ + \mathbf{i}\sin t\ )$$

$$\times\left(\sum_{k=0}^{\infty}\frac{(-1)^k x^{6k}}{(6k)!} + \mathbf{e}_1\mathbf{i}\sum_{k=0}^{\infty}\frac{(-1)^k x^{6k+1}}{(6k+1)!} \ - \mathbf{e}_2\sum_{k=0}^{\infty}\frac{(-1)^k x^{6k+2}}{(6k+2)!} - \mathbf{i}\sum_{k=0}^{\infty}\frac{(-1)^k x^{6k+3}}{(6k+3)!}\right.$$

$$\left. + \mathbf{e}_1\sum_{k=0}^{\infty}\frac{(-1)^k x^{6k+4}}{(6k+4)!} + \mathbf{e}_2\mathbf{i}\sum_{k=0}^{\infty}\frac{(-1)^k x^{6k+5}}{(6k+5)!}\right) e^z.$$

We note that

$$\sum_{k=0}^{\infty}\frac{(-1)^k x^{6k}}{(6k)!} = \frac{14+\sqrt{3}}{48}e^{-\frac{\sqrt{3}}{2}x}\cos\left(\frac{x}{2}\right) + \frac{3-2\sqrt{3}}{48}e^{-\frac{\sqrt{3}}{2}x}\sin\left(\frac{x}{2}\right)$$

$$+\frac{1}{3}e^{\frac{\sqrt{3}}{2}x}\cos\left(\frac{x}{2}\right) + \frac{1}{3}\cos(x).$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k x^{6k+3}}{(6k+3)!} = \frac{14+\sqrt{3}}{48}e^{-\frac{\sqrt{3}}{2}x}\sin\left(\frac{x}{2}\right) + \frac{2\sqrt{3}-3}{48}e^{-\frac{\sqrt{3}}{2}x}\cos\left(\frac{x}{2}\right)$$

$$+\frac{1}{3}e^{\frac{\sqrt{3}}{2}x}\sin\left(\frac{x}{2}\right) - \frac{1}{3}\sin(x).$$

.

Omitting the factor e^z, we obtain three particular solutions of the linearized Korteweg–de-Vries equation. One of them is given by

$$w_1(t,x) = \cos t\left(K_1\, e^{-\frac{\sqrt{3}}{2}x}\cos\left(\frac{x}{2}\right) - K_2\, e^{-\frac{\sqrt{3}}{2}x}\sin\left(\frac{x}{2}\right) + \frac{1}{3}e^{\frac{\sqrt{3}}{2}x}\cos\left(\frac{x}{2}\right) + \frac{1}{3}\cos(x)\right)$$

$$+\sin t\left(K_1\, e^{-\frac{\sqrt{3}}{2}x}\sin\left(\frac{x}{2}\right) + K_2\, e^{-\frac{\sqrt{3}}{2}x}\cos\left(\frac{x}{2}\right) + \frac{1}{3}e^{\frac{\sqrt{3}}{2}x}\sin\left(\frac{x}{2}\right) - \frac{1}{3}\sin(x)\right),$$

where $K_1 = \frac{14+\sqrt{3}}{48}$ and $K_2 = \frac{2\sqrt{3}-3}{48}$.

In the same manner, by computing the pairs

$$\sum_{k=0}^{\infty}\frac{(-1)^k x^{6k+1}}{(6k+1)!}, \qquad \sum_{k=0}^{\infty}\frac{(-1)^k x^{6k+4}}{(6k+4)!}$$

and

$$\sum_{k=0}^{\infty}\frac{(-1)^k x^{6k+2}}{(6k+2)!}, \qquad \sum_{k=0}^{\infty}\frac{(-1)^k x^{6k+5}}{(6k+5)!},$$

we obtain two more solutions of (4.4), respectively, $w_2(t,x)$ and $w_3(t,x)$.

4.4. Fourth-order equation

Let us consider the equation

$$\frac{\partial^2 w}{\partial t^2} + a^2 \frac{\partial^4 w}{\partial x^4} = 0. \tag{4.5}$$

This equation arises in the problems of transverse vibrations of a uniform elastic rod [7].

To use our method, we consider the closely related equation

$$\frac{\partial^4 w}{\partial t^2 \partial z^2} + a^2 \frac{\partial^4 w}{\partial x^4} = 0. \tag{4.6}$$

The corresponding polynomial is as follows:

$$P(\xi_1, \xi_2, \xi_3) = \xi_1^2 \xi_2^2 + a^2 \xi_3^4.$$

In this case, we may consider the bicomplex algebra as $\mathbf{A}$ and $\mathbf{B} = \{a_0 + a_1 \mathbf{i} a + a_2 \mathbf{e}\}$ with the basis $1, \mathbf{i}\sqrt{a}, \mathbf{e}$, which satisfies the equation $P(\xi_1, \xi_2, \xi_3) = 0$.

Consider a function $\overrightarrow{f} : \mathbf{B} \to \mathbf{A}$:

$$\overrightarrow{f}(x_1, x_2, x_3) = u_0(x_1, x_2, x_3) + u_1(x_1, x_2, x_3)\mathbf{i} + u_2(x_1, x_2, x_3)\mathbf{j}$$

$$+ u_3(x_1, x_2, x_3)\mathbf{e},$$

where $u_l : \mathbb{R}^3 \to \mathbb{R}$, $l = 0, 1, \cdots, 5$ are functions continuously differentiable four times.

The components of the exponential function

$$\overrightarrow{f}(x_1, x_2, x_3) = e^{z + \mathbf{i}at + \mathbf{e}x}$$

are solutions of (4.6). It is easily seen that solutions of (4.5) are components of the function $e^{\mathbf{i}at + \mathbf{e}x}$, namely

$$\begin{aligned} u_0(t, x) &= \cos(at) \cosh(x), \\ u_1(t, x) &= \sin(at) \cosh(x), \\ u_2(t, x) &= \cos(at) \sinh(x), \\ u_3(t, x) &= \sin(at) \sinh(x). \end{aligned}$$

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