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Research Article

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The distribution of random motion with Erlang-3 sojourn times

Abstract: In this paper, we study a one-dimensional random motion with switching process having an Erlang-3 distribution for the sojourn times. We obtain the solution of the associated hyperbolic PDE of sixth order through algebraic techniques.

Note 1: Red parts indicate major changes. Please check them carefully.

Keywords: Random motion, Erlang distribution, differentiable functions on commutative algebras, telegraph equation

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1 Introduction

The basic telegraph model of random motion was introduced in the seminal work [3]. Eversince, this model has been extended to deal with many applications. For instance, in financial mathematics [2, 11**?**], for modeling of absorbing processes [7], for processes of population dynamics in ecology [5], and in other sciences. So, the Goldstein model or telegraph process is an alternative to model many practical situations where researches use typically Brownian motion or Wiener processes [12]. Motions in one and two dimensions having finite velocities were analyzed in [**?**] by using higher order differential equations.

Note 2: bibitem with citation key *DiCrescenzo2* is missing!

Note 3: bibitem with citation key *DeGregorio2005* is missing!

A one-dimensional random walk of a single particle with Erlang-m distribution of time intervals between subsequent alternating velocities was studied in [10]. An important finding of the authors was to obtain the following higher order hyperbolic equation:

$$\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x} + \lambda\right)^m \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \lambda\right)^m f(t,x) - \lambda^{2m} f(t,x) = 0, \tag{1.1}$$

where $f(t,x)$ is the *probability density function* (pdf) of the particle position x at the time epoch t, $v > 0$ is the velocity of the particle and λ is the scale parameter of the Erlang-m distribution. For further details related to the derivation of this model we refer the reader to [10].

A method of solution of such a type of PDEs was proposed in [9] for $m = 1$ and $m = 2$ when $v = 1$ and $\lambda = 1$. Such a method is based on the theory of commutative algebras and algebraic decomposition which uses the elementary properties of differentiability of monogenic function [8]. It is not easy to find the general solution of (1.1) and in the present paper this method will be extended to the case $m = 3$ when $v = 1$ and $\lambda = 1$.

2 The probability density function representation

It has been shown that the pdf $f(t,x)$ can be represented as a sum of a singular part $f_s(t,x)$ and an absolute continuous part $f_c(t,x)$ ([9]). Since it has a singular part, we will call it as generalized pdf. The singular part

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has the form

$$f_s\Big(t,\frac{x}{v}\Big)=\delta\Big(t-\frac{x}{v}\Big)e^{-\lambda t}\sum_{i=0}^{m-1}\frac{(\lambda t)^i}{i!}. \tag{2.1}$$

By applying the transformation $f(t,x)=\exp(-\lambda t)g(t,x)$ and changing the variable $y=x/v$, we reduce equation (1.1) to

$$\Big(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}\Big)^m g_c(t,y)-\lambda^{2m}g_c(t,y)=0, \tag{2.2}$$

with singular part

$$g_s(t,y)=f_s(t,y)\exp(\lambda t).$$

In the sequel, we assume without loss of generality that $\lambda=1$. Furthermore, by introducing the function $\mathbf{f}(t,y,z)=\exp(z)g_c(t,y)$ we have that the sum of both the singular and the absolute continuous parts of $\mathbf{f}(t,y,z)$ must be a solution of the following equation:

Note 5: Introduced for line break

$$\Big(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}\Big)^m \mathbf{f}(t,y,z)-\frac{\partial^{2m}}{\partial z^{2m}}\mathbf{f}(t,y,z)=0. \tag{2.3}$$

It is easy to verify that the last equation is obtained from (2.2). We will use methods from theory of algebras and a particular algebraic decomposition [9] to obtain the solution of equation (2.3) when $m=3$.

3 Using the algebraic method to obtain the solution of the telegraph equation

Let A_0 be a six-dimensional commutative algebra over $\mathbb{R}$. We assume that the set $\{e_0,e_1,e_2,e_3,e_4,e_5\}$ is the basis of A_0 with the Cayley table $e_ie_j=e_{i\oplus j}$, where $i\oplus j=i+j \pmod 6$. The algebra A_0 has a matrix representation $e_k\to P_k=P_1^k$, where $P_1=[p_{ij}]_{6\times 6}$, $P_{ii+1}=1$ for $0\le i\le 5$, $P_{60}=1$ and $P_{ij}=0$ for the rest of i and j. For $l=0,1,\dots,5$, we define

$$\tau_0^l:=e_l,\qquad \tau_1^l:=e_l i\sin s,\qquad \tau_2^l:=e_l\cos s,$$
$$\tau_{2k}^l:=e_l\cos ks,\qquad \tau_{2k+1}^l:=e_l i\sin(k+1)s,\qquad k=0,1,\dots,$$

where $0\le s\le 2\pi$. It is easy to see that

$$\begin{aligned}
\tau_{2n}^{l_1}\cdot\tau_{2k}^{l_2}&=\frac12\big(\tau_{2(n-k)}^{l_1\oplus l_2}+\tau_{2(n+k)}^{l_1\oplus l_2}\big), && n\ge k,\\
\tau_{2n+1}^{l_1}\cdot\tau_{2k+1}^{l_2}&=\frac12\big(\tau_{2(n-k)+4}^{l_1\oplus l_2}-\tau_{2(n+k)}^{l_1\oplus l_2}\big), && k=0,\quad n=0,1,\dots,\\
\tau_{2n+1}^{l_1}\cdot\tau_{2k}^{l_2}&=\frac12\big(\tau_{2(n-k)+1}^{l_1\oplus l_2}+\tau_{2(n+k)+1}^{l_1\oplus l_2}\big), && n\ge k,\\
\tau_{2n+1}^{l_1}\cdot\tau_{2k}^{l_2}&=\frac12\big(\tau_{2k+1}^{l_1\oplus l_2}+\tau_{2k-3}^{l_1\oplus l_2}\big), && n=0,\quad k=2,3,\dots,\\
\tau_{2n+1}^{l_1}\cdot\tau_{2k}^{l_2}&=\frac12\tau_{2(n+k)+1}^{l_1\oplus l_2}, && n=0,\quad k=1.
\end{aligned}$$

Now, we introduce the commutative algebra

$$A=\Big\{\sum_{k=0}^{+\infty}\sum_{l=0}^{5}\big(a_{2k}^l\tau_{2k}^l+a_{2k+1}^l\tau_{2k+1}^l\big):a_j^l\in\mathbb{R}\Big\},$$

where

$$\sum_{k=0}^{+\infty}\sum_{l=0}^{5}\big(|a_{2k}^l|+|a_{2k+1}^l|\big)<+\infty.$$

Let

$$B=\{a_0\tau_1^1+a_1\tau_2^1+a_2\tau_0^0:a_i\in\mathbb{R}\}$$

be a subspace of algebra A. Furthermore, let us introduce the function $\mathbf{f}\colon B \to A$ such that

$$\mathbf{f}(t, y, z) = f(e_1(t \cos s + yi \sin s) + z).$$

Then,

$$\mathbf{f}(t, y, z) = \sum_{k=0}^{+\infty} \sum_{l=0}^{5} (v_{2k}^l(t, y, z)\tau_{2k}^l + v_{2k+1}^l(t, y, z)\tau_{2k+1}^l).$$

The function $\mathbf{f}$ is called B/A-differentiable at $\mathbf{x}_0 \in B$ if there exists an $\mathbf{f}'(\mathbf{x}_0) \in A$ in such a manner that for any $\mathbf{h} \in B$,

$$\mathbf{f}'(\mathbf{x}_0)\mathbf{h} = \lim_{\mathbb{R}\ni\varepsilon\to 0} \frac{\mathbf{f}(\mathbf{x}_0 + \varepsilon\mathbf{h}) - \mathbf{f}(\mathbf{x}_0)}{\varepsilon}.$$

We use the following properties of B/A-differentiable functions from [9]:

$$\frac{\partial}{\partial t}\mathbf{f} = e_1 \cos s \frac{\partial}{\partial z}\mathbf{f}, \tag{3.1}$$

$$\frac{\partial}{\partial y}\mathbf{f} = e_1 i \sin s \frac{\partial}{\partial z}\mathbf{f}. \tag{3.2}$$

In this case all the $v_{2k}^l(t, y, z)$ are solutions of equation (2.3). We denote the element e_1 as $\mathbf{e}$ and we will seek for the solution of equation (2.2) in the form

$$g_c(\mathbf{e}(t \cos s + yi \sin s)) = e^{\mathbf{e}(t \cos s + yi \sin s)}. \tag{3.3}$$

Since

$$f(\mathbf{e}(t \cos s + yi \sin s) + z) = g_c(\mathbf{e}(t \cos s + yi \sin s))e^z,$$

we have

$$v_k^l(t, y, z) = u_k^l(t, y)e^z, \quad l = 0, 1, \dots, 5, \; k = 0, 1, \dots,$$

where

$$g_c(t, y) = \sum_{k=0}^{+\infty} \sum_{l=0}^{5} (u_{2k}^l(t, y)\tau_{2k}^l + u_{2k+1}^l(t, y)\tau_{2k+1}^l).$$

Therefore, we obtain the functions $u_0^l(t, y)$ for $t \ge |y|$ from the equation

$$\begin{aligned} \sum_{l=0}^{5} u_0^l(t, y)\tau_0^l &= \sum_{l=0}^{5} u_0^l(t, y)\mathbf{e}^l \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\mathbf{e}(t \cos s + yi \sin s)}\, ds = I_0\Big(\mathbf{e}\sqrt{t^2 - y^2}\Big), \end{aligned}$$

where I_k is the modified Bessel function of the first kind and k-th order. Equations (3.1) and (3.2) yield the following Cauchy–Riemann-type conditions (we skip bulky intermediate conversions):

$$\begin{aligned} &\frac{\partial}{\partial t}u_0^{l\oplus 1} = \frac{1}{2}u_2^l, \quad \frac{\partial}{\partial t}u_1^{l\oplus 1} = \frac{1}{2}u_3^l, \quad \frac{\partial}{\partial t}u_2^{l\oplus 1} = u_0^l + \frac{1}{2}u_4^l, \\ &\frac{\partial}{\partial t}u_{2k-1}^{l\oplus 1} = \frac{1}{2}(u_{2k-3}^l + u_{2k+1}^l), \quad \frac{\partial}{\partial t}u_{2k}^{l\oplus 1} = \frac{1}{2}(u_{2(k-1)}^l + u_{2(k+1)}^l), \quad k = 2, 3, \dots, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\frac{\partial}{\partial y}u_0^{l\oplus 1} = -\frac{1}{2}u_1^l, \quad \frac{\partial}{\partial y}u_1^{l\oplus 1} = u_0^l + \frac{1}{2}u_4^l, \quad \frac{\partial}{\partial y}u_2^{l\oplus 1} = -\frac{1}{2}u_3^l, \\ &\frac{\partial}{\partial y}u_{2k+1}^{l\oplus 1} = \frac{1}{2}(u_{2k}^l + u_{2(k+2)}^l), \quad \frac{\partial}{\partial y}u_{2k+2}^{l\oplus 1} = \frac{1}{2}(u_{2k-1}^l - u_{2k+3}^l), \quad k = 1, 2, \dots. \end{aligned} \tag{3.5}$$

It is worth noting that last equations are basically the same ones as in [9]. By using equations (3.4) and (3.5) and functions $u_0^l(t, y)$ we can obtain the functions $u_k^l(t, y)$ for any $k \ge 1$ in a recursive manner.

4 Main results

The main result of this work is stated in the following theorem.

Theorem 4.1. *The function $f(t,x)$ is the generalized pdf of the particle position at time t when $m=3$, for $v=\lambda=1$, and for $t \geq |x|$ it is given by*

$$\begin{aligned}
f(t,x) = {} & \frac{(t^2+x^2)e^{-t}}{2(t^2-x^2)} I_0(\Lambda_x) + \Big(\frac{tx\,e^{-t}}{3(t^2-x^2)} - \frac{2(3tx^2+t^3)e^{-t}}{3(t^2-x^2)^2}\Big)\Big(I_0(\Lambda_x) - \omega_3 I_0(\omega_3\Lambda_x) + \omega_3^2 I_0(\omega_3^2\Lambda_x)\Big) \\
& - \frac{(t^2+x^2)e^{-t}}{\sqrt{(t^2-x^2)^3}} I_1(\Lambda_x) + \Big(\frac{4(3tx^2+t^3)e^{-t}}{3\sqrt{(t^2-x^2)^5}} - \frac{2txe^{-t}}{3\sqrt{(t^2-x^2)^3}}\Big)\Big(I_1(\Lambda_x) - I_1(\omega_3\Lambda_x) + I_1(\omega_3^2\Lambda_x)\Big) \\
& + \frac{(3tx^2+t^3)e^{-t}}{6\sqrt{(t^2-x^2)^3}}\Big(I_1(\Lambda_x) - \omega_3^2 I_1(\omega_3\Lambda_x) - \omega_3 I_1(\omega_3^2\Lambda_x)\Big) \\
& + \frac{te^{-t}}{6\Lambda_x}\Big(2I_1(\Lambda_x) + \omega_3^2 I_1(\omega_3\Lambda_x) + \omega_3 I_1(\omega_3^2\Lambda_x)\Big) + e^{-t}\Big(1+t+\frac{t^2}{2}\Big)\delta(t-x),
\end{aligned}$$

where $\Lambda_x = \sqrt{t^2-x^2}$.

Proof. By using equation (2.1) when $m=3$, and $v=\lambda=1$, we obtain

$$f_s(t,x) = e^{-t}\Big(1+t+\frac{t^2}{2}\Big)\delta(t-x)$$

and equivalently

$$g_s(t,y) = \Big(1+t+\frac{t^2}{2}\Big)\delta(t-y).$$

The algebra A_0 has the form

$$A_0 = \{a + e_1 b + e_2 c + e_3 d + e_4 p + e_5 q : a,b,c,d,p,q \in \mathbb{R}\}$$

and its basis is $e_l = \mathbf{e}^l$, $l = 0,1,2,3,4,5$, where $\mathbf{e}$ has the matrix representation

$$\mathbf{e} \to P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$e_0 = \mathbf{e}^0 = 1, \quad e_1 = \mathbf{e}^1 = \mathbf{e}, \quad e_2 = \mathbf{e}^2, \quad e_3 = \mathbf{e}^3, \quad e_4 = \mathbf{e}^4, \quad e_5 = \mathbf{e}^5, \quad e_6 = \mathbf{e}^0 = 1.$$

By using equation (3.3) we obtain

$$\begin{aligned}
u_0^0(t,y) + \mathbf{e}u_0^1(t,y) + \mathbf{e}^2 u_0^2(t,y) + \mathbf{e}^3 u_0^3(t,y) + \mathbf{e}^4 u_0^4(t,y) + \mathbf{e}^5 u_0^5(t,y) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathbf{e}(t\cos s + yi\sin s)}\,ds \\
&= I_0\Big(\mathbf{e}\sqrt{t^2-y^2}\Big).
\end{aligned}$$

Thus, by denoting as $\Lambda_y = \sqrt{t^2-y^2}$ we see that

$$\begin{aligned}
I_0\Big(\mathbf{e}\sqrt{t^2-y^2}\Big) = {} & \frac{I_0(\Lambda_y) + I_0(\omega_3\Lambda_y) + I_0(\omega_3^2\Lambda_y)}{3} + \mathbf{e}^2\Big(\frac{I_0(\Lambda_y) - \omega_3 I_0(\omega_3\Lambda_y) + \omega_3^2 I_0(\omega_3^2\Lambda_y)}{3}\Big) \\
& + \mathbf{e}^4\Big(\frac{I_0(\Lambda_y) + \omega_3^2 I_0(\omega_3\Lambda_y) - \omega_3 I_0(\omega_3^2\Lambda_y)}{3}\Big),
\end{aligned}$$

where $\omega_3 = (1 + i\sqrt{3})/2$. For $t \geqslant |y|$, we have that

$$u_0^1(t,y) = u_0^3(t,y) = u_0^5(t,y) = 0,$$

$$u_0^0(t,y) = \frac{I_0(\Lambda_y) + I_0(\omega_3\Lambda_y) + I_0(\omega_3^2\Lambda_y)}{3},$$

$$u_0^2(t,y) = \frac{I_0(\Lambda_y) - \omega_3 I_0(\omega_3\Lambda_y) + \omega_3^2 I_0(\omega_3^2\Lambda_y)}{3},$$

$$u_0^4(t,y) = \frac{I_0(\Lambda_y) + \omega_3^2 I_0(\omega_3\Lambda_y) - \omega_3 I_0(\omega_3^2\Lambda_y)}{3}.$$

From the first equation in (3.5) it follows that

$$u_1^l(t,y) = -2\frac{\partial}{\partial y}u_0^{l\oplus 1}(t,y).$$

Therefore,

$$u_1^0(t,y) = -2\frac{\partial}{\partial y}u_0^1(t,y) = 0, \quad u_1^2(t,y) = -2\frac{\partial}{\partial y}u_0^3(t,y) = 0, \quad u_1^4(t,y) = -2\frac{\partial}{\partial y}u_0^5(t,y) = 0,$$

and

$$\begin{aligned} u_1^1(t,y) &= -2\frac{\partial}{\partial y}u_0^2(t,y) = -2\frac{\partial}{\partial y}\left[\frac{I_0(\Lambda_y) - \omega_3 I_0(\omega_3\Lambda_y) + \omega_3^2 I_0(\omega_3^2\Lambda_y)}{3}\right] \\ &= \frac{2y}{3\Lambda_y}(I_1(\Lambda_y) - \omega_3^2 I_1(\omega_3\Lambda_y) - \omega_3 I_1(\omega_3^2\Lambda_y)), \\ u_1^3(t,y) &= -2\frac{\partial}{\partial y}u_0^4(t,y) = -2\frac{\partial}{\partial y}\left[\frac{I_0(\Lambda_y) + \omega_3^2 I_0(\omega_3\Lambda_y) - \omega_3 I_0(\omega_3^2\Lambda_y)}{3}\right] \\ &= \frac{2y}{3\Lambda_y}(I_1(\Lambda_y) - I_1(\omega_3\Lambda_y) + I_1(\omega_3^2\Lambda_y)), \\ u_1^5(t,y) &= -2\frac{\partial}{\partial y}u_0^6(t,y) = -2\frac{\partial}{\partial y}u_0^0(t,y) = -2\frac{\partial}{\partial y}\left[\frac{I_0(\Lambda_y) + I_0(\omega_3\Lambda_y) + I_0(\omega_3^2\Lambda_y)}{3}\right] \\ &= \frac{2y}{3\Lambda_y}(I_1(\Lambda_y) + \omega_3 I_1(\omega_3\Lambda_y) + \omega_3^2 I_1(\omega_3^2\Lambda_y)). \end{aligned}$$

Similarly, from the first equation in (3.4) we have that

$$u_2^l(t,y) = 2\frac{\partial}{\partial t}u_0^{l\oplus 1}(t,y).$$

Therefore,

$$u_2^0(t,y) = 2\frac{\partial}{\partial t}u_0^1(t,y) = 0, \quad u_2^2(t,y) = 2\frac{\partial}{\partial t}u_0^3(t,y) = 0, \quad u_2^4(t,y) = 2\frac{\partial}{\partial t}u_0^5(t,y) = 0,$$

and

$$\begin{aligned} u_2^1(t,y) &= 2\frac{\partial}{\partial t}u_0^2(t,y) = 2\frac{\partial}{\partial t}\left[\frac{I_0(\Lambda_y) - \omega_3 I_0(\omega_3\Lambda_y) + \omega_3^2 I_0(\omega_3^2\Lambda_y)}{3}\right] \\ &= \frac{2t}{3\Lambda_y}(I_1(\Lambda_y) - \omega_3^2 I_1(\omega_3\Lambda_y) - \omega_3 I_1(\omega_3^2\Lambda_y)), \\ u_2^3(t,y) &= 2\frac{\partial}{\partial t}u_0^4(t,y) = 2\frac{\partial}{\partial t}\left[\frac{I_0(\Lambda_y) + \omega_3^2 I_0(\omega_3\Lambda_y) - \omega_3 I_0(\omega_3^2\Lambda_y)}{3}\right] \\ &= \frac{2t}{3\Lambda_y}(I_1(\Lambda_y) - I_1(\omega_3\Lambda_y) + I_1(\omega_3^2\Lambda_y)), \\ u_2^5(t,y) &= 2\frac{\partial}{\partial t}u_0^0(t,y) = 2\frac{\partial}{\partial t}\left[\frac{I_0(\Lambda_y) + I_0(\omega_3\Lambda_y) + I_0(\omega_3^2\Lambda_y)}{3}\right] \\ &= \frac{2t}{3\Lambda_y}(I_1(\Lambda_y) + \omega_3 I_1(\omega_3\Lambda_y) + \omega_3^2 I_1(\omega_3^2\Lambda_y)). \end{aligned}$$

Next, from the second equation in (3.4), it follows that

$$u_3^l(t,y) = 2\frac{\partial}{\partial t}u_1^{l\oplus 1}(t,y).$$

Hence,

$$u_3^1(t,y) = 2\frac{\partial}{\partial t}u_1^2(t,y) = 0, \quad u_3^3(t,y) = 2\frac{\partial}{\partial t}u_1^4(t,y) = 0, \quad u_3^5(t,y) = 2\frac{\partial}{\partial t}u_1^0(t,y) = 0,$$

and

$$\begin{aligned}
u_3^0(t,y) &= 2\frac{\partial}{\partial t}u_1^1(t,y) = 2\frac{\partial}{\partial t}\Big[\frac{2y}{3\Lambda_y}(I_1(\Lambda_y) - \omega_3^2 I_1(\omega_3\Lambda_y) - \omega_3 I_1(\omega_3^2\Lambda_y))\Big] \\
&= \frac{4ty}{3(t^2-y^2)}(I_0(\Lambda_y) + I_0(\omega_3\Lambda_y) + I_0(\omega_3^2\Lambda_y)) \\
&\qquad - \frac{8ty}{3\sqrt{(t^2-y^2)^3}}(I_1(\Lambda_y) - \omega_3^2 I_1(\omega_3\Lambda_y) - \omega_3 I_1(\omega_3^2\Lambda_y)), \\
u_3^2(t,y) &= 2\frac{\partial}{\partial t}u_1^3(t,y) = 2\frac{\partial}{\partial t}\Big[\frac{2y}{3\Lambda_y}(I_1(\Lambda_y) - I_1(\omega_3\Lambda_y) + I_1(\omega_3^2\Lambda_y))\Big] \\
&= \frac{4ty}{3(t^2-y^2)}(I_0(\Lambda_y) - \omega_3 I_0(\omega_3\Lambda_y) + \omega_3^2 I_0(\omega_3^2\Lambda_y)) \\
&\qquad - \frac{8ty}{3\sqrt{(t^2-y^2)^3}}(I_1(\Lambda_y) - I_1(\omega_3\Lambda_y) + I_1(\omega_3^2\Lambda_y)), \\
u_3^4(t,y) &= 2\frac{\partial}{\partial t}u_1^5(t,y) = 2\frac{\partial}{\partial t}\Big[\frac{2y}{3\Lambda_y}(I_1(\Lambda_y) + \omega_3 I_1(\omega_3\Lambda_y) + \omega_3^2 I_1(\omega_3^2\Lambda_y))\Big] \\
&= \frac{4ty}{3(t^2-y^2)}(I_0(\Lambda_y) + \omega_3^2 I_0(\omega_3\Lambda_y) - \omega_3 I_0(\omega_3^2\Lambda_y)) \\
&\qquad - \frac{8ty}{3\sqrt{(t^2-y^2)^3}}(I_1(\Lambda_y) + \omega_3 I_1(\omega_3\Lambda_y) + \omega_3^2 I_1(\omega_3^2\Lambda_y)).
\end{aligned}$$

From the third equation in (3.4) we have that

$$u_4^l(t,y) = 2\frac{\partial}{\partial t}u_2^{l\oplus 1}(t,y) - 2u_0^l(t,y).$$

Hence,

$$\begin{aligned}
&u_4^1(t,y) = u_4^3(t,y) = u_4^5(t,y) = 0, \\
&u_4^0(t,y) = 2\frac{\partial}{\partial t}u_2^1(t,y) - 2u_0^0(t,y) \\
&\quad = 2\frac{\partial}{\partial t}\Big[\frac{2t}{3\Lambda_y}(I_1(\Lambda_y) - \omega_3^2 I_1(\omega_3\Lambda_y) - \omega_3 I_1(\omega_3^2\Lambda_y))\Big] - \frac{2}{3}(I_0(\Lambda_y) + I_0(\omega_3\Lambda_y) + I_0(\omega_3^2\Lambda_y)) \\
&\quad = \frac{2(t^2+y^2)}{3(t^2-y^2)}(I_0(\Lambda_y) + I_0(\omega_3\Lambda_y) + I_0(\omega_3^2\Lambda_y)) \\
&\qquad\quad - \frac{4(t^2+y^2)}{3\sqrt{(t^2-y^2)^3}}(I_1(\Lambda_y) - \omega_3^2 I_1(\omega_3\Lambda_y) - \omega_3 I_1(\omega_3^2\Lambda_y)), \\
&u_4^2(t,y) = 2\frac{\partial}{\partial t}u_2^3(t,y) - 2u_0^2(t,y) \\
&\quad = 2\frac{\partial}{\partial t}\Big[\frac{2t}{3\Lambda_y}(I_1(\Lambda_y) - I_1(\omega_3\Lambda_y) + I_1(\omega_3^2\Lambda_y))\Big] - \frac{2}{3}(I_0(\Lambda_y) - \omega_3 I_0(\omega_3\Lambda_y) + \omega_3^2 I_0(\omega_3^2\Lambda_y)) \\
&\quad = \frac{2(t^2+y^2)}{3(t^2-y^2)}(I_0(\Lambda_y) - \omega_3 I_0(\omega_3\Lambda_y) + \omega_3^2 I_0(\omega_3^2\Lambda_y)) \\
&\qquad\quad - \frac{4(t^2+y^2)}{3\sqrt{(t^2-y^2)^3}}(I_1(\Lambda_y) - I_1(\omega_3\Lambda_y) + I_1(\omega_3^2\Lambda_y)),
\end{aligned}$$

$$
\begin{aligned}
u_4^4(t,y) &= 2\frac{\partial}{\partial t}u_2^5(t,y) - 2u_0^4(t,y)\\
&= 2\frac{\partial}{\partial t}\left[\frac{2t}{3\Lambda_y}(I_1(\Lambda_y)+\omega_3 I_1(\omega_3\Lambda_y)+\omega_3^2 I_1(\omega_3^2\Lambda_y))\right] - \frac{2}{3}(I_0(\Lambda_y)+\omega_3^2 I_0(\omega_3\Lambda_y)-\omega_3 I_0(\omega_3^2\Lambda_y))\\
&= \frac{2(t^2+y^2)}{3(t^2-y^2)}(I_0(\Lambda_y)+\omega_3^2 I_0(\omega_3\Lambda_y)-\omega_3 I_0(\omega_3^2\Lambda_y))\\
&\qquad - \frac{4(t^2+y^2)}{3\sqrt{(t^2-y^2)^3}}(I_1(\Lambda_y)+\omega_3 I_1(\omega_3\Lambda_y)+\omega_3^2 I_1(\omega_3^2\Lambda_y)).
\end{aligned}
$$

Also, for $k=2$, the fourth equation in (3.4) yields

$$
u_5^l(t,y) = 2\frac{\partial}{\partial t}u_3^{l\oplus 1}(t,y) - u_1^l(t,y).
$$

Then,

$$
\begin{aligned}
u_5^0(t,y) &= u_5^2(t,y) = u_5^4(t,y) = 0,\\
u_5^1(t,y) &= 2\frac{\partial}{\partial t}u_3^2(t,y) - u_1^1(t,y)\\
&= 2\frac{\partial}{\partial t}\left[\frac{4ty}{3(t^2-y^2)}(I_0(\Lambda_y)-\omega_3 I_0(\omega_3\Lambda_y)+\omega_3^2 I_0(\omega_3^2\Lambda_y))\right.\\
&\qquad\left. - \frac{8ty}{3\sqrt{(t^2-y^2)^3}}(I_1(\Lambda_y)-I_1(\omega_3\Lambda_y)+I_1(\omega_3^2\Lambda_y))\right]\\
&\qquad - \frac{2y}{3\Lambda_y}(I_1(\Lambda_y)-\omega_3^2 I_1(\omega_3\Lambda_y)-\omega_3 I_1(\omega_3^2\Lambda_y))\\
&= -\frac{8(3t^2y+y^3)}{3(t^2-y^2)^2}(I_0(\Lambda_y)-\omega_3 I_0(\omega_3\Lambda_y)+\omega_3^2 I_0(\omega_3^2\Lambda_y))\\
&\qquad + \frac{16(3t^2y+y^3)}{3\sqrt{(t^2-y^2)^5}}(I_1(\Lambda_y)-I_1(\omega_3\Lambda_y)+I_1(\omega_3^2\Lambda_y))\\
&\qquad + \frac{2(3t^2y+y^3)}{3\sqrt{(t^2-y^2)^3}}(I_1(\Lambda_y)-\omega_3^2 I_1(\omega_3\Lambda_y)-\omega_3 I_1(\omega_3^2\Lambda_y)),\\
u_5^3(t,y) &= 2\frac{\partial}{\partial t}u_3^4(t,y) - u_1^3(t,y)\\
&= 2\frac{\partial}{\partial t}\left[\frac{4ty}{3(t^2-y^2)}(I_0(\Lambda_y)+\omega_3^2 I_0(\omega_3\Lambda_y)-\omega_3 I_0(\omega_3^2\Lambda_y))\right.\\
&\qquad\left. - \frac{8ty}{3\sqrt{(t^2-y^2)^3}}(I_1(\Lambda_y)+\omega_3 I_1(\omega_3\Lambda_y)+\omega_3^2 I_1(\omega_3^2\Lambda_y))\right]\\
&\qquad - \frac{2y}{3\Lambda_y}(I_1(\Lambda_y)-I_1(\omega_3\Lambda_y)+I_1(\omega_3^2\Lambda_y))\\
&= -\frac{8(3t^2y+y^3)}{3(t^2-y^2)^2}(I_0(\Lambda_y)+\omega_3^2 I_0(\omega_3\Lambda_y)-\omega_3 I_0(\omega_3^2\Lambda_y))\\
&\qquad + \frac{16(3t^2y+y^3)}{3\sqrt{(t^2-y^2)^5}}(I_1(\Lambda_y)+\omega_3 I_1(\omega_3\Lambda_y)+\omega_3^2 I_1(\omega_3^2\Lambda_y))\\
&\qquad + \frac{2(3t^2y+y^3)}{3\sqrt{(t^2-y^2)^3}}(I_1(\Lambda_y)-I_1(\omega_3\Lambda_y)+I_1(\omega_3^2\Lambda_y)),\\
u_5^5(t,y) &= 2\frac{\partial}{\partial t}u_3^0(t,y) - u_1^5(t,y)\\
&= 2\frac{\partial}{\partial t}\left[\frac{4ty}{3(t^2-y^2)}(I_0(\Lambda_y)+I_0(\omega_3\Lambda_y)+I_0(\omega_3^2\Lambda_y))\right.\\
&\qquad\left. - \frac{8ty}{3\sqrt{(t^2-y^2)^3}}(I_1(\Lambda_y)-\omega_3^2 I_1(\omega_3\Lambda_y)-\omega_3 I_1(\omega_3^2\Lambda_y))\right]\\
&\qquad - \frac{2y}{3\Lambda_y}(I_1(\Lambda_y)+\omega_3 I_1(\omega_3\Lambda_y)+\omega_3^2 I_1(\omega_3^2\Lambda_y))
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{8(3t^2y+y^3)}{3(t^2-y^2)^2}(I_0(\Lambda_y)+I_0(\omega_3\Lambda_y)+I_0(\omega_3^2\Lambda_y))\\
&\qquad + \frac{16(3t^2y+y^3)}{3\sqrt{(t^2-y^2)^5}}(I_1(\Lambda_y)-\omega_3^2I_1(\omega_3\Lambda_y)-\omega_3I_1(\omega_3^2\Lambda_y))\\
&\qquad + \frac{2(3t^2y+y^3)}{3\sqrt{(t^2-y^2)^3}}(I_1(\Lambda_y)+\omega_3I_1(\omega_3\Lambda_y)+\omega_3^2I_1(\omega_3^2\Lambda_y)).
\end{aligned}
$$

Similarily, for $k = 2$, the last equation in (3.4) yields

$$u_6^l = 2\frac{\partial}{\partial t}u_4^{l\oplus 1}(t,y) - u_2^l(t,y).$$

Thus, we can obtain

$$
\begin{aligned}
u_6^0(t,y) &= u_6^2(t,y) = u_6^4(t,y) = 0,\\
u_6^1(t,y) &= 2\frac{\partial}{\partial t}u_4^2(t,y) - u_2^1(t,y)\\
&= 2\frac{\partial}{\partial t}\Big[\frac{2(t^2+y^2)}{3(t^2-y^2)}(I_0(\Lambda_y)-\omega_3I_0(\omega_3\Lambda_y)+\omega_3^2I_0(\omega_3^2\Lambda_y))\\
&\qquad - \frac{4(t^2+y^2)}{3\sqrt{(t^2-y^2)^3}}(I_1(\Lambda_y)-I_1(\omega_3\Lambda_y)+I_1(\omega_3^2\Lambda_y))\Big]\\
&\qquad - \frac{2t}{3\Lambda_y}(I_1(\Lambda_y)-\omega_3^2I_1(\omega_3\Lambda_y)-\omega_3I_1(\omega_3^2\Lambda_y))\\
&= -\frac{8(3ty^2+t^3)}{3(t^2-y^2)^2}(I_0(\Lambda_y)-\omega_3I_0(\omega_3\Lambda_y)+\omega_3^2I_0(\omega_3^2\Lambda_y))\\
&\qquad + \frac{16(3ty^2+t^3)}{3\sqrt{(t^2-y^2)^5}}(I_1(\Lambda_y)-I_1(\omega_3\Lambda_y)+I_1(\omega_3^2\Lambda_y))\\
&\qquad + \frac{2(3ty^2+t^3)}{3\sqrt{(t^2-y^2)^3}}(I_1(\Lambda_y)-\omega_3^2I_1(\omega_3\Lambda_y)-\omega_3I_1(\omega_3^2\Lambda_y)),\\
u_6^3(t,y) &= 2\frac{\partial}{\partial t}u_4^4(t,y) - u_2^3(t,y)\\
&= 2\frac{\partial}{\partial t}\Big[\frac{2(t^2+y^2)}{3(t^2-y^2)}(I_0(\Lambda_y)+\omega_3^2I_0(\omega_3\Lambda_y)-\omega_3I_0(\omega_3^2\Lambda_y))\\
&\qquad - \frac{4(t^2+y^2)}{3\sqrt{(t^2-y^2)^3}}(I_1(\Lambda_y)+\omega_3I_1(\omega_3\Lambda_y)+\omega_3^2I_1(\omega_3^2\Lambda_y))\Big]\\
&\qquad - \frac{2t}{3\Lambda_y}(I_1(\Lambda_y)-I_1(\omega_3\Lambda_y)+I_1(\omega_3^2\Lambda_y))\\
&= -\frac{8(3ty^2+t^3)}{3(t^2-y^2)^2}(I_0(\Lambda_y)+\omega_3^2I_0(\omega_3\Lambda_y)-\omega_3I_0(\omega_3^2\Lambda_y))\\
&\qquad + \frac{16(3ty^2+t^3)}{3\sqrt{(t^2-y^2)^5}}(I_1(\Lambda_y)+\omega_3I_1(\omega_3\Lambda_y)+\omega_3^2I_1(\omega_3^2\Lambda_y))\\
&\qquad + \frac{2(3ty^2+t^3)}{3\sqrt{(t^2-y^2)^3}}(I_1(\Lambda_y)-I_1(\omega_3\Lambda_y)+I_1(\omega_3^2\Lambda_y)),\\
u_6^5(t,y) &= 2\frac{\partial}{\partial t}u_4^0(t,y) - u_2^5(t,y)\\
&= 2\frac{\partial}{\partial t}\Big[\frac{2(t^2+y^2)}{3(t^2-y^2)}(I_0(\Lambda_y)+I_0(\omega_3\Lambda_y)+I_0(\omega_3^2\Lambda_y))\\
&\qquad - \frac{4(t^2+y^2)}{3\sqrt{(t^2-y^2)^3}}(I_1(\Lambda_y)-\omega_3^2I_1(\omega_3\Lambda_y)-\omega_3I_1(\omega_3^2\Lambda_y))\Big]\\
&\qquad - \frac{2t}{3\Lambda_y}(I_1(\Lambda_y)+\omega_3I_1(\omega_3\Lambda_y)+\omega_3^2I_1(\omega_3^2\Lambda_y))
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{8(3ty^2 + t^3)}{3(t^2 - y^2)^2}(I_0(\Lambda_y) + I_0(\omega_3\Lambda_y) + I_0(\omega_3^2\Lambda_y)) \\
&\quad + \frac{16(3ty^2 + t^3)}{3\sqrt{(t^2 - y^2)^5}}(I_1(\Lambda_y) - \omega_3^2 I_1(\omega_3\Lambda_y) - \omega_3 I_1(\omega_3^2\Lambda_y)) \\
&\quad + \frac{2(3ty^2 + t^3)}{3\sqrt{(t^2 - y^2)^3}}(I_1(\Lambda_y) + \omega_3 I_1(\omega_3\Lambda_y) + \omega_3^2 I_1(\omega_3^2\Lambda_y)).
\end{aligned}
$$

By using well-known integrals of Bessel functions (see [1] and [4]), we can obtain

$$
\begin{aligned}
\int_{-t}^{t} u_0^0(t, y)\,dy &= \frac{2}{3}(\sinh t - \omega_3^2 \sinh \omega_3 t - \omega_3 \sinh \omega_3^2 t), \\
\int_{-t}^{t} u_0^2(t, y)\,dy &= \frac{2}{3}(\sinh t - \sinh \omega_3 t + \sinh \omega_3^2 t), \\
\int_{-t}^{t} u_0^4(t, y)\,dy &= \frac{2}{3}(\sinh t + \omega_3 \sinh \omega_3 t + \omega_3^2 \sinh \omega_3^2 t), \\
\int_{-t}^{t} u_1^1(t, y)\,dy &= \int_{-t}^{t} u_1^3(t, y)\,dy = \int_{-t}^{t} u_1^5(t, y)\,dy = 0, \\
\int_{-t}^{t} u_2^1(t, y)\,dy &= \int_{-t}^{t} 2\frac{\partial}{\partial t} u_0^2(t, y)\,dy = 2\bigg(\frac{\partial}{\partial t}\int_{-t}^{t} u_0^2(t, y)\,dy - u_0^2(t, t) - u_0^2(t, -t)\bigg) \\
&= \frac{4}{3}(\cosh t - \omega_3 \cosh \omega_3 t + \omega_3^2 \cosh \omega_3^2 t), \\
\int_{-t}^{t} u_2^3(t, y)\,dy &= 2\int_{-t}^{t} \frac{\partial}{\partial t} u_0^4(t, y)\,dy = 2\bigg(\frac{\partial}{\partial t}\int_{-t}^{t} u_0^4(t, y)\,dy - u_0^4(t, t) - u_0^4(t, -t)\bigg) \\
&= \frac{4}{3}(\cosh t + \omega_3^2 \cosh \omega_3 t - \omega_3 \cosh \omega_3^2 t), \\
\int_{-t}^{t} u_2^5(t, y)\,dy &= 2\int_{-t}^{t} \frac{\partial}{\partial t} u_0^0(t, y)\,dy = 2\bigg(\frac{\partial}{\partial t}\int_{-t}^{t} u_0^0(t, y)\,dy - u_0^0(t, t) - u_0^0(t, -t)\bigg) \\
&= \frac{4}{3}(\cosh t + \cosh \omega_3 t + \cosh \omega_3^2 t) - 4.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\int_{-t}^{t} u_4^0(t, y)\,dy &= 2\int_{-t}^{t}\Big(\frac{\partial}{\partial t} u_2^1(t, y) - u_0^0(t, y)\Big)\,dy = \frac{4}{3}(\sinh t - \omega_3^2 \sinh \omega_3 t - \omega_3 \sinh \omega_3^2 t) - 4t, \\
\int_{-t}^{t} u_4^2(t, y)\,dy &= 2\int_{-t}^{t}\Big(\frac{\partial}{\partial t} u_2^3(t, y) - u_0^2(t, y)\Big)\,dy = \frac{4}{3}(\sinh t - \sinh \omega_3 t + \sinh \omega_3^2 t), \\
\int_{-t}^{t} u_4^4(t, y)\,dy &= 2\int_{-t}^{t}\Big(\frac{\partial}{\partial t} u_2^5(t, y) - u_0^4(t, y)\Big)\,dy = \frac{4}{3}(\sinh t + \omega_3 \sinh \omega_3 t + \omega_3^2 \sinh \omega_3^2 t), \\
\int_{-t}^{t} u_6^1(t, y)\,dy &= \int_{-t}^{t}\Big(2\frac{\partial}{\partial t} u_4^2(t, y) - u_2^1(t, y)\Big)\,dy = \frac{4}{3}(\cosh t - \omega_3 \cosh \omega_3 t + \omega_3^2 \cosh \omega_3^2 t) - 2t^2, \\
\int_{-t}^{t} u_6^3(t, y)\,dy &= \int_{-t}^{t}\Big(2\frac{\partial}{\partial t} u_4^4(t, y) - u_2^3(t, y)\Big)\,dy = \frac{4}{3}(\cosh t + \omega_3^2 \cosh \omega_3 t - \omega_3 \cosh \omega_3^2 t),
\end{aligned}
$$

$$\int_{-t}^{t} u_6^5(t,y)\,dy = \int_{-t}^{t}\Big(2\frac{\partial}{\partial t}u_4^0(t,y) - u_2^5(t,y)\Big)\,dy = \frac{4}{3}(\cosh t + \cosh \omega_3 t + \cosh \omega_3^2 t) - 4.$$

All remaining integrals of the functions $u_k^l(t,y)$ are equal to zero.

Next, for $t \le |y|$ we consider the function $g(t,y) = g_c(t,y) + g_s(t,y)$, where

$$\begin{aligned} g_c(t,y) &= \frac{1}{4}(u_2^3(t,y) + u_2^5(t,y) + u_3^2(t,y) + u_4^0(t,y) + u_4^2(t,y) + u_4^4(t,y) + u_6^1(t,y)),\\ g_s(t,y) &= \delta(t-y) + t\delta(t-y) + \frac{t^2}{2}\delta(t-y). \end{aligned} \tag{4.1}$$

It is easily seen that the function $g_c(t,y)$ is a solution of the equation

$$\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\Big)^3 g_c(t,y) - g_c(t,y) = 0. \tag{4.2}$$

Moreover, the function $f_c(t,x) = e^{-t}g_c(t,x)$ is the solution of equation (1.1) in case when $m = 3$ and $\lambda = \nu = 1$. Hence, we assume that $f(t,x) = e^{-t}(g_c(t,x) + g_s(t,x))$. Furthermore, keeping in mind the values of integrals of functions that appear in the expression for $g_c(t,y)$, we have

$$\int_{-t}^{t} f(t,x)\,dx = 1 \quad \text{for all } t \ge 0.$$

Now, let us prove that

$$\lim_{x\uparrow t} f_c(t,x) = \lim_{\varepsilon\downarrow 0}\frac{P\{0 < t - x(t) < \varepsilon\}}{\varepsilon}$$

and

$$\lim_{x\downarrow -t} f_c(t,x) = \lim_{\varepsilon\downarrow 0}\frac{P\{t + x(t) < \varepsilon\}}{\varepsilon}.$$

Applying [9, Lemma 1.2] for $m = 3$,

$$\begin{aligned} \lim_{\varepsilon\downarrow 0}\frac{P\{0 < t - x(t) < \varepsilon\}}{\varepsilon} &= \frac{1}{4}t^2 e^{-t},\\ \lim_{\varepsilon\downarrow 0}\frac{P\{t + x(t) < \varepsilon\}}{\varepsilon} &= 0. \end{aligned}$$

It is easy to verify that

$$\begin{aligned} \lim_{y\uparrow t} g_c(t,y) &= \frac{1}{4}\Big(0 + 0 + \frac{t^2}{2} + 0 + \frac{t^2}{2} + 0 + 0\Big) = \frac{t^2}{4},\\ \lim_{y\downarrow -t} g_c(t,y) &= \frac{1}{4}\Big(0 + 0 - \frac{t^2}{2} + 0 + \frac{t^2}{2} + 0 + 0\Big) = 0. \end{aligned} \tag{4.3}$$

Then,

$$\begin{aligned} \lim_{x\uparrow t} f_c(t,x) &= \frac{1}{4}t^2 e^{-t} = \lim_{\varepsilon\downarrow 0}\frac{P\{t - x(t) < \varepsilon\}}{\varepsilon},\\ \lim_{x\downarrow -t} f_c(t,x) &= 0 = \lim_{\varepsilon\downarrow 0}\frac{P\{t + x(t) < \varepsilon\}}{\varepsilon}. \end{aligned}$$

Conditions (4.3) in addition to the condition

$$\int_{-t}^{t} g(t,y)e^{-t}\,dy = 1$$

insure the uniqueness of the solution $g_c(t,y)$ for equation (4.2), and consequently the uniqueness of the solution $f_c(t,x)$ of equation (1.1). Moreover, all the solutions of these equations, when $m = 1$ and $m = 2$ ([9]), correspond to the solution of equation (4.2) when $m = 3$. □

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