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On the Cauchy theorem for hyperholomorphic functions of spatial variable

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Abstract. We proved a theorem about the integral of quaternionic-differentiable functions of spatial variable over the closed surface. It is an analog of the Cauchy theorem from complex analysis.

Keywords. Quaternion, Dirac operator, differentiable function.

1. Introduction

Several researchers (see, e.g., [1,2]) tried to generalize methods of complex analysis onto the analysis of functions acting in several-dimensional algebras. At that, generalizations of different but mutually equivalent definitions of holomorphy in complex analysis generate diverse classes of hyperholomorphic functions in several-dimensional algebras.

Hypercomplex analysis in the space $\mathbb{R}^3$ was launched in the work of G. Moisil and N. Theodoresco [3], where a three-dimensional analog of the Cauchy–Riemann system was posed for the first time. R. Fueter [4] first introduced a class of "regular" quaternion functions by means of a four-dimensional generalization of the Moisil–Theodoresco system. He proved quaternion analogs of the Cauchy theorem, integral Cauchy formula, and Liouville theorem and constructed an analog of the Laurent series.

Now, quaternion analysis gained a wide evolution (more details can be found in [1, 5–7]) due to its physical applications. In most works, it was usual to consider functions having continuous partial derivatives in a domain and satisfying the above Cauchy–Riemann-type system. In particular in [1], a spatial analog of the Cauchy theorem was proved, by using the quaternion Stokes formula for bounded domains with a piecewise-smooth boundary and for functions having continuous partial derivatives in the closure of the domain.

In the survey paper [6], the continuity of partial derivatives was replaced by a weaker condition of real-differentiability for components of the quaternion function. In [8], we considered the same class of functions defined in a three-dimensional domain with a piecewise-smooth boundary and requiring only the componentwise real-differentiability and satisfying the Cauchy–Riemann-type conditions, like the class of holomorphic functions in complex analysis (see, e.g., [9]).

In the present work, we will extend the result in [8] onto a wider class of surfaces, by using methods of work [10], where a similar theorem was proved for functions taking values in finite-dimensional commutative associative algebras.

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2. Quaternion hyperholomorphic functions

Let $\mathbb{H}(\mathbb{C})$ be the associative algebra of *complex quaternions*

$$a = \sum_{k=0}^{3} a_k \boldsymbol{i}_k,$$

where $\{a_k\}_{k=0}^3 \subset \mathbb{C}$, $\boldsymbol{i}_0 = 1$ and $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$ be the imaginary quaternion units with the multiplication rule $\boldsymbol{i}_1^2 = \boldsymbol{i}_2^2 = \boldsymbol{i}_3^2 = \boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3 = -1$. The module of a quaternion is defined by the formula

$$|a| := \sqrt{\sum_{k=0}^{3} |a_k|^2}.$$

Lemma 2.1 ([11])**.** *$|ab| \leqslant \sqrt{2}\,|a|\,|b|$ for all $\{a; b\} \subset \mathbb{H}(\mathbb{C})$.*

For $\{z_k\}_{k=1}^3 \subset \mathbb{R}$, consider the vector quaternions $z := z_1\boldsymbol{i}_1 + z_2\boldsymbol{i}_2 + z_3\boldsymbol{i}_3$ as points of the Euclidean space $\mathbb{R}^3$ with the basis $\{\boldsymbol{i_k}\}_{k=1}^3$. Let Ω be a domain of $\mathbb{R}^3$. For functions $f : \Omega \to \mathbb{H}(\mathbb{C})$ having first-order partial derivatives, consider the differential operators

$$D_l[f] := \sum_{k=1}^{3} \boldsymbol{i}_k \frac{\partial f}{\partial z_k},$$

$$D_r[f] := \sum_{k=1}^{3} \frac{\partial f}{\partial z_k} \boldsymbol{i}_k.$$

Definition 2.1. *The function $f := f_0 + f_1\boldsymbol{i}_1 + f_2\boldsymbol{i}_2 + f_3\boldsymbol{i}_3$ is called* left- *or* right-$\mathbb{H}$-differentiable *at a point $z^{(0)} \in \mathbb{R}^3$, if its components f_0, f_1, f_2, and f_3 are $\mathbb{R}^3$-differentiable functions in $z^{(0)}$, and if the condition*

$$D_l[f](z^{(0)}) = 0 \tag{2.1}$$

or

$$D_r[f](z^{(0)}) = 0$$

holds true, respectively.

There is the notion of $\mathbb{C}$-differentiability of a function $f(\zeta) = u(x, y) + v(x, y)\boldsymbol{i}$, $\zeta = x + y\boldsymbol{i}$, in complex analysis (see [9, p. 33–34]). It is equivalent to $\mathbb{R}^2$-differentiability at the point (x_0, y_0) of the components $u(x, y)$ and $v(x, y)$ and to the validity of the condition

$$\frac{\partial f(\zeta_0)}{\partial x} + \frac{\partial f(\zeta_0)}{\partial y}\boldsymbol{i} = 0.$$

Thus, the above-defined notion of $\mathbb{H}$-differentiability is the exact analog of $\mathbb{C}$-differentiability from complex analysis.

It is well known (see [9, p. 35]) that the $\mathbb{C}$-differentiability of a complex function is equivalent to the existence of its derivative. But only the linear functions of a special form have a derivative in quaternion analysis (see [12]).

The operator D_l is called the Dirac operator (see [13]) or the Moisil–Theodoresco operator (see [14]) and equality (2.1) is equivalent to the Moisil–Theodoresco system [3].

Definition 2.2. *A function f is called* left- *or* right-hyperholomorphic *in a domain Ω, if it is left- or right-$\mathbb{H}$-differentiable at every point of the domain.*

3. Quaternion surface integral

Consider the notions of surface and closed surface like those defined in work [10].

Definition 3.1. *A* surface $\Gamma \subset \mathbb{R}^3$ *is an image of the closed set* $G \subset \mathbb{R}^2$ *under a homeomorphic mapping* $\varphi : G \to \mathbb{R}^3$

$$\varphi(u,v) := (z_1(u,v), z_2(u,v), z_3(u,v)),\ (u,v) \in G,$$

such that the Jacobians

$$A := \frac{\partial z_2}{\partial u}\frac{\partial z_3}{\partial v} - \frac{\partial z_2}{\partial v}\frac{\partial z_3}{\partial u},\ B := \frac{\partial z_3}{\partial u}\frac{\partial z_1}{\partial v} - \frac{\partial z_3}{\partial v}\frac{\partial z_1}{\partial u},\ C := \frac{\partial z_1}{\partial u}\frac{\partial z_2}{\partial v} - \frac{\partial z_1}{\partial v}\frac{\partial z_2}{\partial u}$$

exist almost everywhere on the set G *and are summable on* G.

The area of the surface Γ is calculated by the formula

$$\mathcal{L}(\Gamma) = \iint\limits_{G} \sqrt{A^2 + B^2 + C^2}\,dudv,$$

where the integral is understood in the Lebesgue sense.

A surface Γ is called *quadrable* (see [10]), if $\mathcal{L}(\Gamma) < +\infty$.

Let $\Gamma \subset \mathbb{R}^3$ be an image of the sphere $S \subset \mathbb{R}^3$ under such homeomorphic mapping $\psi : S \to \mathbb{R}^3$ that the image of a great circle γ on the sphere S is a closed Jordan rectifiable curve $\widetilde{\gamma}$ on the set Γ. The sphere S is the union of two half-spheres S_1 and S_2 with the common edge γ. It is easy to see that there exist continuously differentiable mappings $\varphi_1 : K \to S_1$, $\varphi_2 : K \to S_2$ of the disk $K := \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leqslant 1\}$. So the set Γ is the union of two sets $\Gamma_1 = \psi(\varphi_1(K))$, $\Gamma_2 = \psi(\varphi_2(K))$ with the intersection $\widetilde{\gamma} = \psi(\varphi_1(\partial K)) = \psi(\varphi_2(\partial K))$.

Definition 3.2. *A set* Γ *is called a* closed surface, *if there exist such homeomorphic mapping* $\psi : S \to \mathbb{R}^3$ *that the sets* Γ_1, Γ_2 *are surfaces in the sense of Definition 3.1, and the orientation of the circle* ∂K *induces two mutually opposite orientations of the curve* $\widetilde{\gamma}$ *under the mappings* $\psi \circ \varphi_1$ *and* $\psi \circ \varphi_2$, *respectively.*

Let $\Gamma^\varepsilon := \left\{z \in \mathbb{R}^3 : \rho(z,\Gamma) \leqslant \varepsilon\right\}$ (ρ denotes the Euclidean distance) be a closed ε-neighborhood of the surface Γ, let $V(\Gamma^\varepsilon)$ be the space Lebesgue measure of the set Γ^ε, and let $\mathcal{M}^*(\Gamma) := \overline{\lim\limits_{\varepsilon\to 0}} \frac{V(\Gamma^\varepsilon)}{2\varepsilon}$ be the two-dimensional upper Minkowski content (see [15, p. 79]) of the surface Γ. For the functions $f : \Gamma \to \mathbb{H}(\mathbb{C})$ and $g : \Gamma \to \mathbb{H}(\mathbb{C})$ in the case of non-closed quadrable surface Γ, the *quaternion surface integral* is defined by the formula

$$\iint\limits_{\Gamma} f(z)\,\sigma\,g(z) := \iint\limits_{G} f(\varphi(u,v))(A\boldsymbol{i}_1 + B\boldsymbol{i}_2 + C\boldsymbol{i}_3)g(\varphi(u,v))du\,dv,$$

where $\sigma := dz_2dz_3\boldsymbol{i}_1 + dz_3dz_1\boldsymbol{i}_2 + dz_1dz_2\boldsymbol{i}_3$, and, in the case of a closed surface, by the formula

$$\iint\limits_{\Gamma} f(z)\,\sigma\,g(z) := \iint\limits_{\Gamma_1} f(z)\,\sigma\,g(z) + \iint\limits_{\Gamma_2} f(z)\,\sigma\,g(z).$$

In particular, $\iint\limits_{\Gamma} |\sigma| = \mathcal{L}(\Gamma)$.

Theorem 3.1 ([8]). *Let P be the surface of a closed cube contained in a simply connected domain $\Omega \subset \mathbb{R}^3$, let a function $f : \Omega \to \mathbb{H}(\mathbb{C})$ be right-hyperholomorphic, and let a function $g : \Omega \to \mathbb{H}(\mathbb{C})$ be left-hyperholomorphic. Then*

$$\iint\limits_P f(z)\,\sigma\,g(z) = 0.$$

Let $\delta > 0$, let $\omega_\Gamma(f,\delta) := \sup\limits_{\substack{|z_1 - z_2| \leqslant \delta \\ z_1, z_2 \in \Gamma}} |f(z_1) - f(z_2)|$ be the module of continuity of a function f on Γ, and let $d(\Gamma)$ be the diameter of Γ.

Lemma 3.1 ([10]). *Let Γ be a quadrable closed surface. Then*

$$\iint\limits_\Gamma \sigma = 0. \tag{3.1}$$

Lemma 3.2. *Let Γ be a quadrable closed surface, and let $f : \Omega \to \mathbb{H}(\mathbb{C})$ and $g : \Omega \to \mathbb{H}(\mathbb{C})$ be continuous functions. Then*

$$\left| \iint\limits_\Gamma f(z)\,\sigma\,g(z) \right| \leqslant 2\,\mathcal{L}(\Gamma) \left(\omega_\Gamma(f, d(\Gamma)) \max_{z\in\Gamma} |g(z)| + \omega_\Gamma(g, d(\Gamma)) \max_{z\in\Gamma} |f(z)| \right). \tag{3.2}$$

Proof. In view of formula (3.1), we have

$$\iint\limits_\Gamma f(z_0)\,\sigma\,g(z_0) = 0$$

for any point $z_0 \in \Gamma$. Therefore,

$$\iint\limits_\Gamma f(z)\,\sigma\,g(z) = \iint\limits_\Gamma (f(z) - f(z_0))\,\sigma\,g(z_0) + \iint\limits_\Gamma f(z)\,\sigma\,(g(z) - g(z_0)),$$

which yields estimate (3.2) with regard for Lemma 2.1. □

Theorem 3.2. *Let $\mathbb{R}^3 \supset \Omega$ be a bounded simply connected domain with the quadrable closed boundary Γ, for which*

$$\mathcal{M}^*(\Gamma) < +\infty, \tag{3.3}$$

let Ω have Jordan measurable intersections with planes perpendicular to coordinate axes, let a function $f : \overline{\Omega} \to \mathbb{H}(\mathbb{C})$ be right-hyperholomorphic in Ω and continuous in the closure $\overline{\Omega}$, and let a function $g : \overline{\Omega} \to \mathbb{H}(\mathbb{C})$ be left-hyperholomorphic in Ω and continuous in $\overline{\Omega}$. Then

$$\iint\limits_\Gamma f(z)\,\sigma\,g(z) = 0. \tag{3.4}$$

Proof. Let us use the method proposed in work [10] in the proof of Theorem 6.1. Due to condition (3.3), there exists such constant $c > 0$ that, for all sufficiently small $\varepsilon > 0$, the following inequality holds:

$$V(\Gamma^\varepsilon) \leqslant c\varepsilon. \tag{3.5}$$

Decompose the space by planes perpendicular to the coordinate axes onto closed cubes with the edge $\frac{\varepsilon}{\sqrt{3}}$ in length. Let $\{K_j\}$, $j \in J$, be a finite set of formed cubes having a nonempty intersection with the surface Γ.

The integral (3.4) is representable in the form

$$\iint_{\Gamma} f(z)\,\sigma\, g(z) = \sum_{j\in J} \iint_{\partial(\Omega\cap K_j)} f(z)\,\sigma\, g(z) + \sum_{K_j\subset\Omega} \iint_{\partial K_j} f(z)\,\sigma\, g(z). \tag{3.6}$$

By Theorem 3.1, the second sum in equality (3.6) is equal to zero.

Every set $\Omega\cap K_j$ consists of a finite or infinite totality of connected components. Applying estimate (3.2) to the boundary of the every component, we obtain

$$\left| \iint_{\partial(\Omega\cap K_j)} f(z)\,\sigma\, g(z) \right| \leqslant 2(\mathcal{L}(\Gamma\cap K_j) + 2\varepsilon^2) \left(\omega_\Gamma(f,\varepsilon) \max_{z\in\overline{\Omega}} |g(z)| + \omega_\Gamma(g,\varepsilon) \max_{z\in\overline{\Omega}} |f(z)| \right). \tag{3.7}$$

Substituting inequality (3.7) into equality (3.6), we obtain

$$\left| \iint_{\Gamma} f(z)\,\sigma\, g(z) \right| \leqslant 2 \left(\mathcal{L}(\Gamma) + 2\sum_{j\in J} \varepsilon^2 \right) \left(\omega_\Gamma(f,\varepsilon) \max_{z\in\overline{\Omega}} |g(z)| + \omega_\Gamma(g,\varepsilon) \max_{z\in\overline{\Omega}} |f(z)| \right).$$

Since $\bigcup_{j\in J} K_j \subset \Gamma^\varepsilon$, we obtain from inequality (3.5) that

$$\frac{1}{3\sqrt{3}} \sum_{j\in J} \varepsilon^3 \leqslant V(\Gamma^\varepsilon) \leqslant c\varepsilon.$$

Therefore,

$$\left| \iint_{\Gamma} f(z)\,\sigma\, g(z) \right| \leqslant 2(\mathcal{L}(\Gamma) + 6\sqrt{3}c) \left(\omega_\Gamma(f,\varepsilon) \max_{z\in\overline{\Omega}} |g(z)| + \omega_\Gamma(g,\varepsilon) \max_{z\in\overline{\Omega}} |f(z)| \right),$$

and equality (3.4) can be obtained from here by passing to the limit as $\varepsilon \to 0$. □

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