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Abstract	We consider Schwartz-type boundary value problems for monogenic functions in a commutative algebra $\mathbb{B}$ over the field of complex numbers with the bases $\{e_1, e_2\}$ satisfying the conditions , . The algebra is associated with the biharmonic equation, and considered problems have relations to the plane elasticity. We develop methods of its solving which are based on expressions of solutions by hypercomplex integrals analogous to the classic Schwartz and Cauchy integrals.	
Keywords (separated by "-")	Biharmonic equation - Biharmonic algebra - Biharmonic plane - Monogenic function - Schwartz-type boundary value problem	
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Schwartz-Type Boundary Value Problems for Monogenic Functions in a Biharmonic Algebra

S. V. Gryshchuk and S. A. Plaksa

Dedicated to Professor Heinrich G.W. Begher on the occasion of his 80th birthday

Abstract We consider Schwartz-type boundary value problems for monogenic functions in a commutative algebra $\mathbb{B}$ over the field of complex numbers with the bases $\{e_1, e_2\}$ satisfying the conditions $(e_1^2 + e_2^2)^2 = 0$, $e_1^2 + e_2^2 \neq 0$. The algebra $\mathbb{B}$ is associated with the biharmonic equation, and considered problems have relations to the plane elasticity. We develop methods of its solving which are based on expressions of solutions by hypercomplex integrals analogous to the classic Schwartz and Cauchy integrals.

Keywords Biharmonic equation · Biharmonic algebra · Biharmonic plane · Monogenic function · Schwartz-type boundary value problem

Mathematics Subject Classification (2010) Primary 30G35; Secondary 31A30

1 Monogenic Functions in a Biharmonic Algebra

Definition 1.1 An associative commutative two-dimensional algebra $\mathbb{B}$ with the unit 1 over the field of complex numbers $\mathbb{C}$ is called *biharmonic* (see [1, 2]) if in $\mathbb{B}$ there exists a basis $\{e_1, e_2\}$ satisfying the conditions

$$(e_1^2 + e_2^2)^2 = 0, \qquad e_1^2 + e_2^2 \neq 0\,.$$

Such a basis $\{e_1, e_2\}$ is also called *biharmonic*.

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In the paper [2] I.P. Mel'nichenko proved that there exists the unique biharmonic algebra $\mathbb{B}$, and he constructed all biharmonic bases in $\mathbb{B}$. Note that the algebra $\mathbb{B}$ is isomorphic to four-dimensional over the field of real numbers $\mathbb{R}$ algebras considered by A. Douglis [3] and L. Sobrero [4].

In what follows, we consider a biharmonic basis $\{e_1, e_2\}$ with the following multiplication table (see [1]):

$$e_1 = 1, \qquad e_2^2 = e_1 + 2ie_2\,, \tag{1.1}$$

where i is the imaginary complex unit. We consider also a basis $\{1, \rho\}$ (see [2]), where a nilpotent element

$$\rho = 2e_1 + 2ie_2 \tag{1.2}$$

satisfies the equality $\rho^2 = 0$.

We use the Euclidean norm $\|a\| := \sqrt{|z_1|^2 + |z_2|^2}$ in the algebra $\mathbb{B}$, where $a = z_1e_1 + z_2e_2$ and $z_1, z_2 \in \mathbb{C}$.

Consider a *biharmonic plane* $\mu_{e_1,e_2} := \{\zeta = x\,e_1 + y\,e_2 : x, y \in \mathbb{R}\}$ which is a linear span of the elements e_1, e_2 of the biharmonic basis (1.1) over the field $\mathbb{R}$.

With a domain D of the Cartesian plane xOy we associate the congruent domain $D_\zeta := \{\zeta = xe_1 + ye_2 \in \mu_{e_1,e_2} : (x, y) \in D\}$ in the biharmonic plane μ_{e_1,e_2} and the congruent domain $D_z := \{z = x + iy : (x, y) \in D\}$ in the complex plane $\mathbb{C}$. Its boundaries are denoted by ∂D, ∂D_ζ and ∂D_z, respectively. Let $\overline{D_\zeta}$ (or $\overline{D_z}$, $\overline{D}$) be the closure of domain D_ζ (or D_z, D, respectively).

In what follows, $\zeta = x\,e_1 + y\,e_2$, $z = x + iy$, where $(x, y) \in D$, and $\zeta_0 = x_0\,e_1 + y_0\,e_2$, $z_0 = x_0 + iy_0$, where $(x_0, y_0) \in \partial D$.

Any function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ has an expansion of the type

$$\Phi(\zeta) = U_1(x, y)\,e_1 + U_2(x, y)\,ie_1 + U_3(x, y)\,e_2 + U_4(x, y)\,ie_2\,, \tag{1.3}$$

where $U_l\colon D \longrightarrow \mathbb{R}$, $l = 1, 2, 3, 4$, are real-valued component-functions. We shall use the following notation: $\mathrm{U}_l\,[\Phi] := U_l$, $l = 1, 2, 3, 4$.

Definition 1.2 A function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ is *monogenic* in a domain D_ζ if it has the classical derivative $\Phi'(\zeta)$ at every point $\zeta \in D_\zeta$:

$$\Phi'(\zeta) := \lim_{h\to 0,\, h\in\mu_{e_1,e_2}} \big(\Phi(\zeta + h) - \Phi(\zeta)\big)\, h^{-1}\,.$$

It is proved in [1] that a function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ is monogenic in D_ζ if and only if its each real-valued component-function in (1.3) is real differentiable in D and the following analog of the Cauchy–Riemann condition is fulfilled:

$$\frac{\partial\Phi(\zeta)}{\partial y} = \frac{\partial\Phi(\zeta)}{\partial x}\, e_2. \tag{1.4}$$

Rewriting the condition (1.4) in the extended form, we obtain the system of four equations (cf., e.g., [1, 5]) with respect to component-functions U_k, $k = \overline{1,4}$, in (1.3):

$$\begin{aligned} \frac{\partial U_1(x,y)}{\partial y} &= \frac{\partial U_3(x,y)}{\partial x}, \\ \frac{\partial U_2(x,y)}{\partial y} &= \frac{\partial U_4(x,y)}{\partial x}, \\ \frac{\partial U_3(x,y)}{\partial y} &= \frac{\partial U_1(x,y)}{\partial x} - 2\frac{\partial U_4(x,y)}{\partial x}, \\ \frac{\partial U_4(x,y)}{\partial y} &= \frac{\partial U_2(x,y)}{\partial x} + 2\frac{\partial U_3(x,y)}{\partial x}. \end{aligned} \tag{1.5}$$

All component-functions U_l, $l = 1, 2, 3, 4$, in the expansion (1.3) of any monogenic function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ are biharmonic functions (cf., e.g., [5, 6]), i.e., satisfy the biharmonic equation in D:

$$\Delta^2 U(x,y) \equiv \frac{\partial^4 U(x,y)}{\partial x^4} + 2\frac{\partial^4 U(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 U(x,y)}{\partial y^4} = 0.$$

At the same time, every biharmonic in a simply-connected domain D function $U(x,y)$ is the first component $U_1 \equiv U$ in the expression (1.3) of a certain function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ monogenic in D_ζ and, moreover, all such functions Φ are found in [5, 6] in an explicit form.

Every monogenic function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ is expressed via two corresponding analytic functions $F\colon D_z \longrightarrow \mathbb{C}$, $F_0\colon D_z \longrightarrow \mathbb{C}$ of the complex variable z in the form (cf., e.g., [5, 6]):

$$\Phi(\zeta) = F(z)e_1 - \left(\frac{iy}{2}\,F'(z) - F_0(z)\right)\rho \quad \forall\, \zeta \in D_\zeta\,. \tag{1.6}$$

The equality (1.6) establishes one-to-one correspondence between monogenic functions Φ in the domain D_ζ and pairs of complex-valued analytic functions F, F_0 in the domain D_z.

Using the equality (1.2), we rewrite the expansion (1.6) for all $\zeta \in D_\zeta$ in the basis $\{e_1, e_2\}$:

$$\Phi(\zeta) = \Big(F(z) - iyF'(z) + 2F_0(z)\Big)e_1 + i\Big(2F_0(z) - iyF'(z)\Big)e_2\,. \tag{1.7}$$

2 Schwartz-Type BVP's for Monogenic Functions

Consider a boundary value problem on finding a function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ which is monogenic in a domain D_ζ when limiting values of two component-functions in (1.3) are given on the boundary ∂D_ζ, i.e., the following boundary conditions are satisfied:

$$U_k(x_0, y_0) = u_k(\zeta_0)\,, \quad U_m(x_0, y_0) = u_m(\zeta_0) \qquad \forall\, \zeta_0 \in \partial D_\zeta$$

for $1 \le k < m \le 4$, where

$$U_l(x_0, y_0) = \lim_{\zeta \to \zeta_0, \zeta \in D_\zeta} \mathrm{U}_l\left[\Phi\left(\zeta\right)\right], \qquad l \in \{k, m\},$$

and u_k, u_m are given continuous functions.

We demand additionally the existence of finite limits

$$\lim_{\|\zeta\| \to \infty,\, \zeta \in D_\zeta} \mathrm{U}_l\left[\Phi(\zeta)\right], \qquad l \in \{k, m\},$$

in the case where the domain D_ζ is unbounded as well as the assumption that every given function u_l, $l \in \{k, m\}$, has a finite limit

$$u_l(\infty) := \lim_{\|\zeta\| \to \infty,\, \zeta \in \partial D_\zeta} u_l(\zeta) \tag{2.1}$$

if ∂D_ζ is unbounded.

We shall call such a problem by the (k-m)-problem.

V.F. Kovalev [7] considered (k-m)-problems with additional assumptions that the sought-for function $\Phi : \overline{D_\zeta} \longrightarrow \mathbb{B}$ is continuous in $\overline{D_\zeta}$ and has the limit

$$\lim_{\|\zeta\| \to \infty,\, \zeta \in D_\zeta} \Phi(\zeta) =: \Phi(\infty) \in \mathbb{B}$$

in the case where the domain D_ζ is unbounded. He named such problems as *biharmonic Schwartz problems* owing to their analogy with the classic Schwartz problem on finding an analytic function of a complex variable when values of its real part are given on the boundary of domain. We shall call problems of such a type as (k-m)-*problems in the sense of Kovalev*.

Note, that in previous papers [5, 6, 8–13] we interpret the (k-m)-problem as the (k-m)-problem in the sense of Kovalev.

It was established in [7] that all (k-m)-problems are reduced to the main three problems: with $k = 1$ and $m \in \{2, 3, 4\}$, respectively.

It is shown (see [7–9]) that the main biharmonic problem is reduced to the (1–3)-problem. In [8], we investigated the (1–3)-problem for cases where D_ζ is either a half-plane or a unit disk in the biharmonic plane. Its solutions were found in explicit forms with using of some integrals analogous to the classic Schwartz integral.

In [9, 10], using a hypercomplex analog of the Cauchy type integral, we reduced the (1–3)-problem to a system of integral equations and established sufficient conditions under which this system has the Fredholm property. It was made for the case where the boundary of domain belongs to a class being wider than the class of Lyapunov curves that was usually required in the plane elasticity theory (cf., e.g., [14–18]). The similar is done for the (1–4)-problem in [12].

In [12, 13], there is considered a relation between (1–4)-problem and boundary value problems of the plane elasticity theory. Namely, there is considered a problem on finding an elastic equilibrium for isotropic body occupying D with given limiting values of partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$ for displacements $u = u(x, y)$, $v = v(x, y)$ on the boundary ∂D. In particular, it is shown in [13] that such a problem is reduced to (1–4)-problem.

3 (1–3)-Problem and a Biharmonic Problem

A *biharmonic problem* (cf., e.g., [14, p. 13]) is a boundary value problem on finding a biharmonic function $V : D \longrightarrow \mathbb{R}$ with the following boundary conditions:

$$\begin{aligned} &\lim_{(x,y)\to(x_0,y_0),\,(x,y)\in D} \frac{\partial V(x,y)}{\partial x} = u_1(x_0,y_0)\,, \\ &\lim_{(x,y)\to(x_0,y_0),\,(x,y)\in D} \frac{\partial V(x,y)}{\partial y} = u_3(x_0,y_0) \qquad \forall\,(x_0,y_0)\in\partial D\,. \end{aligned} \tag{3.1}$$

It is well-known a great importance of the biharmonic problem in the plane elasticity theory (see, e.g., [14, 19]).

Let Φ_1 be monogenic in D_ζ function having the sought-for function $V(x, y)$ of the problem (3.1) as the first component:

$$\Phi_1(\zeta) = V(x,y)\,e_1 + V_2(x,y)\,ie_1 + V_3(x,y)\,e_2 + V_4(x,y)\,ie_2\,.$$

Differentiating the previous equality with respect to x and using a condition of the type (1.5) for the monogenic function Φ_1, we obtain

$$\Phi_1'(\zeta) = \frac{\partial V(x,y)}{\partial x}\,e_1 + \frac{\partial V_2(x,y)}{\partial x}\,ie_1 + \frac{\partial V(x,y)}{\partial y}\,e_2 + \frac{\partial V_4(x,y)}{\partial x}\,ie_2$$

and, as consequence, we conclude that the biharmonic problem with boundary conditions (3.1) is reduced to the (1–3)-problem for monogenic functions with the same boundary data.

In what follows, let us agree to use the same denomination u for functions $u\colon \partial D \longrightarrow \mathbb{R}$, $u\colon \partial D_z \longrightarrow \mathbb{R}$, $u\colon \partial D_\zeta \longrightarrow \mathbb{R}$ taking the same values at corresponding points of boundaries ∂D, ∂D_z, ∂D_ζ, respectively, i.e., $u(x_0, y_0) = u(z_0) = u(\zeta_0)$ for all $(x_0, y_0) \in \partial D$.

A necessary condition of solvability of the (1–3)-problem as well as the biharmonic problem (3.1) is the following (cf., e.g., [9]):

$$\int\limits_{\partial D} u_1(x, y)\, dx + u_3(x, y)\, dy = 0. \tag{3.2}$$

Below, we state assumptions, under which the condition (3.2) is also sufficient for the solvability of the (1–3)-problem.

4 Boundary Value Problems Associated with a (1–4)-Problem

Now, we assume that D is a bounded simply connected domain in the Cartesian plane xOy. For a function $u\colon D \longrightarrow \mathbb{R}$ we denote a limiting value at a point $(x_0, y_0) \in \partial D$ by

$$u(x, y)\Big|_{(x_0, y_0)} := \lim_{(x,y)\in D, (x,y)\to(x_0,y_0)} u(x, y)\,,$$

if there exists such a finite limit.

Consider a boundary value problem: to find in D partial derivatives $\mathcal{V}_1 := \frac{\partial u}{\partial x}$, $\mathcal{V}_2 := \frac{\partial v}{\partial y}$ for displacements $u = u(x, y)$, $v = v(x, y)$ of an elastic isotropic body occupying D, when their limiting values are given on the boundary ∂D:

$$\mathcal{V}_k(x, y)\Big|_{(x_0, y_0)} = h_k(x_0, y_0) \quad \forall\, (x_0, y_0) \in \partial D, \quad k = 1, 2, \tag{4.1}$$

where $h_k\colon \partial D \longrightarrow \mathbb{R}$, $k = 1, 2$, are given functions.

We shall call this problem as the (u_x, v_y)-*problem*. This problem has been posed in [13].

For a biharmonic function $W\colon D \longrightarrow \mathbb{R}$ we denote

$$\mathrm{C}_k[W](x, y) := -W_k(x, y) + \kappa_0\, W_0(x, y) \qquad \forall\, (x, y) \in D, \quad k = 1, 2,$$

where

$$W_1(x, y) := \frac{\partial^2 W(x, y)}{\partial x^2}\,, \quad W_2(x, y) := \frac{\partial^2 W(x, y)}{\partial y^2}\,,$$

$$W_0(x, y) := W_1(x, y) + W_2(x, y)\,,$$

$\kappa_0 := \frac{\lambda+2\mu}{2(\lambda+\mu)}$, μ and λ are Lamé constants (cf., e.g., [19, p. 2]).

The following equalities are valid in D (cf., e.g., [19, pp. 8–9],[14, p. 5]):

$$2\mu\, \mathcal{V}_k(x, y) = \mathrm{C}_k[W](x, y) \qquad \forall\,(x, y) \in D, \quad k = 1, 2.$$

Then solving the (u_x, v_y)-problem is reduced to finding the functions $\mathrm{C}_k[W]$, $k = 1, 2,$ in D with an unknown biharmonic function $W\colon D \longrightarrow \mathbb{R}$, when their limiting values satisfy the system

$$C_k[W](x, y)\Big|_{(x_0, y_0)} = 2\mu\, h_k(x_0, y_0) \qquad \forall\,(x_0, y_0) \in \partial D, \quad k = 1, 2. \tag{4.2}$$

Consider some auxiliary statements.

Lemma 4.1 *([13]) Let W be a biharmonic function in a domain D and Φ_* be a monogenic in D_ζ function such that $\mathrm{U}_1[\Phi_*] = W$. Then the following equalities are true*:

$$\frac{\partial^2 W(x, y)}{\partial x^2} = \mathrm{U}_1[\Phi(\zeta)], \qquad \frac{\partial^2 W(x, y)}{\partial y^2} = \mathrm{U}_1[\Phi(\zeta)] - 2\mathrm{U}_4[\Phi(\zeta)], \tag{4.3}$$

for every $(x, y) \in D$, where $\Phi := \Phi_*''$.

Lemma 4.2 *([13]) The (u_x, v_y)-problem is equivalent to a boundary value problem on finding in D the second derivatives $\frac{\partial^2 W(x,y)}{\partial x^2}$, $\frac{\partial^2 W(x,y)}{\partial y^2}$ of a biharmonic function W, which have limiting values at all $(x_0, y_0) \in \partial D$ and satisfy the boundary data*:

$$\frac{\partial^2 W(x, y)}{\partial x^2}\Big|_{(x_0, y_0)} = \lambda\, h_1(x_0, y_0) + (\lambda + 2\mu)\, h_2(x_0, y_0),$$

$$\frac{\partial^2 W(x, y)}{\partial y^2}\Big|_{(x_0, y_0)} = (\lambda + 2\mu)\, h_1(x_0, y_0) + \lambda\, h_2(x_0, y_0).$$

Then the general solution of (u_x, v_y)-problem is expressed by the formula:

$$\mathcal{V}_k(x, y) = \frac{1}{2\mu}\mathrm{C}_k[W](x, y) \qquad \forall (x, y) \in D, \quad k = 1, 2. \tag{4.4}$$

The following theorem establishes relations between solutions of (u_x, v_y)-problem and corresponding (1–4)-problem.

Theorem 4.3 *Let W be a biharmonic function satisfying the boundary conditions (4.2). Then W rebuilds the general solution of (u_x, v_y)-problem with boundary data (4.1) by the formula (4.4). The general solution Φ of (1–4)-problem with boundary data*

$$u_1 = \lambda\, h_1 + (\lambda + 2\mu)\, h_2, \quad u_4 = -\mu\, h_1 + \mu\, h_2,$$

generates the second order derivatives $\frac{\partial^2 W}{\partial x^2}$, $\frac{\partial^2 W}{\partial y^2}$ *in* D *by the formulas* (4.3). *The general solution of* (u_x, v_y)*-problem is expressed for every* $(x, y) \in D$ *by the equalities*

$$2\mu \frac{\partial u(x,y)}{\partial x} = \frac{\mu}{\lambda+\mu} \mathrm{U}_1 \left[\Phi(\zeta)\right] - \frac{\lambda+2\mu}{\lambda+\mu} \mathrm{U}_4 \left[\Phi(\zeta)\right],$$

$$2\mu \frac{\partial v(x,y)}{\partial y} = \frac{\mu}{\lambda+\mu} \mathrm{U}_1 \left[\Phi(\zeta)\right] + \frac{\lambda}{\lambda+\mu} \mathrm{U}_4 \left[\Phi(\zeta)\right].$$

A theorem analogous to Theorem 4.3 is proved in [13, Theorem 4] in assumption that the (1–4)-problem is understood in the sense of Kovalev. But it is still valid with the same proof for the (1–4)-problem formulated in this paper. It happens due to Lemmas 4.1, 4.2 and the fact that the left-hand sides of (4.3) have limiting values on ∂D if and only if $\mathrm{U}_1[\Phi]$, $\mathrm{U}_4[\Phi]$ have limiting values on ∂D_ζ.

The elastic equilibrium in terms of displacements and stresses can be found by use of the generalized Hooke's law and solutions $\mathcal{V}_1$, $\mathcal{V}_2$ of the (u_x, v_y)-problem (see [13, sect. 5]).

5 Solving Process of (1–4)-Problem via Analytic Functions of a Complex Variable

A method for solving the (1–4)-problem by means of its reduction to classic Schwartz boundary value problems for analytic functions of a complex variable is delivered in [11]. Let us formulate some results of such a kind.

In what follows, we assume that the domain D_z is simply connected (bounded or unbounded), and in this case we shall say that the domains D and D_ζ are also simply connected.

For a function $F\colon D_z \longrightarrow \mathbb{C}$ we denote its limiting value at a point $z_0 \in \partial D_z$ by $F^+(z_0)$ if it exists.

The *classic Schwartz problem* is a problem on finding an analytic function $F\colon D_z \longrightarrow \mathbb{C}$ of a complex variable when values of its real part are given on the boundary of domain, i.e.,

$$(\mathrm{Re}\, F)^+(t) = u(t) \quad \forall\, t \in \partial D_z, \tag{5.1}$$

where $u\colon \partial D_z \longrightarrow \mathbb{R}$ is a given continuous function.

Theorem 5.1 *Let* $u_l\colon \partial D_\zeta \longrightarrow \mathbb{R}$, $l \in \{1, 4\}$, *be continuous functions and* F *be a solution of the classic Schwartz problem with boundary condition*:

$$(\mathrm{Re}\, F)^+(t) = u_1(t) - u_4(t) \quad \forall\, t \in \partial D_z. \tag{5.2}$$

and, furthermore, the function

$$\widetilde{F}_*(z) := \mathrm{Re}\,\big(-iyF'(z)\big) \quad \forall z \in D_z$$

have continuous limiting values on ∂D_z*. Then a solution of the* (1–4)*-problem is expressed by the formula* (1.6) *or, the same, by the formula* (1.7)*, where the function* F_0 *is a solution of the classic Schwartz problem with boundary condition*:

$$(\mathrm{Re}\, F_0)^+(t) = \frac{1}{2}\Big(u_4(t) - \big(\widetilde{F}_*\big)^+(t)\Big) \quad \forall t \in \partial D_z. \tag{5.3}$$

Proof It follows from the expression (1.7) that the (1–4)-problem is reduced to finding a pair of analytic in D_z functions F, F_0 satisfying the following boundary conditions:

$$\begin{cases} \big(\mathrm{Re}\,\big(F + \widetilde{F}_* + 2F_0\big)\big)^+(t) = u_1(t) & \forall t \in \partial D_z\,, \\ \big(\mathrm{Re}\,\big(\widetilde{F}_* + 2F_0\big)\big)^+(t) \qquad = u_4(t) & \forall t \in \partial D_z\,. \end{cases} \tag{5.4}$$

In the case where the function $\widetilde{F}_*$ has continuous limiting values on ∂D_z, the conditions (5.4) are equivalent to the boundary conditions (5.2), (5.3) of classic Schwartz problems. □

Theorem 5.2 *The general solution of the homogeneous* (1–4)*-problem for an arbitrary simply connected domain* D_ζ *is expressed by the formula*

$$\Phi(\zeta) = a_1 i e_1 + a_2 e_2, \tag{5.5}$$

where a_1*,* a_2 *are any real constants*

Proof By Theorem 5.1, a solving process of the homogeneous (1–4)-problem consists of consecutive finding of solutions of two homogeneous classic Schwartz problems, viz.:

a) to find an analytic in D_z function F satisfying the boundary condition $(\mathrm{Re}\, F)^+(t) = 0$ for all $t \in \partial D_z$. As a result, we have $F(z) = ai$, where a is an arbitrary real constant;
b) to find similarly an analytic in D_z function F_0 satisfying the boundary condition $(\mathrm{Re}\, F_0)^+(t) = 0$ for all $t \in \partial D_z$.

Consequently, getting a general solution of the homogeneous (1–4)-problem in the form (1.7), we can rewrite it in the form (5.5). □

Remark 5.3 A statement similar to Theorem 5.2 is proved for homogeneous (1–4)-problem in the sense of Kovalev in [11], where the formula of solutions is the same as (5.5).

Remark 5.4 Considering the functions

$$\widetilde{F}(z) := -iyF'(z) \quad \forall z \in D_z,$$

$$\widetilde{\Phi}(\zeta) := -iy\Phi'(\zeta)\,\rho \quad \forall \zeta \in D_\zeta$$

and taking into account the equalities (1.6), (1.2), we obtain the relations

$$\widetilde{\Phi}(\zeta) = -iy\frac{\partial \Phi(\zeta)}{\partial x}\,\rho = -iy\left(F'(z)e_1 - \left(\frac{iy}{2}F''(z) - F_0'(z)\right)\rho\right)\rho = \widetilde{F}(z)\rho =$$

$$= 2\widetilde{F}(z)e_1 + 2i\widetilde{F}(z)e_2 \quad \forall \zeta \in D_\zeta\,.$$

Thus,

$$\widetilde{F}_*(z) = \frac{1}{2}\,\mathrm{U}_1\left[\widetilde{\Phi}(\zeta)\right] \quad \forall z \in D_z,$$

and limiting values $(\widetilde{F}_*)^+$ exist and continuous on ∂D_z if and only if $\mathrm{U}_1\left[\widetilde{\Phi}\right]$ is continuously extended on ∂D.

Remark 5.5 Theorem 5.2 shows an example of the (1–4)-problem (with $u_1 = u_4 \equiv 0$ and with no extra assumptions on a domain D_ζ) when the condition on existence of continuous limiting values $\left(\widetilde{F}_*\right)^+$ is satisfied. Evidently, we have another similar trivial case, where u_1 and u_4 are real constants. In the next section we consider else a nontrivial case of the (1–4)-problem when the continuous limiting values $\left(\widetilde{F}_*\right)^+$ exist.

6 (1–4)-Problem for a Half-Plane

Consider the (1–4)-problem in the case where the domain D_ζ is the half-plane $\Pi^+ := \{\zeta = xe_1 + ye_2 : y > 0\}$.

Consider the *biharmonic Schwartz integral* for the half-plane Π^+:

$$S_{\Pi^+}[u](\zeta) := \frac{1}{\pi i}\int\limits_{-\infty}^{+\infty} \frac{u(t)(1+t\zeta)}{(t^2+1)}(t-\zeta)^{-1}\,dt \quad \forall \zeta \in \Pi^+\,.$$

Here and in what follows, all integrals along the real axis are understood in the sense of their Cauchy principal values, i.e.

$$\int\limits_{-\infty}^{+\infty} g(t,\cdot)\,dt := \lim_{N\to+\infty}\int\limits_{-N}^{N} g(t,\cdot)\,dt\,,$$

The function $S_{\Pi^+}[u](\zeta)$ is the principal extension (see [20, p. 165]) into the half-plane Π^+ of the complex Schwartz integral

$$S[u](z) := \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{u(t)(1+tz)}{(t^2+1)(t-z)}\, dt\,,$$

which determines a holomorphic function in the half-plane $\{z = x + iy : y > 0\}$ of the complex plane $\mathbb{C}$ with the given boundary values $u(t)$ of real part on the real line $\mathbb{R}$. Furthermore, the equality

$$S_{\Pi^+}[u](\zeta) = S[u](z)e_1 - \frac{y}{2\pi}\,\rho \int_{-\infty}^{\infty} \frac{u(t)}{(t-z)^2}dt \qquad \forall\, \zeta \in \Pi^+ \tag{6.1}$$

holds.

The following relations were proved within the proof of Theorem 1 in [8]:

$$y \int_{-\infty}^{\infty} \frac{u(t)}{(t-z)^2}dt \le 4\,\omega_{\mathbb{R}}(u, 2y) + 2\,y \int_{2y}^{\infty} \frac{\omega_{\mathbb{R}}(u,\eta)}{\eta^2}d\eta \to 0, \quad z \to \xi\,, \quad \forall\, \xi \in \mathbb{R}\,, \tag{6.2}$$

where

$$\omega_{\mathbb{R}}(u, \varepsilon) = \sup_{t_1,t_2 \in \mathbb{R}:|t_1-t_2| \le \varepsilon} |u(t_1) - u(t_2)|$$

is the modulus of continuity of the function u .

In addition,

$$y \int_{-\infty}^{\infty} \frac{u(t)}{(t-z)^2}dt \to 0, \quad z \to \infty\,. \tag{6.3}$$

It follows from the equality (6.1) and the relations (6.2), (6.3) that

$$U_1\Big[S_{\Pi^+}[u](\zeta)\Big] \to u(\xi)\,, \quad z \to \xi, \quad \forall\, \xi \in \mathbb{R} \cup \{\infty\} \tag{6.4}$$

and

$$\big(\widetilde{F}_*\big)^+(\xi) = 0 \quad \forall\, \xi \in \mathbb{R} \cup \{\infty\}$$

for the function $\widetilde{F}_*$ defined in Theorem 5.1.

Theorem 6.1 *Let every function $u_l : \mathbb{R} \longrightarrow \mathbb{R}$, $l \in \{1, 4\}$, have a finite limit of the type* (2.1)*. Then the general solution of the* (1–4)*-problem for the half-plane Π^+ is expressed by the formula*

$$\Phi(\zeta) = S_{\Pi^+}[u_1](\zeta)\, e_1 + S_{\Pi^+}[u_4](\zeta)\, ie_2 + a_1 ie_1 + a_2 e_2\,, \tag{6.5}$$

where a_1, a_2 are any real constants.

Proof It follows from the relation (6.4) that the function

$$\Phi_{1,4}(\zeta) = S_{\Pi^+}[u_1](\zeta)\, e_1 + S_{\Pi^+}[u_4](\zeta)\, ie_2 \tag{6.6}$$

is a solution of the (1–4)-problem for the half-plane Π^+.

The general solution of the (1–4)-problem in the form (6.5) is obtained by summarizing the particular solution (6.6) of the inhomogeneous (1–4)-problem and the general solution (5.5) of the homogeneous (1–4)-problem. □

Remark 6.2 In Theorem 3 [11] we obtain the general solution of (1–4)-problem in the sense of Kovalev in the form (6.5) but under complementary assumptions that for every given function $u_l : \mathbb{R} \longrightarrow \mathbb{R}$, $l \in \{1, 4\}$, its modulus of continuity and the local centered (with respect to the infinitely remote point) modulus of continuity satisfy Dini conditions.

7 Solving Process of (1–4)-Problem for Bounded Simply Connected Domain with Use of the Complex Green Function

Now, for solving the (1–4)-problem we shall use solutions of the classic Schwartz problem for analytic functions of a complex variable in the form of an appropriate Schwartz operator involving the complex Green function.

Here we assume that D_z is a bounded simply connected domain with a smooth boundary ∂D_z. Let $g(z, z_0)$ be the Green function of D_z for the Laplace operator (cf, e.g., [21, p. 22]).

It is well-known that the general solution of the Schwartz boundary value problem for analytic functions with boundary datum (5.1) is expressed in the form (cf, e.g., [21, p. 52])

$$F(z) = (Su)(z) + ia_0, \tag{7.1}$$

with an arbitrary real number a_0 and the Schwartz operator $(Su)(z)$ having the form

$$(Su)(z) := -\frac{1}{2\pi} \int\limits_{\partial D_z} u(t)\, \frac{\partial M(t, z)}{\partial \mathbf{n}_t}\, ds_t \quad \forall z \in D_z, \tag{7.2}$$

where the complex Green function M (cf, e.g., [21, p. 32]) is of the form $M(w, z) = g(w, z) + ih(w, z)$, h is a conjugate harmonic function to the Green function g with respect to $w \in D_z$: $w \neq z$, $\mathbf{n}_t$ is the outward normal unit vector at the point $t \in \partial D_z$, s_t is an arc coordinate of the point t .

Now, the function $\widetilde{F}_*$ defined in Theorem 5.1 takes the form

$$\widetilde{F}_*(z) = \mathrm{Re}\,\Big(-iy\big(S(u_1 - u_4)\big)'(z)\Big) \quad \forall\, z \in D_z, \tag{7.3}$$

where u_1 and u_4 are given functions of the (1–4) problem.

Therefore, using expressions of solutions of the classic Schwartz problems with the boundary conditions (5.2), (5.3) in the form (7.1) via appropriate Schwartz operators of the type (7.2), by Theorem 5.1 we obtain the following statement.

Theorem 7.1 *Let the function* (7.3) *have the continuous limiting values* $\big(\widetilde{F}_*\big)^+$ *on* ∂D_z. *Then the general solution of* (1–4)*-problem is expressed in the form*

$$\Phi(\zeta) = \Big(F(z) - iyF'(z) + 2F_0(z)\Big)e_1 + i\Big(2F_0(z) - iyF'(z)\Big)e_2 +$$

$$+a_1 i e_1 + a_2 e_2 \qquad \forall\, z \in D_z\,,$$

where

$$F(z) = \big(S(u_1 - u_4)\big)(z), \quad F_0(z) = \frac{1}{2}\Big(S\big(u_4 - \big(\widetilde{F}_*\big)^+\big)\Big)(z)$$

and a_1, a_2 *are any real constants.*

In the next section we develop a method for solving the inhomogeneous (1–4)-problem without an essential in Theorem 5.1 assumption that the function $\widetilde{F}_*$ have continuous limiting values on the boundary ∂D_z.

8 Solving BVP's by Means Hypercomplex Cauchy-Type Integrals

Let the boundary ∂D_ζ of the bounded domain D_ζ be a closed smooth Jordan curve.

Below, we show a method for reducing (1–3)-problem and (1–4)-problem to systems of the Fredholm integral equations. Such a method was developed in [9, 12]. Obtained results are appreciably similar for the mentioned problems, however, in contrast to (1–3)-problem, which is solvable in a general case if and only if a certain natural condition is satisfied, the (1–4)-problem is solvable unconditionally.

We use the modulus of continuity of a continuous function φ given on ∂D_ζ:

$$\omega(\varphi, \varepsilon) := \sup_{\tau_1, \tau_2 \in \partial D_\zeta :\, \|\tau_1 - \tau_2\| \le \varepsilon} \|\varphi(\tau_1) - \varphi(\tau_2)\|\,.$$

We assume that $\omega(\varphi, \varepsilon)$ satisfies the Dini condition:

$$\int_0^1 \frac{\omega(\varphi, \eta)}{\eta}\, d\eta < \infty. \tag{8.1}$$

Consider the *biharmonic* Cauchy type integral:

$$\mathcal{B}[\varphi](\zeta) := \frac{1}{2\pi i} \int_{\partial D_\zeta} \varphi(\tau)(\tau - \zeta)^{-1} d\tau \quad \forall \zeta \in \mu_{e_1,e_2} \setminus \partial D_\zeta. \tag{8.2}$$

It is proved in Theorem 4.2 [9] that the integral (8.2) has limiting values

$$\mathcal{B}^+[\varphi](\zeta_0) := \lim_{\zeta \to \zeta_0,\, \zeta \in D_\zeta} \Phi(\zeta), \qquad \mathcal{B}^-[\varphi](\zeta_0) := \lim_{\zeta \to \zeta_0,\, \zeta \in \mu_{e_1,e_2} \setminus \overline{D_\zeta}} \Phi(\zeta)$$

in every point $\zeta_0 \in \partial D_\zeta$ that are represented by the Sokhotski–Plemelj formulas:

$$\mathcal{B}^+[\varphi](\zeta_0) = \frac{1}{2}\, \varphi(\zeta_0) + \frac{1}{2\pi i} \int_{\partial D_\zeta} \varphi(\tau)(\tau - \zeta_0)^{-1} d\tau\,, \tag{8.3}$$

$$\mathcal{B}^-[\varphi](\zeta_0) = -\frac{1}{2}\, \varphi(\zeta_0) + \frac{1}{2\pi i} \int_{\partial D_\zeta} \varphi(\tau)(\tau - \zeta_0)^{-1} d\tau\,,$$

where a singular integral is understood in the sense of its Cauchy principal value:

$$\int_{\partial D_\zeta} \varphi(\tau)(\tau - \zeta_0)^{-1} d\tau := \lim_{\varepsilon \to 0} \int_{\{\tau \in \partial D_\zeta : \|\tau - \zeta_0\| > \varepsilon\}} \varphi(\tau)(\tau - \zeta_0)^{-1} d\tau.$$

We assume that boundary functions u_k, $k \in \{1, 3\}$ or $k \in \{1, 4\}$, of the (1–3) problem or the (1–4) problem, respectively, satisfy Dini conditions of the type (8.1).

We seek solutions in a class of functions represented in the form

$$\Phi(\zeta) = \mathcal{B}[\varphi](\zeta) \quad \forall\, \zeta \in D_\zeta\,,$$

where

$$\varphi(\zeta) = \varphi_1(\zeta)\, e_1 + \varphi_3(\zeta)\, e_2 \quad \forall\, \zeta \in \partial D_\zeta \tag{8.4}$$

for the (1–3) problem or

$$\varphi(\zeta) = \varphi_1(\zeta)\, e_1 + \varphi_4(\zeta)\, i e_2 \quad \forall\, \zeta \in \partial D_\zeta \tag{8.5}$$

for the (1–4) problem, and every function $\varphi_k \colon \partial D_\zeta \longrightarrow \mathbb{R}$, $k \in \{1, 3, 4\}$, satisfies a Dini condition of the type (8.1).

We use a conformal mapping $z = \tau(t)$ of the upper half-plane $\{t \in \mathbb{C} : \operatorname{Im} t > 0\}$ onto the domain D_z. Denote $\tau_1(t) := \operatorname{Re} \tau(t)$, $\tau_2(t) := \operatorname{Im} \tau(t)$.

Inasmuch as the mentioned conformal mapping is continued to a homeomorphism between the closures of corresponding domains, the function

$$\widetilde{\tau}(s) := \tau_1(s) e_1 + \tau_2(s) e_2 \qquad \forall\, s \in \overline{\mathbb{R}}$$

generates a homeomorphic mapping of the extended real axis $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ onto the curve ∂D_ζ.

Introducing the function

$$g(s) := \varphi\left(\widetilde{\tau}(s)\right) \qquad \forall\, s \in \overline{\mathbb{R}},$$

we rewrite the equality (8.3) in the form (cf. [9])

$$\mathcal{B}^+[\varphi](\zeta_0) = \frac{1}{2}\, g(t) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} g(s) k(t,s)\, ds + \frac{1}{2\pi i} \int_{-\infty}^{\infty} g(s) \frac{1 + st}{(s-t)(s^2+1)}\, ds\,,$$

where $k(t,s) = k_1(t,s) e_1 + i\rho\, k_2(t,s)\,,$

$$k_1(t,s) := \frac{\tau'(s)}{\tau(s) - \tau(t)} - \frac{1+st}{(s-t)(s^2+1)}\,,$$

$$k_2(t,s) := \frac{\tau'(s)\big(\tau_2(s) - \tau_2(t)\big)}{2\big(\tau(s) - \tau(t)\big)^2} - \frac{\tau_2'(s)}{2\big(\tau(s) - \tau(t)\big)}\,,$$

and a correspondence between the points $\zeta_0 \in \partial D_\zeta \setminus \{\widetilde{\tau}(\infty)\}$ and $t \in \mathbb{R}$ is given by the equality $\zeta_0 = \widetilde{\tau}(t)$.

Evidently, $g(s) = g_1(s) e_1 + g_3(s) e_2$ for the (1–3) problem and $g(s) = g_1(s) e_1 + g_4(s) i e_2$ for the (1–4) problem, where $g_l(s) := \varphi_l\left(\widetilde{\tau}(s)\right)$ for all $s \in \overline{\mathbb{R}}$, $l \in \{1, 3, 4\}$.

Now, in the case of (1–3)-problem, we single out components $U_l\big[\mathcal{B}^+[\varphi](\zeta_0)\big]$, $l \in \{1, 3\}$, and after the substitution them into the boundary conditions of the

(1–3)-problem, we shall obtain the following system of integral equations for finding the functions g_1 and g_3:

$$\begin{aligned} U_1\big[\mathcal{B}^+[\varphi](\zeta_0)\big] \equiv \frac{1}{2}\, g_1(t) + \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} g_1(s)\Big(\mathrm{Im}\, k_1(t,s) + 2\mathrm{Re}\, k_2(t,s)\Big)\, ds - \\ -\frac{1}{\pi}\int\limits_{-\infty}^{\infty} g_3(s)\mathrm{Im}\, k_2(t,s)\, ds = \widetilde{u}_1(t), \\ U_3\big[\mathcal{B}^+[\varphi](\zeta_0)\big] \equiv \frac{1}{2}\, g_3(t) + \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} g_3(s)\Big(\mathrm{Im}\, k_1(t,s) - 2\mathrm{Re}\, k_2(t,s)\Big)\, ds - \\ -\frac{1}{\pi}\int\limits_{-\infty}^{\infty} g_1(s)\mathrm{Im}\, k_2(t,s)\, ds = \widetilde{u}_3(t) \qquad \forall\, t \in \mathrm{R}, \end{aligned} \tag{8.6}$$

where $\widetilde{u}_l(t) := u_l\big(\widetilde{\tau}(t)\big)$, $l \in \{1, 3\}$.

Similarly, in the case of (1–4)-problem, we single out components $U_l\big[\mathcal{B}^+[\varphi](\zeta_0)\big]$, $l \in \{1, 4\}$, and after the substitution them into the boundary conditions of the (1–4)-problem, we shall obtain the following system of integral equations for finding the functions g_1 and g_4:

$$\begin{aligned} U_1\big[\mathcal{B}^+[\varphi](\zeta_0)\big] \equiv \frac{1}{2}\, g_1(t) + \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} g_1(s)\Big(\mathrm{Im}\, k_1(t,s) + 2\mathrm{Re}\, k_2(t,s)\Big)\, ds - \\ -\frac{1}{\pi}\int\limits_{-\infty}^{\infty} g_4(s)\mathrm{Re}\, k_2(t,s)\, ds = \widetilde{u}_1(t), \\ U_4\big[\mathcal{B}^+[\varphi](\zeta_0)\big] \equiv \frac{1}{2}\, g_4(t) + \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} g_4(s)\Big(\mathrm{Im}\, k_1(t,s) - 2\mathrm{Re}\, k_2(t,s)\Big)\, ds + \\ +\frac{1}{\pi}\int\limits_{-\infty}^{\infty} g_1(s)\mathrm{Re}\, k_2(t,s)\, ds = \widetilde{u}_4(t) \qquad \forall\, t \in \mathbb{R}, \end{aligned} \tag{8.7}$$

where $\widetilde{u}_l(t) := u_l\big(\widetilde{\tau}(t)\big)$, $l \in \{1, 4\}$.

Let $C(\overline{\mathbb{R}})$ denote the Banach space of functions $g_* : \overline{\mathbb{R}} \longrightarrow \mathbb{C}$ that are continuous on the extended real axis $\overline{\mathbb{R}}$ with the norm $\|g_*\|_{C(\overline{\mathbb{R}})} := \sup\limits_{t\in\mathbb{R}} |g_*(t)|$.

In Theorem 6.13 [9] there are conditions which are sufficient for compactness of integral operators on the left-hand sides of equations of the systems (8.6), (8.7) in the space $C(\overline{\mathbb{R}})$.

To formulate such conditions, consider the conformal mapping $\sigma(T)$ of the unit disk $\{T \in \mathbb{C} : |T| < 1\}$ onto the domain D_z such that $\tau(t) = \sigma\left(\frac{t-i}{t+i}\right)$ for all $t \in \{t \in \mathbb{C} : \operatorname{Im} t > 0\}$.

Thus, it follows from Theorem 6.13 [9] that if the conformal mapping $\sigma(T)$ have the nonvanishing continuous contour derivative $\sigma'(T)$ on the unit circle $\Gamma := \{T \in \mathbb{C} : |T| = 1\}$, and its modulus of continuity

$$\omega_\Gamma(\sigma', \varepsilon) := \sup_{T_1, T_2 \in \Gamma,\, |T_1 - T_2| \le \varepsilon} |\sigma'(T_1) - \sigma'(T_2)|$$

satisfies a condition of the type (8.1), then the integral operators in the systems (8.6), (8.7) are compact in the space $C(\overline{\mathbb{R}})$.

Let $\mathcal{D}(\overline{\mathbb{R}})$ denote the class of functions $g_* \in C(\overline{\mathbb{R}})$ whose the modulus of continuity $\omega_{\mathbb{R}}(g_*, \varepsilon)$ and the local centered (with respect to the infinitely remote point) modulus of continuity

$$\omega_{\mathbb{R},\infty}(g_*, \varepsilon) = \sup_{\tau \in \mathbb{R} : |\tau| \ge 1/\varepsilon} |g_*(\tau) - g_*(\infty)|$$

satisfy the Dini conditions

$$\int_0^1 \frac{\omega_{\mathbb{R}}(g_*, \eta)}{\eta}\, d\eta < \infty, \quad \int_0^1 \frac{\omega_{\mathbb{R},\infty}(g_*, \eta)}{\eta}\, d\eta < \infty.$$

Since the sought-for function φ in (8.2) has to satisfy the condition (8.1), it is necessary to require that the corresponding functions g_1, g_3 in (8.4) or g_1, g_4 in (8.5) should belong to the class $\mathcal{D}(\overline{\mathbb{R}})$. In the next theorems we state a condition on the conformal mapping $\sigma(T)$, under which all solutions of the system (8.6), (8.7) satisfy the mentioned requirement.

Theorem 8.1 *Assume that the functions $u_l : \partial D_\zeta \longrightarrow \mathbb{R}$, $l \in \{1, 3\}$, satisfy conditions of the type* (8.1)*. Also, assume that the conformal mapping $\sigma(T)$ has the nonvanishing continuous contour derivative $\sigma'(T)$ on the circle Γ, and its modulus of continuity $\omega_\Gamma(\sigma', \varepsilon)$ satisfies the condition*

$$\int_0^2 \frac{\omega_\Gamma(\sigma', \eta)}{\eta} \ln \frac{3}{\eta}\, d\eta < \infty. \tag{8.8}$$

Then all functions $g_1, g_3 \in C(\overline{\mathrm{R}})$ satisfying the system of Fredholm integral equations (8.6) *belong to the class $\mathcal{D}(\overline{\mathrm{R}})$, and the corresponding function φ in* (8.4) *satisfies the Dini condition* (8.1).

Assume additionally:

1) *all solutions* $(g_1, g_3) \in C(\overline{\mathbb{R}}) \times C(\overline{\mathbb{R}})$ *of the homogeneous system of equations* (8.6) (*with* $\widetilde{u}_k \equiv 0$ *for* $k \in \{1, 3\}$) *are differentiable on* $\mathbb{R}$;
2) *for every mentioned solution* (g_1, g_3) *of the homogeneous system of equations* (8.6), *the integral* $\mathcal{B}[\varphi']$ *is finite in* D_ζ *and* $\mu_{e_1,e_2} \setminus \overline{D_\zeta}$, *and the functions*

$$\mathrm{U}_1\left[\mathcal{B}\left[\varphi'\right](\zeta)\right] - \mathrm{U}_4\left[\mathcal{B}\left[\varphi'\right](\zeta)\right] \quad \forall\, \zeta \in D_\zeta,$$

$$\mathrm{U}_2\left[\mathcal{B}\left[\varphi'\right](\zeta)\right] + \mathrm{U}_3\left[\mathcal{B}\left[\varphi'\right](\zeta)\right] \quad \forall\, \zeta \in \mu_{e_1,e_2} \setminus \overline{D_\zeta}$$

are bounded, where φ' *is the contour derivative of the corresponding function* φ *in* (8.4), *i.e.,* $\varphi(\zeta) \equiv \varphi(\widetilde{\tau}(s)) := g_1(s)e_1 + g_3(s)e_2$ *for all* $s \in \mathbb{R}$.

Then the following assertions are true:

(i) *the number of linearly independent solutions of the homogeneous system of equations* (8.6) *is equal to* 1;
(ii) *the non-homogeneous system of equations* (8.6) *is solvable if and only if the condition* (3.2) *is satisfied.*

Theorem 8.2 *Assume that the functions* $u_l : \partial D_\zeta \longrightarrow \mathbb{R}$, $l \in \{1, 4\}$, *satisfy conditions of the type* (8.1). *Also, assume that the conformal mapping* $\sigma(T)$ *has the nonvanishing continuous contour derivative* $\sigma'(T)$ *on the circle* Γ, *and its modulus of continuity satisfy the condition* (8.8). *Then the following assertions are true*:

(i) *the system of Fredholm integral equations* (8.7) *has the unique solution in* $C(\overline{\mathbb{R}})$;
(ii) *all functions* $g_1, g_4 \in C(\overline{\mathbb{R}})$ *satisfying the system* (8.7) *belong to the class* $\mathcal{D}(\overline{\mathbb{R}})$, *and the corresponding function* φ *in* (8.5) *satisfies the Dini condition* (8.1).

Remark 8.3 Generalizing Theorem 6.13 [9], Theorem 8.1 is proved similarly. Theorem 8.2 is proved in [12] if a (1–4)-problem is understood in the sense of Kovalev but it is still valid for a (1–4)-problem considered in this paper.

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