ON EQUICONTINUITY OF FAMILIES OF MAPPINGS IN A CASE OF VARIABLE DOMAINS

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Throughout, D and D' are domains in $\mathbb{R}^n$, $n \geqslant 2$. In what follows, by $\Gamma(E, F, D)$ we define a family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}^n} : \gamma(a) \in E$, $\gamma(b) \in F$, $\gamma(t) \in D$, when $t \in [a, b]$. Given $0 < r_1 < r_2 < \infty$, denote $A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$. Given a family of paths Γ in $\mathbb{R}^n$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm}\Gamma$, if $\int\limits_{\gamma} \rho\, ds \geqslant 1$ for all (locally rectifiable) $\gamma \in \Gamma$. The *modulus* of Γ is defined as follows: $M(\Gamma) = \inf\limits_{\rho \in \operatorname{adm}\Gamma} \int\limits_{D} \rho^n(x)\, dm(x)$. Let $Q : D \to [0, \infty]$ be a Lebesgue measurable function. A mapping $f : D \to D'$ is called a *ring Q-mapping at a point* $x_0 \in \overline{D}$, if the inequality $M(f(\Gamma(S(x_0, r_1), S(x_0, r_2), A(x_0, r_1, r_2)))) \leqslant \int\limits_{A(x_0,r_1,r_2)\cap D} Q(x)\cdot\eta^n(|x-x_0|))\, dm(x)$ holds for some $r_0 = r_0(x_0) > 0$, for all $0 < r_1 < r_2 < r_0$ and for any measurable function $\eta : (r_1, r_2) \to [0, \infty]$ with $\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1$. A mapping f of D onto D' is called *closed* if $C(f, \partial D) \subset \partial D'$, where, as usually, $C(f, \partial D)$ is a limit set of f on ∂D. Let $h(x, y)$ denotes the chordal distance between points $x, y \in \overline{\mathbb{R}^n}$, let $h(A, B)$ denotes the chordal distance between the sets $A, B \subset \overline{\mathbb{R}^n}$, and let $h(E)$ be the chordal diameter of the set E in $\overline{\mathbb{R}^n}$. For $E \subset \overline{\mathbb{R}^n}$ and $\delta > 0$, denote by $\mathfrak{R}_{Q,\,\delta,\,E}(D)$ the family of all open discrete closed ring Q-mappings $f : D \to \overline{\mathbb{R}^n}\backslash E$ in $\overline{D}$ with following condition: for every domain $D_f^{'} = f(D)$ there is a continuum $K_f \subset D_f'$ such that $h(K_f) \geqslant \delta$ and $h(f^{-1}(K_f), \partial D) \geqslant \delta > 0$. Let $q_{x_0}(r) := \frac{1}{\omega_{n-1}r^{n-1}} \int\limits_{|x-x_0|=r} Q(x)\, dS$, where dS is an area element, $q_b'(r) := \frac{1}{\omega_{n-1}r^{n-1}} \int\limits_{|x-b|=r} Q'(x)\, dS$ and $Q'(x) = \max\{Q(x), 1\}$.

Theorem. *Suppose D is locally connected on ∂D, and $D_f' = f(D)$ are uniformly equicontinuous for all $f \in \mathfrak{R}_{Q,\,\delta,\,E}(D)$. Let E be a set of positive capacity. Suppose that one of the following conditions holds: 1) either $Q \in FMO$ in $\overline{D}$ or 2) $\int\limits_0^{\beta(x_0)} \frac{dt}{t q_{x_0}'^{\frac{1}{n-1}}(t)} = \infty$ for some $\beta(x_0) > 0$ at every point $x_0 \in \overline{D}$. Then every $f \in \mathfrak{R}_{Q,\,\delta,\,E}(D)$ has a continuous extension to $\overline{D}$ and the family $\mathfrak{R}_{Q,\,\delta,\,E}(\overline{D})$ consisting of all extended mappings $\overline{f} : \overline{D} \to \overline{\mathbb{R}^n}$ is equicontinuous in $\overline{D}$.*

Example 1. The family $f_n(z) = z^n$ of the unit disc onto itself, $n = 1, 2, \ldots$, is an example of equicontinuous family of mappings in D, what is not so on ∂D. The reason is violation of conditions $h(K_f) \geqslant \delta$ and $h(f^{-1}(K_f), \partial D) \geqslant \delta > 0$ in the definition of the class $\mathfrak{R}_{Q,\,\delta,\,E}(\mathbb{D})$ and in conditions of the theorem. Worth noting that this family consist of ring 1-mappings.

Example 2. To obtain a similar "good" family of mappings, we put $f_n(z) = \left(\frac{z+\frac{1}{n}}{1+\frac{z}{n}}\right)^2, n \in \mathbb{N} \setminus \{1\}$. The mappings f_n are open, discrete, closed and at the same time they are 1-mappings. If we put $A = [0, \frac{1}{2}]$, then $f_n(A) = \left[\frac{1}{n^2}, \left(\frac{n+2}{2n+1}\right)^2\right]$. Then in definition of class $\mathfrak{R}_{Q,\,\delta,\,E}(\mathbb{D})$ we put $Q \equiv 1, E = \mathbb{C}\backslash\mathbb{D}, K_{f_n} = f_n(A)$ and $\delta = \frac{1}{10}$. Now, $f_n \in \mathfrak{R}_{1,\,1/10,\,\mathbb{C}\backslash\mathbb{D}}(\mathbb{D})$ for large enough $n \in \mathbb{N}$.