An algebraic approach for solving fourth-order partial differential equations

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This report is devoted to the study of solutions of the following equation

$$\left(\frac{\partial^4}{\partial x^4} - 2c\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right)u(x,y) = 0, \qquad c > 0. \tag{1}$$

We consider two of three cases depending on the value of c: 1) the case, where $0 < c < 1$, which is said to be the c-biwave equation of the elliptic type; 2) the case, where $c > 1$, which is called the c-biwave equation of the hyperbolic type. If $c = 1$ Eq. (1) is the well-known biwave equation. In order to obtain all solutions of Eq. (1) for $1 \neq c > 0$ we used the method developed in [1].

Hyperbolic case. Consider an associative commutative algebra as follows

$$\mathbb{A}_c = \{x\mathbf{u} + y\mathbf{f} + z\mathbf{e} + v\mathbf{fe} | x, y, z, v \in \mathbb{R}\},$$

where $\{\mathbf{u}, \mathbf{f}, \mathbf{e}, \mathbf{fe}\}$ is a basis of $\mathbb{A}_c$ with the identity element $\mathbf{u}$ and with the following Cayley table: $\mathbf{fe} = \mathbf{ef}$, $\mathbf{f}^2 = \mathbf{u}$, $\mathbf{e}^2 = \mathbf{u} - m\mathbf{fe}$, $m = \sqrt{2(c-1)}$.

Let $\mathbb{B}_c$ be a subspace of $\mathbb{A}_c$ of the following form

$$\mathbb{B}_c = \{x\mathbf{u} + y\mathbf{e} | x, y \in \mathbb{R}\}.$$

It is easily verified that for $c > 1$ algebra $\mathbb{A}_c$ has the following idempotents

$$i_1 = \frac{k_1}{k_1+k_2}\mathbf{u} - \frac{\mathbf{f}\sqrt{2}}{k_1+k_2}\mathbf{e},\ i_2 = \frac{k_2}{k_1+k_2}\mathbf{u} + \frac{\mathbf{f}\sqrt{2}}{k_1+k_2}\mathbf{e},$$

where $k_1 = \sqrt{c+1} - \sqrt{c-1}, k_2 = \sqrt{c+1} + \sqrt{c-1}$.

Passing in $\mathbb{B}_c$ from the basis $\{\mathbf{u}, \mathbf{e}\}$ to the basis $\{i_1, i_2\}$, we have

$$\omega = x\mathbf{u} + y\mathbf{e} = \left(x - \mathbf{f}\frac{k_2}{\sqrt{2}}y\right)i_1 + \left(x + \mathbf{f}\frac{k_1}{\sqrt{2}}y\right)i_2, \forall\omega \in \mathbb{B}_c.$$

Definition. *A function $g: \mathbb{B}_c \to \mathbb{A}_c$ is called differentiable (or monogenic) on $\mathbb{B}_c$ if for any $\mathbb{B}_c \ni \omega = x\boldsymbol{u} + y\boldsymbol{e}$ there exists a unique element $g'(\omega) \in \mathbb{A}_c$ such that for any $h \in \mathbb{B}_c$*

$$\lim_{\mathbb{R}\ni\varepsilon\to 0}\frac{g(\omega + \varepsilon h) - g(\omega)}{\varepsilon} = hg'(\omega),$$

where $hg'(\omega)$ is the product of h and $g'(\omega)$ as elements of $\mathbb{A}_c$.

Lemma. *A function $g: \mathbb{B}_c \to \mathbb{A}_c$, where $c > 1$, is monogenic if and only if it can be represented in the following form*

$$g(\omega) = \alpha(\omega_1)i_1 + \beta(\omega_2)i_2, \tag{2}$$

where $\omega_1 = x - \mathbf{f}\frac{k_2}{\sqrt{2}}y$, $\omega_2 = x + \mathbf{f}\frac{k_1}{\sqrt{2}}y$ and $\alpha(\omega_1), \beta(\omega_2)$ have continuous partial derivatives $\frac{\partial}{\partial x}\alpha(\omega_1), \frac{\partial}{\partial y}\alpha(\omega_1), \frac{\partial}{\partial x}\beta(\omega_2), \frac{\partial}{\partial y}\beta(\omega_2)$ and

$$\frac{\partial}{\partial y}\alpha(\omega_1) = -\mathbf{f}\frac{k_2}{\sqrt{2}}\frac{\partial}{\partial x}\alpha(\omega_1), \frac{\partial}{\partial y}\beta(\omega_2) = \mathbf{f}\frac{k_1}{\sqrt{2}}\frac{\partial}{\partial x}\beta(\omega_2).$$

Hence, considering variables $x, y_1 = -\frac{k_2}{\sqrt{2}}y$ and $x, y_2 = \frac{k_1}{\sqrt{2}}y$, we have

$$\frac{\partial}{\partial y_1}\alpha(\omega_1) = \mathbf{f}\frac{\partial}{\partial x}\alpha(\omega_1), \frac{\partial}{\partial y_2}\beta(\omega_2) = \mathbf{f}\frac{\partial}{\partial x}\beta(\omega_2). \tag{3}$$

Theorem. *A function $u(x,y)$ is a solution of Eq. (1) for $c > 1$ if and only if for some $i, j \in \{1,2\}$ it can be represented in the following form*

$$u(x,y) = \alpha_i(\omega_1) + \beta_j(\omega_2),$$

where $\alpha_i(\omega_1), \beta_j(\omega_2)$ are four times continuous differentiable components of $\alpha(\omega_1)$ and $\beta(\omega_2)$ of monogenic function $g(\omega)$ in the decomposition (2), and $\alpha(\omega_1) = \alpha_1(\omega_1) + \mathbf{f}\alpha_2(\omega_1), \beta(\omega_2) = \beta_1(\omega_2) + \mathbf{f}\beta_2(\omega_2)$ satisfy Eq. (3).

Much in the same way all solutions of Eq. (1) can be obtained for $0 < c < 1$ [2].

References:

[1] Pogorui A. A. Solutions for PDEs with constant coefficients and derivability of functions ranged in commutative algebras / A. A. Pogorui, R. M. Rodríguez-Dagnino, M. Shapiro // Math. Meth. Appl. Sci. – 2014. – Vol. 37, No. 17. – P. 2799–2810.

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