Series expansions for monogenic functions in Clifford algebras and their application

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Abstract. This paper deals with studying some properties of a monogenic function defined on a vector space with values in the Clifford algebra generated by the space. We provide some expansions of a monogenic function and consider its application to study solutions of second order partial differential equations.

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1. Introduction

A Clifford algebra is a unital associative algebra, which is the minimal extension of a finite-dimensional vector space V with a quadratic form. In this case the vector space V is called embedded in the Clifford algebra and its orthogonal basis determines the basis of the algebra. We are interested in properties of a monogenic function defined on the embedded vector space with values in Clifford algebra, which is a null solution of the Dirac operator. A monogenic function is one of the basic concept of Clifford analysis [1]. We obtained an expansion of a left monogenic function into series of Fueter type polynomials and since any right monogenic function can be expanded in a similar manner only left monogenic functions are considered. We also study application of the properties of monogenic functions to solving some second-order partial differential equations.

2. Monogenic functions in Clifford algebras

Suppose E^{d+1} is an $(d+1)$-dimensional linear space over $\mathbb{R}$ with orthogonal basis $\boldsymbol{e}_i$, $i = 0, 1, \ldots, d$. Suppose E^{d+1} is embedded in the

Clifford algebra $Cl_{p,q}$, $p + q = d + 1$ with an identity $I = e_0$ and $e_k e_l + e_l e_k = \mathbf{0}$, $k \neq l$, $1 \leq k, l \leq d$. In addition, $e_k^2 = I$ for $1 \leq k \leq p$ and $e_k^2 = -I$ for $p < k \leq d$.

Denote by $D = \sum_{i=0}^{d} e_i \frac{\partial}{\partial x_i}$ the Dirac operator. It is easily verified that $D^2 = \sum_{i=0}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{i=p+1}^{d} \frac{\partial^2}{\partial x_i^2}$.

We are interested in studing of the following equation

$$D^2 f = 0. \tag{2.1}$$

Definition 2.1. *A function $f : E^{d+1} \to Cl_{p,q}$ is called left monogenic if $Df = 0$, and f is called right monogenic if $fD = 0$.*

In the sequel we will consider only left monogenic functions. Any left monogenic function is a solution of the equation $D^2 f = 0$.

For $E^{d+1} \ni \boldsymbol{x} = \sum_{i=0}^{d} e_i x_i$ let us introduce the following polynomials:

$$p_k(\boldsymbol{x}) = x_k - e_k x_0, \quad 1 \leq k \leq d.$$

Polynomials p_k might be called Fueter type polynomials since in the case of the algebra of quaternions $\mathbb{H} = x_0 + \boldsymbol{i} x_1 + \boldsymbol{j} x_2 + \boldsymbol{k} x_3$ polynomials $x_1 - \boldsymbol{i} x_0$, $x_2 - \boldsymbol{j} x_0$, $x_3 - \boldsymbol{k} x_0$ were introduced by Fueter in [2] and are known today as the Fueter polynomials.

An expansion of hyperholomorphic quaternion functions in Fueter polynomials has been reported in [3] and the similar expansion of hyperholomorphic co-quaternion functions was considered in [4]. The relations between monogenic and hyperholomorphic functions was considered in [5].

Theorem 2.1. *Let $f \in C^{\infty}(E^{d+1})$, $n \in \mathbb{N}$, and the function f with its derivatives*

$$\frac{\partial^n}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_n}} f(\boldsymbol{x})$$

be a left monogenic, then there exists real θ such that

$$f(\boldsymbol{x}) = f(\mathbf{0}) + \sum_{i=1}^{d} p_k(\boldsymbol{x}) \frac{\partial}{\partial x_i} f(\mathbf{0})$$

$$+ \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} p_i(\boldsymbol{x}) p_j(\boldsymbol{x}) \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{0}) + \cdots + \frac{1}{(n-1)!}$$

$$\times \sum_{i_1=1}^{d} \sum_{i_2=1}^{d} \cdots \sum_{i_{n-1}=1}^{d} p_{i_1}(\boldsymbol{x}) p_{i_2}(\boldsymbol{x}) \ldots p_{i_{n-1}}(\boldsymbol{x}) \frac{\partial^{n-1}}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_{n-1}}} f(\mathbf{0})$$

$$+\frac{1}{n!}\sum_{i_1=1}^{d}\sum_{i_2=1}^{d}\cdots\sum_{i_n=1}^{d}p_{i_1}(\boldsymbol{x})\,p_{i_2}(\boldsymbol{x})\ldots p_{i_n}(\boldsymbol{x})\,\frac{\partial^n}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_n}}f(\theta_n\boldsymbol{x}),$$

where $0 \leq \theta_n < 1$.

Proof. Consider function $f(t\boldsymbol{x})$ as a function of $t \in \mathbb{R}$. Taking into account that $f(\boldsymbol{x})$ is left monogenic that is $\frac{\partial}{\partial x_0}f(\boldsymbol{x}) = -\sum_{i=1}^{d}\boldsymbol{e}_i\frac{\partial}{\partial x_i}f(\boldsymbol{x})$, we obtain

$$\frac{d}{dt}f(t\boldsymbol{x}) = \sum_{i=0}^{n}x_i\frac{\partial}{\partial x_i}f(t\boldsymbol{x}) = \sum_{i=1}^{n}(x_i - \boldsymbol{e}_i x_0)\frac{\partial}{\partial x_i}f(t\boldsymbol{x}). \qquad (2.2)$$

Putting $t = 0$, we have

$$\frac{d}{dt}f(\boldsymbol{0}) = \sum_{i=1}^{n}p_i(\boldsymbol{x})\frac{\partial}{\partial x_i}f(\boldsymbol{0}). \qquad (2.3)$$

Now considering that functions

$$\frac{\partial}{\partial x_i}f(\boldsymbol{x}), \quad i = 1,\ldots,n$$

are left monogenic and Eq. (2.2), we get

$$\frac{d^2}{dt^2}f(t\boldsymbol{x}) = \sum_{i=1}^{n}(x_i - \boldsymbol{e}_i x_0)\frac{d}{dt}\left(\frac{\partial}{\partial x_i}f(t\boldsymbol{x})\right)$$

$$= \sum_{i=1}^{n}(x_i - \boldsymbol{e}_i x_0)\sum_{j=1}^{n}(x_j - \boldsymbol{e}_j x_0)\frac{\partial^2}{\partial x_j \partial x_i}f(t\boldsymbol{x}).$$

Hence, by putting $t = 0$

$$\frac{d^2}{dt^2}f(\boldsymbol{0}) = \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - \boldsymbol{e}_i x_0)(x_j - \boldsymbol{e}_j x_0)\frac{\partial^2}{\partial x_i \partial x_j}f(\boldsymbol{0}). \qquad (2.4)$$

Much in the same way one can show that

$$\frac{d^n}{dt^n}f(t\boldsymbol{x})$$

$$= \sum_{i_1=1}^{d}\sum_{i_2=1}^{d}\cdots\sum_{i_n=1}^{d}p_{i_1}(\boldsymbol{x})\,p_{i_2}(\boldsymbol{x})\ldots p_{i_n}(\boldsymbol{x})\,\frac{\partial^n}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_n}}f(t\boldsymbol{x}). \quad (2.5)$$

Now, consider the Taylor series for $f(t\boldsymbol{x})$ w.r.t. t

$$f(t\boldsymbol{x}) = f(\boldsymbol{0}) + \frac{d}{dt}f(\boldsymbol{0})\,t + \ldots \tag{2.6}$$

$$+ \frac{d^{n-1}}{dt^{n-1}}f(\boldsymbol{0})\frac{t^{n-1}}{(n-1)!} + \frac{d^n}{dt^n}f(\theta_n\boldsymbol{x})\frac{t^n}{n!}, \quad 0 \le \theta_n < 1.$$

Substituting Eqs. (2.3)–(2.5) into Eq. (2.6), we obtain Eq. (2.1). If

$$\frac{1}{n!}\sum_{i_1=1}^{d}\sum_{i_2=1}^{d}\cdots\sum_{i_n=1}^{d}p_{i_1}(\boldsymbol{x})p_{i_2}(\boldsymbol{x})\ldots p_{i_n}(\boldsymbol{x})\frac{\partial^n}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_n}}f(\theta_n\boldsymbol{x}) \to \boldsymbol{0},$$

$$n \to \infty,$$

then it follows from Eq. (2.1) that

$$f(\boldsymbol{x}) = f(\boldsymbol{0})$$

$$+ \sum_{n=1}^{\infty}\frac{1}{n!}\left(\sum_{i_1=1}^{d}\sum_{i_2=1}^{d}\cdots\sum_{i_n=1}^{d}p_{i_1}(\boldsymbol{x})p_{i_2}(\boldsymbol{x})\ldots p_{i_n}(\boldsymbol{x})\right. \tag{2.7}$$

$$\left.\times\frac{\partial^n}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_n}}f(\boldsymbol{0})\right).$$

In particular case when

$$\frac{\partial^n}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_n}}f(\boldsymbol{0}) = a_n$$

for all $i_1,\,i_2,\ldots,\,i_n \in \{1,2,\ldots,n\}$, $n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty}\frac{1}{n!}\left(\sum_{i_1=1}^{d}\sum_{i_2=1}^{d}\cdots\sum_{i_n=1}^{d}p_{i_1}(\boldsymbol{x})\,p_{i_2}(\boldsymbol{x})\ldots p_{i_n}(\boldsymbol{x})\,a_n\right)$$

$$= \sum_{n=1}^{\infty}\frac{1}{n!}(p_{i_1}(\boldsymbol{x}) + p_{i_2}(\boldsymbol{x}) + \cdots + p_{i_n}(\boldsymbol{x}))^n\,a_n$$

$$= \sum_{n=1}^{\infty}\frac{1}{n!}\left(\sum_{k=1}^{d}x_k - x_0\sum_{k=1}^{d}\boldsymbol{e}_k\right)^n a_n.$$

Thus, in this case

$$f(\boldsymbol{x}) = a_0 + \sum_{n=1}^{\infty}\frac{1}{n!}\left(\sum_{k=1}^{d}x_k - x_0\sum_{k=1}^{d}\boldsymbol{e}_k\right)^n a_n, \tag{2.8}$$

where $a_0 = f(\boldsymbol{0})$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

3. Studying of hyperbolic and Laplace equations by using monogenic functions

By using Eqs. (2.7) and (2.8) we can construct solutions of Eq. (2.1). For example, putting in Eq. (2.8) $a_n = n!$ for $1 \leq n \leq m$, $m \in \mathbb{N}$ and $a_n = 0$ for all $n > m$ we get the following solution of Eq. (2.1)

$$f(\boldsymbol{x}) = 1 + \sum_{n=1}^{m} \left(\sum_{k=1}^{d} x_k - x_0 \sum_{k=1}^{d} \boldsymbol{e}_k \right)^n = \sum_{n=0}^{m} \left(\sum_{k=1}^{d} x_k - x_0 \sum_{k=1}^{d} \boldsymbol{e}_k \right)^n.$$

Taking into account that $\left(\sum_{k=1}^{d} \boldsymbol{e}_k \right)^{2n} = (p-q)^n$ in $Cl_{p,q}$, we obtain a real solution of Eq. (2.1) as follows

$$u(x_1, x_2, \ldots, x_d)$$

$$= \sum_{n=0}^{m} \left(\left(\sum_{k=1}^{d} x_k \right)^n + C_n^2 \left(\sum_{k=1}^{d} x_k \right)^{n-2} (p-q)^2 x_0^2 \right.$$

$$\left. + C_n^4 \left(\sum_{k=1}^{d} x_k \right)^{n-4} (p-q)^4 x_0^4 + \ldots \right),$$

where we assume that $C_n^k = 0$ for $n < k$.

We can also construct solutions of Eq. (2.1) in the cases of infinite non-zero terms of series (2.7). Indeed, by putting in Eq. (2.7) $a_n = 1$ for all $n \in \mathbb{N}$, we get the following solution of Eq. (2.8)

$$f(\boldsymbol{x}) = \exp \left(\sum_{k=1}^{d} x_k - x_0 \sum_{k=1}^{d} \boldsymbol{e}_k \right) = \exp \left(\sum_{k=1}^{d} x_k \right) \exp \left(-x_0 \sum_{k=1}^{d} \boldsymbol{e}_k \right).$$

Considering that $\left(\sum_{k=1}^{d} \boldsymbol{e}_k \right)^{2n} = (p-q)^n$ and the fact that expressions $\left(\sum_{k=1}^{d} \boldsymbol{e}_k \right)^{2n+1}$ are not real for all $n \in \mathbb{N}$, we obtain the following real solution of Eq. (2.1)

$$u(x_1, x_2, \ldots, x_d) = \exp \left(\sum_{k=1}^{d} x_k \right) \cos \left(x_0 \sqrt{p-q} \right).$$

In the case where $p = 0$ (or $q = 0$) that is, the Clifford algebra is $Cl_{0,d}$ (or $Cl_{d,0}$) from Eq. (2.1) follows that a monogenic function f satisfies the d-dimensional Laplace equation

$$\Delta_d f = \sum_{i=0}^{d} \frac{\partial^2}{\partial x_i^2} f = 0. \tag{3.1}$$

Consider the case $Cl_{0,3}$ with the embedded vector space E^3 and its basis I, e_1, e_2, where I is the identity and $e_k e_l + e_l e_k = -2I\delta_{kl}$, $1 \leq k, l \leq 2$. For this case Eq. (3.1) is the Laplace equation

$$\Delta_3 f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f = 0.$$

Taking into account Eq. (2.8), it is easily seen that

$$f(x, y, z) = \frac{(e_1 + e_2)\, x - y - z}{1 - (e_1 + e_2)\, x + y + z}$$

is monogenic in $E^3 \setminus \{(e_1 + e_2)\, x + y + z = 1\}$.

Let us find the real and imaginary parts of f

$$f(x, y, z) = \frac{(e_1 + e_2)\, x - y - z}{1 - (e_1 + e_2)\, x + y + z}\, \frac{1 + (e_1 + e_2)\, x + y + z}{1 + (e_1 + e_2)\, x + y + z}$$

$$= \frac{y + z - 2x^2 - (y + z)^2}{2x^2 + (1 - y - z)^2} + \frac{x}{2x^2 + (1 - y - z)^2}\, (e_1 + e_2).$$

Thus, we have the following solutions of Eq. (3.1)

$$u_1(x, y, z) = \frac{y + z - 2x^2 - (y + z)^2}{2x^2 + (1 - y - z)^2}$$

and

$$u_2(x, y, z) = \frac{x}{2x^2 + (1 - y - z)^2}.$$

Now consider the case $Cl_{1,3}$ where the embedded vector space E^4 has the basis I, e_1, e_2, e_3. I is the identity and $e_k e_l + e_l e_k = -2I\delta_{kl}$, $1 \leq k, l \leq 3$.

For this case Eq. (2.1) is the following form

$$\frac{\partial^2}{\partial t^2} f - \frac{\partial^2}{\partial x^2} f - \frac{\partial^2}{\partial y^2} f - \frac{\partial^2}{\partial z^2} f = 0. \tag{3.2}$$

Consider the function

$$f(w) = \frac{w}{1 - w},$$

where $w = (e_1 + e_2 + e_3)\, t - x - y - z$.

It follows from Eq. (2.7) that for $w \neq 1$ function $f(w)$ is monogenic. The real and imaginary parts of f can be calculated as follows

$$
\begin{aligned}
f(w) &= \frac{(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)\, t - x - y - z}{1 - (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)\, t + x + y + z} \; \frac{1 + (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)\, t + x + y + z}{1 + (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)\, t + x + y + z} \\
&= \frac{3t^2 - (x + y + z) + x + y + z}{(1 - x - y - z)^2 - 3t^2} + \frac{t\,(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)}{(1 - x - y - z)^2 - 3t^2}.
\end{aligned}
$$

Hence, we obtain two solutions of Eq. (3.2).

$$
u_1\,(t, x, y, z) = \frac{3t^2 - (x + y + z) + x + y + z}{(1 - x - y - z)^2 - 3t^2}
$$

and

$$
u_2\,(t, x, y, z) = \frac{t}{(1 - x - y - z)^2 - 3t^2}.
$$

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