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SOME ALGEBRAIC PROPERTIES OF COMPLEX SEGRE QUATERNOINS

This paper deals with the basic properties the algebra of Segre quaternions over the field of complex numbers. We study idempotents, ideals, matrix representation and the Peirce decomposition of this algebra. We also investigate the structure of zeros of a polynomial in Segre complex quaternions by reducing it to the system of four polynomial equations in the complex field. In addition, Cauchy–Riemann type conditions are obtained for the differentiability of a function on the complex Segre quaternionic algebra.

MSC: 16H05.

Keywords: *quaternions; Segre algebra; Pierce decomposition; polynomials.*

Introduction

The algebra of Segre quaternions or bicomplex numbers was introduced and studied by Italian mathematician C. Segre in 1892 [1]. The advantage of this real four-dimensional algebra, or two-dimensional algebra over complex numbers, is its commutativity, which makes it more applicable to a variety of important problems. For example, in contrast to quaternions, it is not necessary to consider the right and left derivatives of a function separately or to study polynomials with coefficients on special positions. Basic properties of bicomplex numbers and their applications were studied in [2].

In the early of 80-th of the 20-th century, it was understood the possible applications of bicomplex numbers to problems of inertial navigation [3]. A detailed analysis of the algebraic and geometric properties of bicomplex numbers is presented in [4]. In papers [6, 7] the authors developed the theory of monogenic functions in the algebra of Segre quaternions. In [5] a method for solving polynomial equations over bicomplex numbers was developed. At present time the algebra of bicomplex numbers is still the subject of interest of mathematicians.

The main object of the study of this paper is the algebra of Segre complex quaternions, which is a generalization of bicomplex numbers to the algebra of Segre quaternions over the field of complex numbers, similar to the complex generalization of quaternions, which is well studied and has a number of applications in mathematical physics [10]. We study the main algebraic properties of this algebra such as idempotents, ideals, Peirce decomposition and matrix representation. We also develop a method for solution of polynomial equations in Segre complex quaternions and study the Cauchy–D'Alamber

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type conditions for differential functions in this algebra. This algebra has already arisen in our research in the study of solutions of partial differential equations and we believe that it will find its application in study of differential equations of mathematical physics.

1. The basic properties of the algebra of Segre quaternions.

The algebra of Segre quaternions over real numbers is defined as follows

$$\mathbb{B}(\mathbb{R}) = \{a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{f}\},$$

where a_i are real numbers and $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{f}$ are imaginary units of the algebra such that $\boldsymbol{i}^2 = \boldsymbol{j}^2 = -\boldsymbol{f}^2 = -1$, $\boldsymbol{ij} = \boldsymbol{ji} = \boldsymbol{f}$, $\boldsymbol{if} = \boldsymbol{fi} = -\boldsymbol{j}$, $\boldsymbol{jf} = \boldsymbol{fj} = -\boldsymbol{i}$. From these conditions it follows that algebra $\mathbb{B}(\mathbb{R})$ is commutative. In addition, algebra $\mathbb{B}(\mathbb{R})$ has exactly two nontrivial idempotents $\boldsymbol{f}_+ = \frac{1+\boldsymbol{f}}{2}$ and $\boldsymbol{f}_- = \frac{1-\boldsymbol{f}}{2}$, which satisfies the following properties $\boldsymbol{f}_+ + \boldsymbol{f}_- = 1$ and $\boldsymbol{f}_+\boldsymbol{f}_- = 0$.

Algebra $\mathbb{B}(\mathbb{R})$ can be decomposed in the direct sum of principal ideals $I\left(\boldsymbol{f}_+\right)$ and $I\left(\boldsymbol{f}_-\right)$, which are generated by $\boldsymbol{f}_+$ and $\boldsymbol{f}_-$ respectively (the Peirce decomposition)

$$\mathbb{B}(\mathbb{R}) = I\left(\boldsymbol{f}_+\right) \oplus I\left(\boldsymbol{f}_-\right).$$

Elements of the ideals are zero-divisors of the algebra $\mathbb{B}(\mathbb{R})$. The algebra of Segre quaternions $\mathbb{B}(\mathbb{R})$ has the exact regular representation by 4×4 matrices over the field of real numbers as follows

$$\mathcal{B}_4 = \left\{ \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & -x_0 & x_3 & -x_2 \\ x_2 & x_3 & -x_0 & -x_1 \\ x_3 & -x_2 & -x_1 & x_0 \end{bmatrix}, x_0, x_1, x_2, x_3 \in \mathbb{R} \right\},$$

that is, we have the following isomorphism of $\mathbb{B}(\mathbb{R})$ onto $\mathcal{B}_4$

$$\mathbb{B}(\mathbb{R}) \ni \boldsymbol{x} = x_0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{f} \leftrightarrow \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & -x_0 & x_3 & -x_2 \\ x_2 & x_3 & -x_0 & -x_1 \\ x_3 & -x_2 & -x_1 & x_0 \end{bmatrix} \in \mathcal{B}_4.$$

2. The algebra of Segre quaternions over complex numbers.

Let us consider the algebra of bicomplex numbers with complex coefficients, that is, the four-dimensional Segre algebra over the field of complex numbers

$$\mathbb{B}(\mathbb{C}) = \{c_0 + c_1\boldsymbol{j} + c_2\boldsymbol{k} + c_3\boldsymbol{f}\},$$

where c_0, c_1, c_2, c_3 are complex numbers and $\boldsymbol{j}$, $\boldsymbol{k}$, $\boldsymbol{f}$ are imaginary units of the algebra which are defined by their properties $\boldsymbol{j}^2 = \boldsymbol{k}^2 = -\boldsymbol{f}^2 = -1$, $\boldsymbol{jk} = \boldsymbol{kj} = \boldsymbol{f}$, $\boldsymbol{jf} = \boldsymbol{fj} = -\boldsymbol{k}$, $\boldsymbol{kf} = \boldsymbol{fk} = -\boldsymbol{j}$. In addition, $\boldsymbol{j}$, $\boldsymbol{k}$, $\boldsymbol{f}$ commute with the complex imaginary unit $\boldsymbol{i} \in \mathbb{C}$. It is easily verified that $\mathbb{B}(\mathbb{C})$ is a commutative algebra, which is said to be Segre quaternions over complex numbers.

Similarly to the case of bicomplex numbers, the algebra $\mathbb{B}(\mathbb{C})$ is conveniently considered as the algebra $\mathbb{B}_8(\mathbb{R})$ over real numbers, that is

$$\mathbb{B}_8(\mathbb{R}) = \{a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} + a_4\boldsymbol{f} + a_5\boldsymbol{p} + a_6\boldsymbol{q} + a_7\boldsymbol{r}\},$$

where a_l, $l = 0, 1, \ldots 7$ are real numbers and products of the imaginary units are defined by the following Cayley table

	1	*i*	*j*	*k*	*f*	*p*	*q*	*r*
1	*1*	*i*	*j*	*k*	*f*	*p*	*q*	*r*
i	*i*	-*1*	*p*	*q*	*r*	-*j*	-*k*	-*f*
j	*j*	*p*	-*1*	*f*	-*k*	-*i*	*r*	-*q*
k	*k*	*q*	*f*	-*1*	-*j*	*r*	-*i*	-*p*
f	*f*	*r*	-*k*	-*j*	*1*	-*q*	-*p*	*i*
p	*p*	-*j*	-*i*	*r*	-*q*	*1*	-*f*	*k*
q	*q*	-*k*	*r*	-*i*	-*p*	-*f*	*1*	*j*
r	*r*	-*f*	-*q*	-*p*	*i*	*k*	*j*	-*1*

Proposition 1. The $\mathbb{B}_8(\mathbb{R})$ is 8-dimensional commutative algebra over $\mathbb{R}$.

Proof. The commutativity of the multiplication follows directly from the Cayley table for imaginary units. Associativity and distributivity are verified by direct calculations. □

Idempotents of algebra $\mathbb{B}_8(\mathbb{R})$. Let us consider the following elements of $\mathbb{B}_8(\mathbb{R})$

$$\boldsymbol{i}_1 = \frac{1 - \boldsymbol{f} + \boldsymbol{p} + \boldsymbol{q}}{4}; \qquad \boldsymbol{i}_2 = \frac{1 + \boldsymbol{f} - \boldsymbol{p} + \boldsymbol{q}}{4}; \qquad (1)$$
$$\boldsymbol{i}_3 = \frac{1 + \boldsymbol{f} + \boldsymbol{p} - \boldsymbol{q}}{4}; \qquad \boldsymbol{i}_4 = \frac{1 - \boldsymbol{f} - \boldsymbol{p} - \boldsymbol{q}}{4}.$$

Lemma 1. *The elements $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, $\boldsymbol{i}_3$, $\boldsymbol{i}_4$ satisfy the following conditions*

1.

$$\boldsymbol{i}_1^2 = \boldsymbol{i}_1, \boldsymbol{i}_2^2 = \boldsymbol{i}_2, \boldsymbol{i}_3^2 = \boldsymbol{i}_3, \boldsymbol{i}_4^2 = \boldsymbol{i}_4. \qquad (2)$$

That is, $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3, \boldsymbol{i}_4$ are idempotents of $\mathbb{B}_8(\mathbb{R})$.

2. For $k \neq l$

$$\boldsymbol{i}_k \cdot \boldsymbol{i}_l = \boldsymbol{i}_l \cdot \boldsymbol{i}_k = 0, \quad k, l = 1, 2, 3, 4. \qquad (3)$$

3.

$$\boldsymbol{i}_1 + \boldsymbol{i}_2 + \boldsymbol{i}_3 + \boldsymbol{i}_4 = 1. \qquad (4)$$

Proof.

1. By using the Cayley table, we obtain

$$\boldsymbol{i}_1^2 = \left(\frac{1-\boldsymbol{f}+\boldsymbol{p}+\boldsymbol{q}}{4}\right)^2 =$$

$$\frac{1}{16}\left(1+\boldsymbol{f}^2+\boldsymbol{p}^2+\boldsymbol{q}^2-2\boldsymbol{f}+2\boldsymbol{p}+2\boldsymbol{q}-2\boldsymbol{fp}-2\boldsymbol{fq}+2\boldsymbol{pq}\right) =$$

$$\frac{1}{16}\left(4-4\boldsymbol{f}+4\boldsymbol{p}+4\boldsymbol{q}\right) = \frac{1-\boldsymbol{f}+\boldsymbol{p}+\boldsymbol{q}}{4} = \boldsymbol{i}_1.$$

Much in the same way we can consider the rest of cases of Eqs. (2).

2.

$$\boldsymbol{i}_1\cdot\boldsymbol{i}_2 = \frac{1-\boldsymbol{f}+\boldsymbol{p}+\boldsymbol{q}}{4}\cdot\frac{1+\boldsymbol{f}-\boldsymbol{p}+\boldsymbol{q}}{4} =$$

$$\frac{1}{16}\left(1+\boldsymbol{f}-\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{f}-\boldsymbol{f}^2+\boldsymbol{fp}-\boldsymbol{fq}+\boldsymbol{p}+\boldsymbol{pf}-\boldsymbol{p}^2+\boldsymbol{pq}+\boldsymbol{q}+\boldsymbol{qf}-\boldsymbol{qp}+\boldsymbol{q}^2\right) =$$

$$\frac{1}{16}\left(1+\boldsymbol{f}-\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{f}-1-\boldsymbol{q}+\boldsymbol{p}+\boldsymbol{p}-\boldsymbol{q}-1-\boldsymbol{f}+\boldsymbol{q}-\boldsymbol{p}+\boldsymbol{f}+1\right) = \boldsymbol{i}_2\cdot\boldsymbol{i}_1 = 0.$$

Similarly we can prove the rest of Eqs. (3).

3.

$$\boldsymbol{i}_1+\boldsymbol{i}_2+\boldsymbol{i}_3+\boldsymbol{i}_4 =$$

$$\frac{1-\boldsymbol{f}+\boldsymbol{p}+\boldsymbol{q}}{4}+\frac{1+\boldsymbol{f}-\boldsymbol{p}+\boldsymbol{q}}{4}+\frac{1+\boldsymbol{f}+\boldsymbol{p}-\boldsymbol{q}}{4}+\frac{1-\boldsymbol{f}-\boldsymbol{p}-\boldsymbol{q}}{4} = 1.$$

□

Principal ideals of algebra $\mathbb{B}_8(\mathbb{R})$. Consider an element $\boldsymbol{x} = x_0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k} + x_4\boldsymbol{f} + x_5\boldsymbol{p} + x_6\boldsymbol{q} + x_7\boldsymbol{r}$ of algebra $\mathbb{B}_8(\mathbb{R})$. Let us recall that a linear subspace

$$I(\boldsymbol{x}) = \{\boldsymbol{yx} : \boldsymbol{y}\in\mathbb{B}_8(\mathbb{R})\} = \mathbb{B}_8(\mathbb{R})\,\boldsymbol{x}\subset\mathbb{B}_8(\mathbb{R})$$

is called the principal ideal of algebra $\mathbb{B}_8(\mathbb{R})$ generated by $\boldsymbol{x}$.

Denote by $I(\boldsymbol{i}_1)$, $I(\boldsymbol{i}_2)$, $I(\boldsymbol{i}_3)$, $I(\boldsymbol{i}_4)$ the principal ideals of algebra $\mathbb{B}_8(\mathbb{R})$ generating by respective idempotents (1) as follows

$$I(\boldsymbol{i}_k) = \mathbb{B}_8(\mathbb{R})\,\boldsymbol{i}_k = \{\boldsymbol{y}\boldsymbol{i}_k, \boldsymbol{y}\in\mathbb{B}_8(\mathbb{R})\}, k = 1,2,3,4.$$

Proposition 2. If $\boldsymbol{x}\in I(\boldsymbol{i}_l)$, $\boldsymbol{y}\in I(\boldsymbol{i}_m)$, where $l\neq m$, then

$$\boldsymbol{x}\cdot\boldsymbol{y} = 0.$$

Proof. Since $\boldsymbol{x} \in I(i_l), \boldsymbol{y} \in I(i_m)$ there exist $\boldsymbol{x}_0, \boldsymbol{y}_0 \in \mathbb{B}_8(\mathbb{R})$ such that $\boldsymbol{x} = \boldsymbol{x}_0 \boldsymbol{i}_l, \boldsymbol{y} = \boldsymbol{y}_0 \boldsymbol{i}_m$. Taking into account that algebra $\mathbb{B}_8(\mathbb{R})$ is commutative and Lemma 1, we have $\boldsymbol{xy} = \boldsymbol{x}_0 \boldsymbol{y}_0 \boldsymbol{i}_l \boldsymbol{i}_m = 0$. □

Theorem 1. *Algebra*

$$\mathbb{B}_8(\mathbb{R})$$

can be written as the following direct sum (The Peirce decomposition)

$$\mathbb{B}_8(\mathbb{R}) = I(\boldsymbol{i}_1) \oplus I(\boldsymbol{i}_2) \oplus I(\boldsymbol{i}_3) \oplus I(\boldsymbol{i}_4).$$

Proof. Since as linear subspaces of space $\mathbb{B}_8(\mathbb{R})$ ideals $I(\boldsymbol{i}_1)$, $I(\boldsymbol{i}_2)$, $I(\boldsymbol{i}_3)$, $I(\boldsymbol{i}_4)$ are orthogonal and $I(\boldsymbol{i}_k) \cap I(\boldsymbol{i}_l) = 0, k \neq l$ considering Eq. (4) we conclude the proof of the theorem. □

Taking into account properties of idempotents it is easily seen that elements of ideals are zero-divisors of $\mathbb{B}_8(\mathbb{R})$.

Theorem 2. $I(\boldsymbol{i}_l) = \mathbb{C}\boldsymbol{i}_l, l = 1, 2, 3, 4.$

Proof. Let $\boldsymbol{y} \in I(\boldsymbol{i}_1)$. Then there exist $\boldsymbol{y}_i \in \mathbb{R}, i = 0, 1, \dots 7$ such that

$$\boldsymbol{y} = (y_0 + y_1\boldsymbol{i} + y_2\boldsymbol{j} + y_3\boldsymbol{k} + y_4\boldsymbol{f} + y_5\boldsymbol{p} + y_6\boldsymbol{q} + y_7\boldsymbol{r})\boldsymbol{i}_1 =$$

$$\frac{1}{4}(y_0 + y_1\boldsymbol{i} + y_2\boldsymbol{j} + y_3\boldsymbol{k} + y_4\boldsymbol{f} + y_5\boldsymbol{p} + y_6\boldsymbol{q} + y_7\boldsymbol{r}) -$$

$$\frac{1}{4}(y_0\boldsymbol{f} + y_1\boldsymbol{r} - y_2\boldsymbol{k} - y_3\boldsymbol{j} + y_4 - y_5\boldsymbol{q} - y_6\boldsymbol{p} + y_7\boldsymbol{i}) +$$

$$\frac{1}{4}(y_0\boldsymbol{p} - y_1\boldsymbol{j} - y_2\boldsymbol{i} + y_3\boldsymbol{r} - y_4\boldsymbol{q} + y_5 - y_6\boldsymbol{f} + y_7\boldsymbol{k}) +$$

$$\frac{1}{4}(y_0\boldsymbol{q} - y_1\boldsymbol{k} + y_2\boldsymbol{r} - y_3\boldsymbol{i} - y_4\boldsymbol{p} - y_5\boldsymbol{f} + y_6 + y_7\boldsymbol{j}) =$$

$$(y_0 - y_4 + y_5 + y_6)\boldsymbol{i}_1 + \frac{1}{4}(y_1 - y_2 - y_3 - y_7)(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k} - \boldsymbol{r}) =$$

$$((y_0 - y_4 + y_5 + y_6) + (y_1 - y_2 - y_3 - y_7)\boldsymbol{i})\boldsymbol{i}_1 = c\boldsymbol{i}_1,$$

where $c = y_0 - y_4 + y_5 + y_6 + (y_1 - y_2 - y_3 - y_7)\boldsymbol{i}$ is complex. □

Similarly we can prove the cases $l = 2, 3, 4$.

Theorem 1 makes it possible to reduce polynomial equations in $\mathbb{B}_8(\mathbb{R})$ to systems of polynomial equations in the field of complex numbers as follows.

Let us consider a polynomial $p_m(w) = a_m w^m + a_{m-1} w^{m-1} + \ldots + a_0$, where $w, a_k \in \mathbb{B}_8(\mathbb{R})$, $k = 0, 1, ..., m$. Our purpose is to study zeros of

$$p_m(w) = 0. \tag{5}$$

In accordance with Theorem 1, we consider the following decomposition

$$\begin{aligned} a_r &= a_r^{(1)} + \ldots + a_r^{(4)}, r = 0, 1, \ldots m, \\ w &= w_1 + \ldots + w_4, \end{aligned} \tag{6}$$

where $a_r^{(p)}, w_p \in I(\boldsymbol{i}_p)$. It should be noted that $w_i w_j = 0$ and $a_r^{(i)} a_r^{(j)} = 0$ for $i \neq j$.

Substituting (6) into (5), we obtain the following system of polynomial equations

$$\begin{aligned} a_m^{(1)} w_1^m + a_{m-1}^{(1)} w_1^{m-1} + \ldots + a_0^{(1)} &= 0, \\ a_m^{(2)} w_2^m + a_{m-1}^{(2)} w_2^{m-1} + \ldots + a_0^{(2)} &= 0, \\ a_m^{(3)} w_3^m + a_{m-1}^{(3)} w_3^{m-1} + \ldots + a_0^{(3)} &= 0, \\ a_m^{(4)} w_4^m + a_{m-1}^{(4)} w_4^{m-1} + \ldots + a_0^{(4)} &= 0. \end{aligned} \tag{7}$$

Considering Theorem 2, we have $a_r^{(s)} = c_r^{(s)} i_s$, $w_s^r = z_s^r i_s$, where $c_r^{(s)}, z_s \in \mathbb{C}$.

Thus, taking i_s out of the s-th equation $s = 1, 2, 3, 4$ of the system, we obtain the system of four equations in $\mathbb{C}$ as follows

$$\begin{aligned} c_m^{(1)} z_1^m + c_{m-1}^{(1)} z_1^{m-1} + \ldots + c_0^{(1)} &= 0, \\ c_m^{(2)} z_2^m + c_{m-1}^{(2)} z_2^{m-1} + \ldots + c_0^{(2)} &= 0, \\ c_m^{(3)} z_3^m + c_{m-1}^{(3)} z_3^{m-1} + \ldots + c_0^{(3)} &= 0, \\ c_m^{(4)} z_4^m + c_{m-1}^{(4)} z_4^{m-1} + \ldots + c_0^{(4)} &= 0. \end{aligned} \tag{8}$$

Hence, we proved the following theorem

Theorem 3. *Let* $\{r_1^{(l)}, r_2^{(l)}, ..., r_m^{(l)}\}$ *be the set of complex zeros of* $c_m^{(l)} z_l^m + c_{m-1}^{(l)} z_l^{m-1} + \ldots + c_0^{(l)} = 0$, $l = 1, 2, 3, 4$.

Then the set $S = S_1 \oplus S_2 \oplus S_3 \oplus S_4$, *where* $S_k = \left\{ r_1^{(k)} i_k, r_2^{(k)} i_k, ..., r_m^{(k)} i_k \right\}$, $k = 1, 2, 3, 4$ *is the set of solutions of Eq. (5).*

It is easily verified that Eq. (5) has m^4 zeros.

3. The isomorphic matrix algebra.

Let $T_0, T_1, ..., T_7$ be 8×8 matrices as follows

$$T_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \qquad T_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix};$$

$$T_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}; \qquad T_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix};$$

$$T_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \qquad T_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$T_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \qquad T_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easily verified that

	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7
T_0	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7
T_1	T_1	$-T_0$	T_5	T_6	T_7	$-T_2$	$-T_3$	$-T_4$
T_2	T_2	T_5	$-T_0$	T_4	$-T_3$	$-T_1$	T_7	$-T_6$
T_3	T_3	T_6	T_4	$-T_0$	$-T_2$	T_7	$-T_1$	$-T_5$
T_4	T_4	T_7	$-T_3$	$-T_2$	T_0	$-T_6$	$-T_5$	T_1
T_5	T_5	$-T_2$	$-T_1$	T_7	$-T_6$	T_0	$-T_4$	T_3
T_6	T_6	$-T_3$	T_7	$-T_1$	$-T_5$	$-T_4$	T_0	T_2
T_7	T_7	$-T_4$	$-T_6$	$-T_5$	T_1	T_3	T_2	$-T_0$

The result of multiplication is the same as for basis elements (identity and imaginary units) of algebra $\mathbb{B}_8(\mathbb{R})$.

Let us introduce 8-dimensional matrix algebra over $\mathbb{R}$

$$\mathcal{B}_8 = \left\{ \sum_{j=0}^{7} x_j T_j, x_j \in \mathbb{R} \right\}.$$

Theorem 4. *Algebra $\mathcal{B}_8$ is the exact regular representation of* $\mathbb{B}_8(\mathbb{R})$.

Proof. Define the mapping $\tau(\cdot)$ of $\mathbb{B}_8(\mathbb{R})$ onto $\mathcal{B}_8$ by the formula

$$\tau(\boldsymbol{x}) = x_0T_0 + x_1T_1 + x_2T_2 + x_3T_3 + x_4T_4 + x_5T_5 + x_6T_6 + x_7T_7 \tag{9}$$

for $\boldsymbol{x} = x_0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k} + x_4\boldsymbol{f} + x_5\boldsymbol{p} + x_6\boldsymbol{q} + x_7\boldsymbol{r}$.

Let us show that the mapping $\tau(\cdot)$ defines an isomorphism of algebras $\mathbb{B}_8(\mathbb{R})$ and $\mathcal{B}_8$. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{B}_8(\mathbb{R})$ we have

$$\tau(\boldsymbol{x}+\boldsymbol{y}) = (x_0+y_0)T_0 + (x_1+y_1)T_1 + (x_2+y_2)T_2 + (x_3+y_3)T_3 + (x_4+y_4)T_4 +$$

$$(x_5+y_5)T_5 + (x_6+y_6)T_6 + (x_7+y_7)T_7 =$$

$$\sum_{j=0}^{7} x_jT_j + \sum_{j=0}^{7} y_jT_j = \tau(\boldsymbol{x}) + \tau(\boldsymbol{y}).$$

$$\tau(\boldsymbol{x}) \cdot \tau(\boldsymbol{y}) = \left(\sum_{j=0}^{7} x_jT_j\right) \cdot \left(\sum_{j=0}^{7} y_jT_j\right) =$$

$$(x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 + x_4y_4 + x_5y_5 + x_6y_6 - x_7y_7)T_0 +$$

$$(x_0y_1 + x_1y_0 - x_2y_5 - x_3y_6 + x_4y_7 - x_5y_2 - x_6y_3 + x_7y_4)T_1 +$$

$$(x_0y_2 - x_1y_5 + x_2y_0 - x_3y_4 - x_4y_3 - x_5y_1 + x_6y_7 + x_7y_6)T_2 +$$

$$(\dots)T_3 + (\dots)T_4 + (\dots)T_5 + (\dots)T_6 + (\dots)T_7 = \tau(\boldsymbol{x} \cdot \boldsymbol{y}).$$

It is easily seen that matrices $T_j, j = 0, 1, \dots, 7$ are linear independent. This implies that $\tau(\boldsymbol{x}) = 0$ if $\boldsymbol{x} = 0$ and the kernel of homomorphism $\tau(\cdot)$ is trivial, that is $\tau(\cdot)$ is the isomorphism. □

Denote by $\Delta(\boldsymbol{x}) = \det(\tau(\boldsymbol{x}))$. It is easily seen that $\boldsymbol{x}$ is a zero divisor if and only if $\det(\tau(\boldsymbol{x})) = 0$.

4. Differentiation in $\mathbb{B}_8(\mathbb{R})$. Cauchy–D'Alamber conditions.

Let $\boldsymbol{f} : \mathbb{B}_8(\mathbb{R}) \to \mathbb{B}_8(\mathbb{R})$ is as follows

$$\boldsymbol{f}(\boldsymbol{x}) = u_0(\boldsymbol{x}) + u_1(\boldsymbol{x})\boldsymbol{i} + u_2(\boldsymbol{x})\boldsymbol{j} + u_3(\boldsymbol{x})\boldsymbol{k} + u_4(\boldsymbol{x})\boldsymbol{f} + u_5(\boldsymbol{x})\boldsymbol{p} + u_6(\boldsymbol{x})\boldsymbol{q} + u_7(\boldsymbol{x})\boldsymbol{r},$$

where $u_k(\boldsymbol{x}) = u_k(x_0, x_1, \ldots, x_7)$ are some real functions of eight variables $x_j, j = 0, 1, \ldots, 7$.

Definition 1. The function $\boldsymbol{f}(\boldsymbol{x})$ is called differential at $\boldsymbol{x} \in \mathbb{B}_8(\mathbb{R})$ if there exists the limit

$$\lim_{\Delta(\boldsymbol{h})\neq 0, |\boldsymbol{h}|\to 0} \frac{\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x})}{\boldsymbol{h}} = \boldsymbol{f}'(\boldsymbol{x}),$$

and $\boldsymbol{f}'(\boldsymbol{x})$ does not depend on $\boldsymbol{h}$.

Definition 2. A function $\boldsymbol{f} : \mathbb{B}_8(\mathbb{R}) \to \mathbb{B}_8(\mathbb{R})$ is called differential if it is differential at every point $\boldsymbol{x} \in \mathbb{B}_8(\mathbb{R})$.

Note that the differentiability of a function under consideration differs from the concept of a monogenic function, which is studied in [6–9].

Theorem 5. *A function*

$$\boldsymbol{f}(\boldsymbol{x}) = u_0(\boldsymbol{x}) + u_1(\boldsymbol{x})\boldsymbol{i} + u_2(\boldsymbol{x})\boldsymbol{j} + u_3(\boldsymbol{x})\boldsymbol{k} + u_4(\boldsymbol{x})\boldsymbol{f} + u_5(\boldsymbol{x})\boldsymbol{p} + u_6(\boldsymbol{x})\boldsymbol{q} + u_7(\boldsymbol{x})\boldsymbol{r}$$

is differential if and only if functions $u_k(\boldsymbol{x})$ have continuous partial derivatives $\frac{\partial u_k(\boldsymbol{x})}{\partial x_j}$ for all $j, k = 0, 1, \ldots, 7$ and the following Cauchy-D'Alamber type conditions hold

$$\begin{gathered}
\frac{\partial u_0}{\partial x_0} = \frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2} = \frac{\partial u_3}{\partial x_3} = \frac{\partial u_4}{\partial x_4} = \frac{\partial u_5}{\partial x_5} = \frac{\partial u_6}{\partial x_6} = \frac{\partial u_7}{\partial x_7};\\
\frac{\partial u_1}{\partial x_0} = -\frac{\partial u_0}{\partial x_1} = \frac{\partial u_5}{\partial x_2} = \frac{\partial u_6}{\partial x_3} = \frac{\partial u_7}{\partial x_4} = -\frac{\partial u_2}{\partial x_5} = -\frac{\partial u_3}{\partial x_6} = -\frac{\partial u_4}{\partial x_7};\\
\frac{\partial u_2}{\partial x_0} = \frac{\partial u_5}{\partial x_1} = -\frac{\partial u_0}{\partial x_2} = \frac{\partial u_4}{\partial x_3} = -\frac{\partial u_3}{\partial x_4} = -\frac{\partial u_1}{\partial x_5} = \frac{\partial u_7}{\partial x_6} = -\frac{\partial u_6}{\partial x_7};\\
\frac{\partial u_3}{\partial x_0} = \frac{\partial u_6}{\partial x_1} = \frac{\partial u_4}{\partial x_2} = -\frac{\partial u_0}{\partial x_3} = -\frac{\partial u_2}{\partial x_4} = \frac{\partial u_7}{\partial x_5} = -\frac{\partial u_1}{\partial x_6} = -\frac{\partial u_5}{\partial x_7};\\
\frac{\partial u_4}{\partial x_0} = \frac{\partial u_7}{\partial x_1} = -\frac{\partial u_3}{\partial x_2} = -\frac{\partial u_2}{\partial x_3} = \frac{\partial u_0}{\partial x_4} = -\frac{\partial u_6}{\partial x_5} = \frac{\partial u_5}{\partial x_6} = \frac{\partial u_1}{\partial x_7};\\
\frac{\partial u_5}{\partial x_0} = -\frac{\partial u_2}{\partial x_1} = -\frac{\partial u_1}{\partial x_2} = \frac{\partial u_7}{\partial x_3} = -\frac{\partial u_6}{\partial x_4} = \frac{\partial u_0}{\partial x_5} = -\frac{\partial u_4}{\partial x_6} = \frac{\partial u_3}{\partial x_7};\\
\frac{\partial u_6}{\partial x_0} = -\frac{\partial u_3}{\partial x_1} = \frac{\partial u_7}{\partial x_2} = -\frac{\partial u_1}{\partial x_3} = -\frac{\partial u_5}{\partial x_4} = -\frac{\partial u_4}{\partial x_5} = \frac{\partial u_0}{\partial x_6} = \frac{\partial u_2}{\partial x_7};\\
\frac{\partial u_7}{\partial x_0} = -\frac{\partial u_4}{\partial x_1} = -\frac{\partial u_6}{\partial x_2} = -\frac{\partial u_5}{\partial x_3} = \frac{\partial u_1}{\partial x_4} = \frac{\partial u_3}{\partial x_5} = \frac{\partial u_2}{\partial x_6} = -\frac{\partial u_0}{\partial x_7}.
\end{gathered} \tag{10}$$

Proof. Let a function $\boldsymbol{f}$ be differentiable in the above sense.
Denote by $\boldsymbol{e}_0 = 1, \boldsymbol{e}_1 = \boldsymbol{i}, \ldots, \boldsymbol{e}_7 = \boldsymbol{r}$. To obtain Conditions (10) we consider the following limits

$$\lim_{t\to 0} \frac{\boldsymbol{f}(\boldsymbol{x}+t\boldsymbol{e}_l)-\boldsymbol{f}(\boldsymbol{x})}{t\boldsymbol{e}_l} = \sum_{i=0}^{7} \frac{\partial u_i}{\partial x_l}\boldsymbol{e}_i\boldsymbol{e}_l^{-1}. \tag{11}$$

Since the right-hand side of Eq. (11) does not depend on the index l we have

$$\sum_{i=0}^{7} \frac{\partial u_i}{\partial x_0}\boldsymbol{e}_i = \sum_{i=0}^{7} \frac{\partial u_i}{\partial x_1}\boldsymbol{e}_i\boldsymbol{e}_1^{-1} = \ldots = \sum_{i=0}^{7} \frac{\partial u_i}{\partial x_7}\boldsymbol{e}_i\boldsymbol{e}_7^{-1}. \tag{12}$$

Taking into account that $\boldsymbol{e}_1^{-1} = \boldsymbol{i}^{-1} = -\boldsymbol{i}, \ldots, \boldsymbol{e}_4^{-1} = \boldsymbol{f}^{-1} = \boldsymbol{f}, \ldots, \boldsymbol{e}_7^{-1} = \boldsymbol{r}^{-1} = -\boldsymbol{r}$ it follows from Eqs. (12) conditions (10).

On the contrary, let the functions u_i, $i = 0, 1, \ldots, 7$ have continuous firsts partial derivatives and satisfy Cauchy-D'Alamber type conditions (10). Then for every $j = 0, 1, \ldots, 7$, we have

$$u_j(\boldsymbol{x}+\boldsymbol{h}) - u_j(\boldsymbol{x}) = \sum_{i=0}^{7} \frac{\partial u_j(\boldsymbol{x})}{\partial x_i} h_i + o(|\boldsymbol{h}|), \tag{13}$$

where $\frac{o(|\boldsymbol{h}|)}{|\boldsymbol{h}|} \to 0$ as $|\boldsymbol{h}| \to 0$.

Let us put $c_k = \frac{\partial u_k(\boldsymbol{x})}{\partial x_0}$. Considering Cauchy-D'Alamber type conditions (10) it follows from Eq. (13) that

$$\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) = \sum_{j=0}^{7} \left(u_j(\boldsymbol{x}+\boldsymbol{h}) - u_j(\boldsymbol{x})\right)\boldsymbol{e}_j = \sum_{j=0}^{7}\sum_{i=0}^{7} \frac{\partial u_j(\boldsymbol{x})}{\partial x_i} h_i \boldsymbol{e}_j + o(|\boldsymbol{h}|)$$

$$= \left(\sum_{j=0}^{7} \frac{\partial u_j(\boldsymbol{x})}{\partial x_0}\boldsymbol{e}_j\right)\left(\sum_{i=0}^{7} h_i\boldsymbol{e}_i\right) + o(|\boldsymbol{h}|).$$

Thus,

$$\lim_{\Delta(\boldsymbol{h})\neq 0, |\boldsymbol{h}|\to 0} \frac{\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x})}{\boldsymbol{h}} = \sum_{j=0}^{7} \frac{\partial u_j(\boldsymbol{x})}{\partial x_0}\boldsymbol{e}_j = \sum_{j=0}^{7} c_j\boldsymbol{e}_j = \boldsymbol{f}'(\boldsymbol{x}).$$

□

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Деякі алгебраїчні властивості комплексних кватерніонів Сегре.

Алгебра кватерніонів Сегре чи бікомплексних чисел Сегре була введена та вперше вивчалась італійським математиком К. Сегре в 1892 році. Перевага цієї чотиривимірної алгебри над полем дійсних чисел, або двовимірної алгебри Кліфорда над комплексними числами, полягає у комутативності множення її елементів, що сприяє її застосуванню до дослідження різноманітних важливих проблем математики, фізики, навігації тощо. Наприклад, на відміну від кватерніонів, не потрібно розглядати окремо праві та ліві похідні функції чи окремо вивчати поліноми з коефіцієнтами на спеціальних місцях.

Основним об'єктом дослідження цієї роботи є алгебра комплексних кватерніонів Сегре, що є узагальненням бікомплексних чисел до алгебри кватерніонів Сегре над полем комплексних чисел за аналогією узагальнення кватерніонів до комплексних кватерніонів, яке добре вивчене і має ряд застосувань у математичній фізиці. У статті розглянуто основні алгебраїчні та аналітичні властивості алгебри кватерніонів Сеґре над полем комплексних чисел $\mathbb{B}(\mathbb{C})$. Показується, що ця алгебра має зображення у вигляді восьмивимірної комутативної алгебри $\mathbb{B}_8(\mathbb{R})$ над полем дійсних чисел. Для алгебри $\mathbb{B}_8(\mathbb{R})$ складена таблиця множення базисних елементів (таблиця Келі). Знайдені ідемпотенти алгебри та наведені їх основні властивості. За допомогою головних ідеалів, побудованих на ідемпотентах, розглянуто розклад Пірса та визначено дільники нуля алгебри як елементи ідеалів.

Досліджено структуру нулів многочлена в комплексних кватерніонах Сеґре шляхом зведення його до системи чотирьох поліноміальних рівнянь над полем комплексних чисел. Для цього доведено теорему про зображення головних ідеалів у вигляді добутку довільного комплексного числа на відповідний ідемпотент алгебри. У статті наводиться ізоморфне матричне подання $\mathcal{B}_8$ алгебри $\mathbb{B}_8(\mathbb{R})$. Для цього кожен базисний елемент алгебри записаний відповідною восьмивимірною матрицею та таблиця Келі множення цих елементів.

Також у роботі досліджуються умови диференційованості функції на алгебрі $\mathbb{B}_8(\mathbb{R})$, а саме, отримані умови типу Коші–Рімана, які є достатніми для того, щоб функція на алгебрі комплексних кватерніонів Сеґре була диференційованою.

Ключові слова: *кватерніони, алгебра Сегре, розклад Пірса, поліноми.*

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