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On Quaternionic Measures

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Abstract. In this paper we consider some basic properties of quaternionic measures.

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1. Introduction

The notion of a measure is one of the most fundamental objects in mathematics and it would be superfluous to talk much about this. We present now a few lines only in order to explain what we are going to do in the paper, for more details the reader is referred, for instance, to the book of Halmos [10], but for many other sources as well.

Let X be a non-empty set and let $\mathfrak{M}$ be a σ-algebra of subsets of X. A measure (sometimes called a positive measure) is a function μ defined on the measurable space $(X, \mathfrak{M})$ whose range is in $[0, \infty] =: \overline{\mathbb{R}}_+$ and which is countably additive, i.e., if $\{A_i\}$ is a disjoint countable family of elements of $\mathfrak{M}$ then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu\left(A_i\right). \tag{1.1}$$

This definition includes tacitly that the series on the right-hand side converges to a non-negative number or to ∞.

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We assume that there exists at least one $A \in \mathfrak{M}$ for which $\mu(A) < \infty$. This excludes the trivial situation of the measure identically equal to ∞.

Some important properties are:

(a) $\mu(\emptyset) = 0$.

(b) Any measure is finite additive, i.e., (1.1) holds for a finite number of pair-wise disjoint elements of $\mathfrak{M}$.

(c) Any measure is monotone: if A, B are in $\mathfrak{M}$ and $A \subset B$ then $\mu(A) \leq \mu(B)$.

(d) If $\{A_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}$, $A = \bigcup_{n=1}^{\infty} A_n$, $A_1 \subset A_2 \subset \ldots \subset A_n \ldots$, then $\mu(A_n) \longrightarrow \mu(A)$ as $n \longrightarrow \infty$.

(e) If $\{A_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}$, $A_1 \supset A_2 \supset \ldots \supset A_n \ldots$, $A = \bigcap_{n=1}^{\infty} A_n$, $\mu(A_1) < \infty$, then $\mu(A_n) \longrightarrow \mu(A)$ as $n \longrightarrow \infty$.

Definition 1.1. A measure on a measurable space $(X, \mathfrak{M})$ is called σ-finite if there exists a collection of sets $\{A_n, n \in \mathbb{N}\} \subset \mathfrak{M}$ such that $\cup_{n=1}^{\infty} A_n = X$ and for each $n \geq 1$ it holds that $\mu(A_n) < \infty$.

Let us recall a notion of a signed measure or charge:

Definition 1.2. A signed measure (or a charge) on a measurable space $(X, \mathfrak{M})$ is a function

$$\lambda : \mathfrak{M} \to \mathbb{R} \cup \{-\infty, \infty\} \tag{1.2}$$

such that $\lambda(\emptyset) = 0$ and λ is countably additive.

The origin of the notion of the measure explains why it takes just non-negative values. At the same time the question arises: can the measure be complex-valued?

A *complex measure* ω is a complex-valued countably additive function defined on $\mathfrak{M}$. A good source of basic information may be Chapter 6 of the book Rudin [15].

In accordance with the definition if ω is identically zero then ω is a positive measure. A positive measure is allowed to have $+\infty$ as its value; but it is proved that a complex measure μ has as its values the complex numbers only: any $\mu(E)$ is in $\mathbb{C}$. The *real measures* are defined as σ-additive real-valued functions and they form a subclass of the complex measures. Complex measures are not monotone in general but they verify the other above properties. It is worth noting that for a given σ-algebra the collections of positive and of complex measures have, in general, a non-empty intersection but the former is not necessarily a subcollection of the latter; the same kind of relation exists between the positive and the real measures.

The definition of a complex measure can be rephrased as follows. Consider a countable family $\{E_i\}$ of elements of $\mathfrak{M}$ which are pairwise disjoint and let $E := \bigcup_{i=1}^{\infty} E_i$; the family $\{E_i\}$ is called a partition of E. Then a complex measure ω is a complex function on $\mathfrak{M}$ such that

$$\omega(E) = \sum_{i=1}^{\infty} \omega(E_i) \tag{1.3}$$

for any $E \in \mathfrak{M}$ and for every partition $\{E_i\}$ of E.

Notice that the requirement of being $\{E_i\}$ in (1.3) any partition of E has a strong implication: one can change the order of the enumeration in $\{E_i\}$, thus every rearrangement of the series is convergent to the same complex number; it is known that hence the series in (1.3) converges in fact absolutely.

The main goal and inspiration of this work come from some ideas in the book of W. Rudin [15] which includes measures with values in complex numbers. Rudin shows that such objects are interesting by themselves but they also have interesting relations and are useful for a better understanding of classical positive and real measures. Thus is not surprising that this work extends Rudin's idea and the quaternions instead of the complex numbers have become the candidates to deal with. In this paper it is not pretended to achieve for the highest generality but in contrast it has been chosen an object with rather similar algebraic structure expecting to obtain the results both similar and more sophisticated. This is the motivation. What is more, since the quaternions possess a wide scale of applications both in mathematics and in other areas, there are high possibilities that the results presented here will find their applications. For example, it is known that the usual 3-dimensional vector fields when being embedded into quaternion-valued functions show very nice and new properties which can be hardly seen directly without quaternions.

Quaternionic measures might have applications in probability theory, for example, as an alternative representation of the distribution of a four-component random vector. This representation has the advantage of being an element of the field. Another application is correlation theory, see [17].

We have found very few sources about the general theory of quaternionic measures. In the paper Agrawal and Kulkarni [1] and in the book Colombo, Gantner [6, Section 6.1] one can find some basic information on this subject and in both it is noted, that the corresponding properties are analogous to their complex antecedents. Besides, in both cases the authors are interested, mainly, in the questions for which such properties are just auxiliary. For this reason in these papers the proofs have not been provided. We believe that the proofs are not so trivial and a systematic presentation of them is instructive and useful. There are many fine points and it is worth to give such a presentation with the proofs. Lemma 2.3 is a good illustration of the peculiarities which emerge in the quaternionic situation and which generalize its complex antecedents in a non-trivial way.

Notice that recently a number of papers have been published where a new kind of measures has been introduced, namely, those which take values in the ring of hyperbolic numbers, see, e.g., Ghosh, Biswas and Yasin [9], Kumar and Sharma [12], Alpay, Luna and Shapiro [2]. It turns out that their properties are rather close to their classical real-valued analogues. Of course the hyperbolic theory is totally different to that of quaternionic-valued measures, since it is well known that quaternions do not have zero-divisors.

There exists a long list of works where the notion of a measure extends in a lot of different directions. Some of them are: Ludkowski [13], Diestel and Faires [8], Sun and Yeneng [16], Benci, Horsten and Wenmackers [4], Cutland [7], Ciurea [5], Hofweber and Schindler [11], Artstein [3], Maitland Wright [14].

The paper is divided in five sections of which the first one is Introduction. Section 2 deals with the total variation of a quaternionic measure. In analogy with the complex measures case, in the present situation the total variation keeps being a positive measure. Theorem 2.4 about the finiteness of the total variation of a quaternionic measure is central here, it uses strongly Lemma 2.3 and the peculiarities of the quaternionic situation manifest themselves heavily here. Section 3 describes the properties of the absolute continuity of quaternionic measures. In Sect. 4 the set of quaternionic measures as a quaternionic linear space is discussed. The brief Sect. 5 treats some properties of the derivatives of quaternionic measures.

2. Quaternionic Measure and Its Total Variation

We assume in the sequel that X is a non-empty set.

Definition 2.1. Let $\mathfrak{M}$ be a σ-algebra of subsets of a set X. A quaternionic measure ω on a measurable space $(X, \mathfrak{M})$ is a quaternion-valued function on $\mathfrak{M}$ such that for any collection of sets $\{A_n, n \in \mathbb{N}\} \subset \mathfrak{M}$ with the property that $A_n \cap A_m = \emptyset$ whenever $n \neq m$ we have:

$$\omega\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \omega\,(A_n). \tag{2.1}$$

Since the union of sets A_n is not changed if the subscripts are permuted, every rearrangement of series (2.1) must converge to $\omega\,(\bigcup_{n=1}^{\infty} A_n)$. For this reason, we assume that the series converges absolutely.

Let us ask the question: Is it possible to find a positive measure μ on a measurable space $(X, \mathfrak{M})$ such that $|\omega\,(A)| \leq \mu\,(A)$ for any $A \in \mathfrak{M}$? That is, we ask to find a positive measure μ that dominates the Euclidean modulus of ω. It is easily seen that if there exists such a dominant measure then for any partition $\{A_n, n \in \mathbb{N}\} \subset \mathfrak{M}$ of $A \in \mathfrak{M}$, we have:

$$\sum_{n=1}^{\infty} |\omega\,(A_n)| \leq \sum_{n=1}^{\infty} \mu\,(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right).$$

Let us define the set function $\mathrm{var}\,[\omega]\,(\cdot)$ on $\mathfrak{M}$ as follows:

$$\mathrm{var}\,[\omega]\,(A) := \sup \sum_{n=1}^{\infty} |\omega\,(A_n)|\,,$$

where the supremum is taken over all partitions of A. It is clear that

$$|\omega\,(A)| \leq \mathrm{var}\,[\omega]\,(A) \leq \mu\,(A)\,.$$

We will call the function $\mathrm{var}\,[\omega]$ *the total variation* of the quaternionic measure ω.

Theorem 2.2. *The total variation* $\mathrm{var}\,[\omega]$ *of a quaternionic measure* ω *on a measurable space* $(X, \mathfrak{M})$ *is a positive measure on* $(X, \mathfrak{M})$.

Proof. Suppose $\{A_n, n \in \mathbb{N}\} \subset \mathfrak{M}$ is a partition of A. Let $\{A_{nm}\}$ be a partition of A_n, $n \in \mathbb{N}$. Hence, we have:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\omega(A_{nm})| \leq \mathrm{var}\,[\omega]\,(A)\,.$$

Then, taking into account that $A_n = \bigcup_{m=1}^{\infty} A_{nm}$, we have:

$$\sum_{n=1}^{\infty} \sup \sum_{m=1}^{\infty} |\omega(A_{nm})| \ \leq \mathrm{var}\,[\omega]\,(A)\,.$$

Hence,

$$\sum_{n=1}^{\infty} \mathrm{var}\,[\omega]\,(A_n) \leq \mathrm{var}\,[\omega]\,(A)\,. \tag{2.2}$$

Let us show that

$$\sum_{n=1}^{\infty} \mathrm{var}\,[\omega]\,(A_n) \geq \mathrm{var}\,[\omega]\,(A)\,.$$

Suppose $\{B_m\}$ is a partition of A. Then for a fixed $m \in \mathbb{N}$, the collection $\{B_m \cap A_n\}_{n \in \mathbb{N}}$ is a partition of B_m and for a fixed $n \in \mathbb{N}$, the collection $\{B_m \cap A_n\}_{m \in \mathbb{N}}$ is a partition of A_n. Thus, we have:

$$\begin{aligned}\sum_{m=1}^{\infty} |\omega(B_m)| &= \sum_{m=1}^{\infty} \left| \sum_{n=1}^{\infty} \omega(B_m \cap A_n) \right| \leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\omega(B_m \cap A_n)| \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\omega(B_m \cap A_n)| \leq \sum_{n=1}^{\infty} \mathrm{var}\,[\omega]\,(A_n)\,. \end{aligned} \tag{2.3}$$

Since Eq. (2.3) holds for every partition $\{B_m\}$ of A, it holds that

$$\mathrm{var}\,[\omega]\,(A) \leq \sum_{n=1}^{\infty} \mathrm{var}\,[\omega]\,(A_n)\,.$$

Therefore, together with (2.2) one obtains:

$$\mathrm{var}\,[\omega]\,(A) = \sum_{n=1}^{\infty} \mathrm{var}\,[\omega]\,(A_n).$$

It is easily seen that

$$\mathrm{var}\,[\omega]\,(\emptyset) = 0.$$

□

Now it is necessary to prove a Lemma that is crucial in the proof of Theorem 2.4. This Lemma has its complex antecedent (see Lemma 6.3 in [15]) and the proof that we present here is inspired by the proof given by Rudin. Of course the situation for the quaternions is much more complicated. In order that our proof will be more clear to the reader, let us analyse first

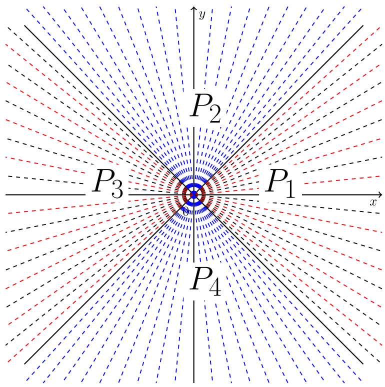

Figure 1. The complex plane is divided in four regions

the proof given for the complex case. Hence, the Lemma and its proof given by Rudin say:

> If $z_1, \ldots, z_n$ are complex numbers, there is a subset S of $\{1, \ldots, n\}$ such that

$$\left| \sum_{j \in S} z_j \right| \geq \frac{1}{6} \sum_{j=1}^{n} |z_j|. \tag{2.4}$$

It is worth noting that, in the third edition of its book, Rudin proved the above inequality with the coefficient $\frac{1}{\pi}$ and the ideas of the proof are completely different. For this paper it is convenient to take the ideas of the second edition of the book because they are more conveniently extended to the quaternionic case. Let us start reproducing the proof given by Rudin of the above Lemma explaining in a more detailed way each step. The author starts the proof writing $w = |z_1| + \cdots + |z_n|$. Then the complex plane is divided in the four closed quadrants bounded by the lines $y = \pm x$: P_1, P_2, P_3, P_4. See Fig. 1.

Denote by S_j the set of subindexes $\ell \in \{1, 2, \ldots, n\}$ such that $z_\ell \in P_j$, $j \in \{1, 2, 3, 4\}$ and write $A_j = \sum_{\ell \in S_j} |z_\ell|$. Set $M := \max\{A_1, A_2, A_3, A_4\}$. After this, it is claimed that there is no loss of generality to assume that $M = A_1$. Let us see why this is true.

(I) If $M = A_1$ it follows that

$$4A_1 \geq w \quad \text{i.e.,} \quad A_1 \geq \frac{w}{4}. \tag{2.5}$$

Note that for any $z \in P_1$, $z = |z|e^{\mathbf{i}\theta}$, with $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$, hence $\frac{1}{\sqrt{2}} \leq \cos\theta \leq \frac{1}{\sqrt{2}}$. Besides, $|z| \geq \Re e(z) = |z|\cos\theta \geq \frac{|z|}{\sqrt{2}}$, which implies that

$$\left|\sum_{\ell\in S_1} z_\ell\right| \geq \sum_{\ell\in S_1} \Re e(z_\ell) \geq \frac{1}{\sqrt{2}} \sum_{\ell\in S_1} |z_\ell| \geq \frac{w}{4\sqrt{2}}.$$

Since $4\sqrt{2} < 6$ we conclude that

$$\left|\sum_{\ell\in S_1} z_\ell\right| \geq \frac{w}{6}.$$

(II) If $M = A_2$ then, as before, one has that $4A_2 \geq w$. Given $z \in P_2$, $z = |z|e^{\mathbf{i}\theta}$ with $\frac{\pi}{4} \leq \theta \leq \frac{3}{4}$, hence $\frac{1}{\sqrt{2}} \leq \sin\theta \leq 1$. Thus, $\Im m(z) = |z|\sin\theta \geq \frac{|z|}{\sqrt{2}}$, hence

$$\left|\sum_{\ell\in S_2} z_\ell\right| \geq \sum_{\ell\in S_2} \Im m(z_\ell) \geq \frac{1}{\sqrt{2}} \sum_{\ell\in S_2} |z_\ell| \geq \frac{w}{4\sqrt{2}}.$$

(III) Assume that $M = A_3$. Note that if the equality (2.4) is true for the given $z_1, \ldots, z_n$, then it is also true for $-z_1, \ldots, -z_n$. Since $P_3 = -P_1$, this case is reduced to Case (I).

(IV) Similarly to the previous item, if $M = A_4$, we can reduce this situation to Case (II).

The above proof says that for an analogous result in $\mathbb{H}$, we need to subdivide the whole space in such a way that we can control the values of the angles and the symmetries between the parts.

Recall which are the angles associated to quaternions. Given $q \in \mathbb{H}$, $q = q_0 + \overrightarrow{q}$, with $q_0 \in \mathbb{R}$, $\overrightarrow{q} \in \mathbb{R}^3$. Assume that q is not a real number, i.e., $\overrightarrow{q} \neq 0$, then

$$q = |q|\left(\frac{q_0}{|q|} + \frac{\overrightarrow{q}}{|q|}\right) = |q|\left(\frac{q_0}{|q|} + \frac{\overrightarrow{q}}{|\overrightarrow{q}|}\frac{|\overrightarrow{q}|}{|q|}\right),$$

i.e.,

$$q = |q|\left(\frac{q_0}{|q|} + \widehat{u}\frac{|\overrightarrow{q}|}{|q|}\right),$$

with a vector $\widehat{u}$ in $\mathbb{R}^3$ of modulus 1. It happens also that

$$\frac{q_0^2}{|q|^2} + \frac{|\overrightarrow{q}|^2}{|q|^2} = 1,$$

thus q can be expressed as

$$q = |q|\left(\cos\alpha + \widehat{u}\sin\alpha\right). \tag{2.6}$$

A geometric interpretation of α is that it is the angle that the quaternion $q \in \mathbb{H} \cong_{\mathbb{R}} \mathbb{R}^4$ makes with the positive real axis. Thus we conclude that $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Since the vector $\widehat{u}$ belongs to $\mathbb{R}^3$, let us write it in spherical coordinates: $\widehat{u} = (\sin\eta\cos\theta, \sin\eta\sin\theta, \cos\eta)$, with $\theta \in [-\pi, \pi]$, $\eta \in [0, \pi]$. Thus (2.6) written as an element in $\mathbb{R}^4$ becomes:

$$q = |q| \left(\cos\alpha, \sin\alpha\sin\eta\cos\theta, \sin\alpha\sin\eta\sin\theta, \sin\alpha\cos\eta\right). \tag{2.7}$$

Now we make a partition of the whole $\mathbb{R}^4 \cong_{\mathbb{R}} \mathbb{H}$ into the following sets (it is clear that in the quaternionic case the description of the partition is not as simple and "natural" as in the complex case):

$$\begin{aligned}
P_1 &:= \left\{q \mid \alpha \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right],\ \eta \in \left[0, \frac{\pi}{4}\right]\right\};\\
P_2 &:= \left\{q \mid \alpha \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right],\ \eta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right],\ \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]\right\};\\
P_3 &:= \left\{q \mid \alpha \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right],\ \eta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right],\ \theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]\right\};\\
P_4 &:= \left\{q \mid \alpha \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right],\ \eta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right],\ \theta \in \left[-\pi, -\frac{3\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]\right\};\\
P_5 &:= \left\{q \mid \alpha \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right],\ \eta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right],\ \theta \in \left[-\frac{3\pi}{4}, -\frac{\pi}{4}\right]\right\};\\
P_6 &:= \left\{q \mid \alpha \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right],\ \eta \in \left[\frac{3\pi}{4}, \pi\right]\right\};\\
P_7 &:= \left\{q \mid \alpha \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]\right\};\\
P_8 &:= \left\{q \mid \alpha \in \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right],\ \eta \in \left[0, \frac{\pi}{4}\right]\right\};\\
P_9 &:= \left\{q \mid \alpha \in \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right],\ \eta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right],\ \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]\right\};\\
P_{10} &:= \left\{q \mid \alpha \in \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right],\ \eta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right],\ \theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]\right\};\\
P_{11} &:= \left\{q \mid \alpha \in \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right],\ \eta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right],\ \theta \in \left[-\pi, -\frac{3\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]\right\};\\
P_{12} &:= \left\{q \mid \alpha \in \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right],\ \eta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right],\ \theta \in \left[-\frac{3\pi}{4}, -\frac{\pi}{4}\right]\right\};\\
P_{13} &:= \left\{q \mid \alpha \in \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right],\ \eta \in \left[\frac{3\pi}{4}, \pi\right]\right\}.
\end{aligned}$$

Lemma 2.3. *Given the quaternions $q_1, q_2, \ldots, q_n$, there is a subset S of $\{1, 2, \ldots, n\}$ such that*

$$\left|\sum_{\ell \in S} q_\ell\right| \geq \frac{1}{39} \sum_{\ell=1}^{n} |q_\ell|. \tag{2.8}$$

Proof. Write $w = \sum_{\ell=1}^{n} |q_\ell|$, and consider the partition of $\mathbb{H} \cong_{\mathbb{R}} \mathbb{R}^4$ given before. Let S_j be the set of subindexes $\ell \in \{1, \dots, n\}$ such that $q_\ell \in P_j$, and define $A_j := \left|\sum_{\ell \in S_j} q_\ell\right|$. Take $M := \max\{A_1, \dots, A_{13}\}$, and assume that $M = A_{\ell_0}$, $\ell \in \{1, \dots, 13\}$. We claim that the set S we are looking for in formula (2.8) is precisely S_{ℓ_0}. Indeed, first note that

$$13M \geq w, \quad \text{i.e.,} \quad M \geq \frac{w}{13}. \tag{2.9}$$

Now consider the following cases.

Case 1 $M = A_1$. Given $q \in P_1$ then $\sin\alpha \geq \frac{1}{\sqrt{2}}$ and $\cos\eta \geq \frac{1}{\sqrt{2}}$. Considering the z components of these quaternions one gets:

$$\begin{aligned}
\left|\sum_{\ell \in S_1} q_\ell\right| &\geq \left(\sum_{\ell \in S_1} q_\ell\right)_z = \sum_{\ell \in S_1} (q_\ell)_z = \sum_{\ell \in S_1} |q_\ell| \sin\alpha_\ell \cos\eta_\ell \\
&\geq \frac{1}{2} \sum_{\ell \in S_1} |q_\ell| = \frac{1}{2} A_1 = \frac{1}{2} M \geq \\
&\geq \frac{w}{2 \cdot 13} \geq \frac{w}{39} = \frac{1}{39} \sum_{\ell=1}^{n} |q_\ell|.
\end{aligned}$$

Case 2 $M = A_2$. For any $q \in P_2$ one has $\sin\alpha \geq \frac{1}{\sqrt{2}}$, $\sin\eta \geq \frac{1}{\sqrt{2}}$, $\cos\theta \geq \frac{1}{\sqrt{2}}$. Now consider the x component of these quaternions:

$$\begin{aligned}
\left|\sum_{\ell \in S_2} q_\ell\right| &\geq \left(\sum_{\ell \in S_2} q_\ell\right)_x = \sum_{\ell \in S_2} (q_\ell)_x = \sum_{\ell \in S_2} |q_\ell| \sin\alpha_\ell \sin\eta_\ell \cos\theta_\ell \\
&\geq \sum_{\ell \in S_2} |q_\ell| \frac{1}{2\sqrt{2}} \geq \frac{w}{2\sqrt{2} \cdot 13} \geq \frac{1}{39} \sum_{\ell=1}^{n} |q_\ell|.
\end{aligned}$$

Case 3 This case is managed similarly as Cases 1 and 2 but using the components Q_y.

Case 4 If $q \in P_4$, then $\sin\alpha \geq \frac{1}{\sqrt{2}}$, $\sin\eta \geq \frac{1}{\sqrt{2}}$, $\cos\theta \in \left[-1, -\frac{1}{\sqrt{2}}\right]$. Let us consider the $\mathbb{R}$-linear map $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ given by

$$T(q) = |q| \left(\cos\alpha, \sin\alpha \sin\eta \cos(\theta + \pi), \sin\alpha \sin\eta \sin(\theta + \pi), \sin\alpha \cos\eta\right).$$

This map is a reflexion in $\mathbb{R}^4$ with respect to the hyperplane generated by $(1, 0, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$. Hence $|T(q)| = |q|$. In particular the Lemma will be true for $q_1, \dots, q_n$ if and only if it is true for $T(q_1), \dots, T(q_n)$. Thus, assuming that $M = A_4$, after applying T the situation is reduced to Case 2.

All the other cases follow a similar scheme, and in all the cases it is obtained that, if $M = A_\ell$, then take $S = S_\ell$ and the equality (2.8) follows. $\square$

Theorem 2.4. *If ω is a quaternionic measure on a measurable space $(X, \mathfrak{M})$, then*

$$\operatorname{var}[\omega](X) < \infty.$$

Proof. Suppose that there is a set $A \in \mathfrak{M}$ such that $\operatorname{var}[\omega](A) = \infty$. Put $t = 39(1 + |\omega(A)|)$. From the definition of $\operatorname{var}[\omega](A)$, there is a partition $\{A_i\}$ of A such that

$$\sum_{i=1}^{n} |\omega(A_i)| > t$$

for some n. Let us apply Lemma with $q_i = \omega(A_i)$ to conclude that there is a set $E \subset A$ which is a union of some sets A_i and

$$|\omega(E)| > \frac{1}{39} t > 1.$$

Considering $F = A \setminus E$, it follows that

$$|\omega(F)| = |\omega(A) - \omega(E)| \geq |\omega(E)| - |\omega(A)| > \frac{1}{39} t - |\omega(A)| = 1.$$

Thus, we have split A into disjoint sets E and F such that $|\omega(E)| > 1$ and $|\omega(F)| > 1$.

Now, if $\operatorname{var}[\omega](X) = \infty$ then we can split X into sets E_1 and F_1 with $|\omega(E_1)| > 1$ and $\operatorname{var}[\omega](F_1) = \infty$. Then we split F_1 into E_2 and F_2 with $|\omega(E_2)| > 1$ and $\operatorname{var}[\omega](F_2) = \infty$. Continuing in this way, we obtain a countably infinite disjoint collection $\{E_n\}$ with $|\omega(E_n)| > 1$ for all n. The countable additivity of ω implies that

$$\omega\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \omega(E_n).$$

But this series cannot converge since $\omega(E_n)$ does not tend to 0 as $n \to \infty$. This contradiction shows that $\operatorname{var}[\omega](X) < \infty$. $\square$

Remark 2.5. The general term measure includes $+\infty$ as an admissible value. Thus the real measures do not form a subclass of the quaternionic measures.

3. Absolute Continuity of Quaternionic Measures

Let μ be a positive measure on a measurable space $(X, \mathfrak{M})$ and ω be a quaternionic measure on $(X, \mathfrak{M})$.

Definition 3.1. We say that ω is **absolutely continuous** with respect to μ if $\mu(A) = 0$ implies $\omega(A) = 0$ for $A \in \mathfrak{M}$. We write $\omega \ll \mu$.

Definition 3.2. Given a quaternionic measure ω on a measurable space $(X, \mathfrak{M})$, assume that there is a set $F \in \mathfrak{M}$ such that $\omega(A) = \omega(A \cap F)$ for every $A \in \mathfrak{M}$, we say that ω is concentrated on F. This is equivalent to say that $\omega(A) = 0$ whenever $A \cap F = \emptyset$.

Let ω_1, ω_2 be quaternionic measures on $(X, \mathfrak{M})$ and suppose there exists a pair of disjoint sets F, G such that ω_1 is concentrated on F and ω_2 is concentrated on G. Then we say that ω_1 and ω_2 are mutually singular, and write $\omega_1 \perp \omega_2$.

Theorem 3.3. (Properties of mutually singular quaternionic measures) *Suppose ω, ω_1, ω_2 are quaternionic measures and μ is a positive measure, then:*

1. *If ω is concentrated on F, so is* $\operatorname{var}[\omega]$.
2. *If $\omega_1 \perp \omega_2$ then* $\operatorname{var}[\omega_1] \perp \operatorname{var}[\omega_2]$.
3. *If $\omega_1 \perp \mu$ and $\omega_2 \perp \mu$, then $(\omega_1 + \omega_2) \perp \mu$.*
4. *If $\omega_1 \ll \mu$ and $\omega_2 \ll \mu$, then $(\omega_1 + \omega_2) \ll \mu$.*
5. *If $\omega \ll \mu$, then* $\operatorname{var}[\omega] \ll \mu$.
6. *If $\omega_1 \ll \mu$ and $\omega_2 \perp \mu$, then $\omega_1 \perp \omega_2$.*
7. *If $\omega \ll \mu$ and $\omega \perp \mu$ then $\omega = 0$ identically.*

Proof.
1. If $A \cap F = \emptyset$ then for any partition $\{A_n, n \in \mathbb{N}\}$ of A we have $\omega(A_n) = 0$ for every $n \in \mathbb{N}$ and hence $\operatorname{var}[\omega](A) = 0$ for any A.
2. This follows from 1.
3. There is a set $B \in \mathfrak{M}$ on which μ is concentrated. There are F, $G \in \mathfrak{M}$ such that ω_1 is concentrated on F and ω_2 is concentrated on G. If $A \subset (F \cup G)^c = F^c \cap G^c$ then $(\omega_1 + \omega_2)(A) = \omega_1(A) + \omega_2(A) = 0$. This means that $\omega_1 + \omega_2$ is concentrated on $F \cup G$, but it is clear that $B \subset (F \cup G)^c$, hence $(\omega_1 + \omega_2) \perp \mu$.
4. Follows directly from the definitions.
5. Suppose $\mu(A) = 0$ and $\{A_n, n \in \mathbb{N}\}$ is a partition of A. Then $\mu(A_n) = 0$ and since $\omega \ll \mu$ then $\omega(A_n) = 0$ for every $n \in \mathbb{N}$; hence $\sum_{n=1}^{\infty} |\omega(A_n)| = 0$. This implies that $\operatorname{var}[\omega](A) = 0$.
6. Since $\omega_2 \perp \mu$ there is a set $E \in \mathfrak{M}$ such that $\mu(E) = 0$ and ω_2 is concentrated on E. Since $\omega_1 \ll \mu$, then $\omega_1(A) = 0$ for every $A \in \mathfrak{M}$ such that $A \subset E$ and hence ω_1 is concentrated on E^c.
7. It follows from *6.* that $\omega \perp \omega$. Hence $\omega = 0$.

□

Theorem 3.4. (Lebesgue) Decomposition of a quaternionic measure. *Let λ be a signed real σ-finite measure on a measurable space $(X, \mathfrak{M})$ and let ω be a quaternionic measure on $(X, \mathfrak{M})$. Then there exists a unique pair of quaternionic measures ω_a and ω_s such that*

$$\omega = \omega_a + \omega_s, \quad \omega_a \ll \lambda, \quad \omega_s \perp \lambda. \tag{3.1}$$

The pair ω_a, ω_s is called the Lebesgue decomposition of the quaternionic measure ω w.r.t. λ, where ω_a is the absolutely continuous part and ω_s is the singular part of the decomposition.

Proof. Since ω is a quaternionic finite measure on $(X, \mathfrak{M})$, we have $\omega = \lambda_0 + \mathbf{i}\lambda_1 + \mathbf{j}\lambda_2 + \mathbf{k}\lambda_3$, with λ_k, $k = 0, 1, 2, 3$ real finite signed measures. By applying Lebesgue's decomposition theorem to each λ_k, we obtain $\lambda_k = \lambda_a^{(k)} + \lambda_s^{(k)}$, where $\lambda_a^{(k)} \ll \lambda$ and $\lambda_s^{(k)} \perp \lambda$. By putting $\omega_a = \lambda_a^{(0)} + \mathbf{i}\lambda_a^{(1)} + \mathbf{j}\lambda_a^{(2)} + \mathbf{k}\lambda_a^{(3)}$ and $\omega_s = \lambda_s^{(0)} + \mathbf{i}\lambda_s^{(1)} + \mathbf{j}\lambda_s^{(2)} + \mathbf{k}\lambda_s^{(3)}$ we conclude the proof of the existence of the pair ω_a, ω_s. Suppose that there is another pair ω_a', ω_s', which satisfies (3.1), then

$$\omega_a' - \omega_a = \omega_s - \omega_s'.$$

It is easily seen that $\omega_a' - \omega_a \ll \lambda$ and $\omega_s - \omega_s' \perp \lambda$. Hence, considering item 7 of Theorem 3.3, we have $\omega_a' - \omega_a = \omega_s - \omega_s' = 0$. □

Theorem 3.5. (Radon–Nikodym theorem for quaternionic measures) *Let μ be a positive σ-finite measure on a measurable space $(X, \mathfrak{M})$, let ω be a quaternionic measure on $(X, \mathfrak{M})$ and let ω_a be absolutely continuous part of the Lebesgue decomposition of ω w.r.t. μ. Then there is a measurable quaternionic function h on X such that for every set $A \in \mathfrak{M}$*

$$\omega_a(A) = \int_A h d\mu,$$

where h is uniquely defined up to a μ-null set.

Remark 3.6. Recall that a quaternionic function is measurable if the preimage of any borelian set belongs to $\mathfrak{M}$.

Proof. Since $\omega_a \ll \mu$, taking into account that $\omega_a(\cdot) := \lambda_a^{(0)}(\cdot) + \mathbf{i}\lambda_a^{(1)}(\cdot) + \mathbf{j}\lambda_a^{(2)}(\cdot) + \mathbf{k}\lambda_a^{(3)}(\cdot)$, where $\lambda_a^{(k)}$ are signed measures, we have that $\lambda_a^{(k)} \ll \mu$ for each $k = 0, 1, 2, 3$. Taking into account Radon–Nikodym Theorem for signed measures there exist measurable functions h_k such that

$$\lambda_a^{(k)}(A) = \int_A h_k d\mu, \quad \forall A \in \mathfrak{M}, \; k = 0, 1, 2, 3.$$

Hence

$$\omega_a(A) = \int_A (h_0(x) + \mathbf{i}h_1(x) + \mathbf{j}h_2(x) + \mathbf{k}h_3(x))\, d\mu(x).$$

$\square$

Remark 3.7. The quaternionic function $h(x) := h_0(x) + \mathbf{i}h_1(x) + \mathbf{j}h_2(x) + \mathbf{k}h_3(x)$ will be called *the Radon–Nikodym derivative* of the quaternionic measure ω_a w.r.t. μ and it is denoted by $d\omega_a/d\mu$.

In the quaternionic case the Radon–Nikodym theorem has also many corollaries and we give one of them.

Theorem 3.8. *Let ω be a quaternionic measure on a measurable space $(X, \mathfrak{M})$. Then there exists a measurable function h such that $|h(x)| = 1$ for all $x \in X$ and*

$$\frac{d\omega}{d\mathrm{var}[\omega]} = h.$$

Proof. Since $\omega \ll \mathrm{var}[\omega]$ it follows from Theorem 3.5 that there is a measurable function h such that $d\omega/d\mathrm{var}[\omega] = h$.

For a positive real p let us consider $S_p := \{x \in X : |h(x)| < p\}$. Then for any partition $\{A_n\}$ of S_p we have:

$$\sum_{n=1}^{\infty} |\omega(A_n)| = \sum_{n=1}^{\infty} \left| \int_{A_n} h(x) d\mathrm{var}[\omega](x) \right|$$

$$\leq p \sum_{n=1}^{\infty} \mathrm{var}[\omega](A_n) = p\mathrm{var}[\omega](S_p).$$

Hence $\mathrm{var}[\omega](S_p) \leq p\mathrm{var}[\omega](S_p)$. If $p < 1$ then $\mathrm{var}[\omega](S_p) = 0$. Therefore, $|h(x)| \geq 1$ a.e. On the other hand for $A \in \mathfrak{M}$ such that $\mathrm{var}[\omega](A) > 0$ we have:

$$\frac{1}{\mathrm{var}[\omega](A)} \left| \int_A h(x) d\mathrm{var}[\omega](x) \right| = \frac{|\omega(A)|}{\mathrm{var}[\omega](A)} \leq 1.$$

Thus, the integral $I_A(h) = \frac{1}{\mathrm{var}[\omega](A)} \int_A h(x) d\mathrm{var}[\omega](x)$ lies in a 4-dimensional ball $B_1(0)$ of radius 1 for each $A \in \mathfrak{M}$ such that $\mathrm{var}[\omega](A) > 0$. Suppose $B_r(a)$ is a ball of radius r and with center at the point a such that $B_r(a) \cap B_1(0) = \emptyset$. Let us show that $\mathrm{var}[\omega](C) = 0$, where $C = h^{-1}(B_r(a))$.

Indeed, if $\mathrm{var}[\omega](C) > 0$ then

$$|I_C(h) - a| = \frac{1}{\mathrm{var}[\omega](C)} \left| \int_C (h(x) - a) d\mathrm{var}[\omega](x) \right|$$

$$\leq \frac{1}{\mathrm{var}[\omega](C)} \int_C |h(x) - a| d\mathrm{var}[\omega](x) \leq r,$$

which is impossible since $I_C(f) \in B_1(0)$ and we conclude that $|h(x)| \leq 1$ (a. e.). Therefore, $|h(x)| = 1$ (a. e.).

Let $N := \{x \in X : |h(x)| \neq 1\}$. Since it was shown that $\mathrm{var}[\omega](N) = 0$ we redefine h on N so that $h(x) = 1$ for all $x \in N$ and obtain a function with the desired properties. □

4. The Set of Quaternionic Measures as a Quaternionic Linear Space

Suppose ω and μ are quaternionic measures on the same measurable space $(X, \mathfrak{M})$. For any $c \in \mathbb{H}$ define quaternionic measures $c\mu$, μc and $\omega + \mu$ by

$$\begin{aligned} (c\mu)(A) &:= c\mu(A), \\ (\mu c)(A) &:= \mu(A)c, \\ (\omega + \mu)(A) &:= \omega(A) + \mu(A) \end{aligned} \tag{4.1}$$

for any $A \in \mathfrak{M}$.

The set of all quaternionic measures on $(X, \mathfrak{M})$ endowed with these operations forms a two-sided quaternionic module although the term “linear quaternionic space” is used instead also, which we will use as well.

Let us put

$$\|\omega\| := \mathrm{var}[\omega](X). \tag{4.2}$$

It is direct to check that Formula (4.2) satisfies the definition of norm on left-quaternionic (or right-quaternionic) linear space.

Theorem 4.1. *The left-linear space of quaternionic measures on* $(X, \mathfrak{M})$ *with the norm* (4.2) *is a quaternionic Banach space.*

Proof. Suppose $A \in \mathfrak{M}$ and $\{\omega_n\}$ is a Cauchy sequence of quaternionic measures, that is, for every $\varepsilon > 0$ there exists a number N such that $n > N$ and $m > N$ imply

$$\|\omega_n - \omega_m\| < \varepsilon.$$

It follows from the theorem of completion of a normed space that there exists the following limit $\omega := \lim_{n\to\infty} \omega_n$. Let us show that ω is a quaternionic measure. Suppose $\{B_m\}$ is a partition of A and put $C_m = A \setminus \bigcup_{k=1}^{m} B_k$.

$$\begin{aligned} \omega(C_m) = \lim_{n\to\infty} \omega_n(C_m) &= \lim_{n\to\infty} \left(\omega_n(A) - \sum_{k=1}^{m} \omega_n(B_k) \right) \\ &= \omega(A) - \sum_{k=1}^{m} \omega(B_k). \end{aligned} \tag{4.3}$$

Hence, $\omega(A) = \omega(C_m) + \sum_{k=1}^{m} \omega(B_k)$ and this implies finite additivity of ω.

Since $\operatorname{var}[\omega]$ is bounded and $C_m \downarrow \emptyset$ as $m \to \infty$ we have $\operatorname{var}[\omega](C_m) \to 0$, $m \to \infty$. Considering $|\omega(C_m)| \le \operatorname{var}[\omega](C_m)$ it follows that $\omega(C_m) \to 0$, $m \to \infty$.

From Eq. (4.3) it follows that

$$\omega(A) - \sum_{k=1}^{\infty} \omega(B_k) = 0.$$

Therefore, ω is a quaternionic measure and the linear space of quaternionic measures on $(X, \mathfrak{M})$ with the norm $\|\cdot\|$ is a quaternionic Banach space. □

5. Derivatives of Measures

Denote by m_k Lebesgue measure on $\mathbb{R}^k$ and denote the open ball with center $x_0 \in \mathbb{R}^k$ and radius $r > 0$ by

$$B(x_0, r) := \left\{ x \in \mathbb{R}^k : |x - x_0| < r \right\}.$$

Definition 5.1. Suppose ω is a quaternionic Borel measure on $\mathbb{R}^k$. The symmetric derivative of ω at $x_0 \in \mathbb{R}^k$ w.r.t. m_k is defined to be

$$(D\omega)(x_0) = \lim_{r\to 0} \frac{\omega(B(x_0, r))}{m_k(B(x_0, r))} .$$

Let us put

$$(M\omega)(x_0) := \sup_{0<r<\infty} \frac{\operatorname{var}[\omega](B(x_0, r))}{m_k(B(x_0, r))}.$$

The function $(M\omega)(x)$ is said to be *the maximal function* of the quaternionic measure ω.

Let us give without proof the following well-known statement [15].

Lemma 5.2. *Consider a collection $B_{r_i}(x_i), i = 1,\ldots,n$ of balls in $\mathbb{R}^k$ and denote by $U = \bigcup_{i=1}^n B_{r_i}(x_i)$. Then there is a subset $S \subset \{1,\ldots,n\}$ such that*

- *the balls $B_{r_j}(x_j), j \in S$ are disjoint,*
- *$U = \bigcup_{j\in S} B_{3r_j}(x_j)$,*
- *$m_k(U) \le 3^k \sum_{j\in S} m_k(B_{r_j}(x_j))$.*

Theorem 5.3. *If ω is a quaternionic Borel measure on $\mathbb{R}^k$ then for any $\varepsilon > 0$*

$$m_k\left(\left\{x \in \mathbb{R}^k : (M\omega)(x) > \varepsilon\right\}\right) \le 3^k \frac{\operatorname{var}[\omega]\left(\mathbb{R}^k\right)}{\varepsilon}. \tag{5.1}$$

Proof. Taking into account that the measure m_k is regular, it is sufficient to prove (5.1) for any compact subset of $\left\{x \in \mathbb{R}^k : (M\omega)(x) > \varepsilon\right\}$. Suppose $C \subset \left\{x \in \mathbb{R}^k : (M\omega)(x) > \varepsilon\right\}$ is a compact set. From the definition of $(M\omega)(x)$ it follows that for any $x \in C$ there exists a ball $B_r(x)$ such that $\operatorname{var}[\omega](B_r(x)) > \varepsilon m_k(B_r(x))$. Compactness of C implies a finite number of balls $B_{r_j}(x_j)$, $j = 1,\ldots,m$ needed to cover it. By Lemma 5.2 there exists a subcollection $B_{r_{j_l}}(x_{j_l})$, $l = 1,\ldots,s$, $s \le m$ of pairwise disjoint sets such that

$$\begin{aligned} m_k(C) &\le m_k\left(\bigcup_{l=1}^s B_{r_{j_l}}(x_{j_l})\right) \le 3^k \sum_{l=1}^s m_k(B_{r_{j_l}}(x_{j_l})) \\ &\le 3^k \sum_{l=1}^s \frac{\operatorname{var}[\omega]\left(B_{r_{j_l}}(x_{j_l})\right)}{\varepsilon} \le \frac{3^k}{\varepsilon}\|\omega\|. \end{aligned}$$

Thus, since this is true for any compact subset C hence considering the regularity of measure m_k this is true for $\left\{x \in \mathbb{R}^k : (M\omega)(x) > \varepsilon\right\}$. ☐

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