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AN ALGEBRAIC APPROACH FOR SOLVING FOURTH-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. It is well-known that any solution of the Laplace equation is a real or imaginary part of a complex holomorphic function. In this paper, in some sense, we extend this property into four order hyperbolic and elliptic type PDEs. To be more specific, the extension is for a c-biwave PDE with constant coefficients, and we show that the components of a differentiable function on the associated hypercomplex algebras provide solutions for the equation.

1. Introduction

In this paper we are interested in finding the solution of the following equation

$$\left(\frac{\partial^4}{\partial x^4} - 2c\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) u\left(x, y\right) = 0,\ \ c > 0. \tag{1.1}$$

Depending on the value of c we may consider three cases. Namely, the case where $0 < c < 1$ and we call it as the c-biwave equation of the elliptic type, the case where $c > 1$ and we call it as the c-biwave equation of hyperbolic type, and in the case where $c = 1$ Eq.(1.1) is the well-known biwave equation. The biwave equation has been used in modeling of d-wave superconductors (see for instance [1], and references therein) or in probability theory [2, 3]. In [4] the author studied Eq.(1.1) in the case where $c < -1$ and considered its application to theory of plain orthotropy.

It is easily verified that any equation of the form

$$\left(A\frac{\partial^4}{\partial x^4} + 2B\frac{\partial^4}{\partial x^2 \partial y^2} + C\frac{\partial^4}{\partial y^4}\right) u\left(x, y\right) = 0,$$

where $AC > 0$ and $AB < 0$ can be reduced to Eq.(1.1) by changing variables. To obtain all solutions of Eq. (1.1) for $1 \neq c > 0$ we will use the method developed in [7]. According to such approach we need a commutative algebra with basis containing e_1, e_2 such that

$$e_1^4 - 2c\, e_1^2 e_2^2 + e_2^4 = 0. \tag{1.2}$$

Then, we study monogenic functions on the subspace of this algebra containing e_1, e_2 and show that any solution of Eq. (1.1) can be obtained as a component of such monogenic functions.

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2. Hyperbolic case

Firstly we study Eq. (1.1) in the case where $c > 1$, which is said to be hyperbolic. Let us consider an associative commutative algebra over the real field $\mathbb{R}$

$$A_c = \{x\mathbf{u} + y\mathbf{f} + z\mathbf{e} + v\mathbf{fe} : x, y, z, v \in \mathbb{R}\}$$

with a basis $\mathbf{u}$, $\mathbf{f}$, $\mathbf{e}$, $\mathbf{fe}$, where $\mathbf{u}$ is the identity element of A_c and the following Cayley table holds $\mathbf{fe} = \mathbf{ef}$, $\mathbf{f}^2 = \mathbf{u}$, $\mathbf{e}^2 = \mathbf{u} - m\mathbf{fe}$, where $m = \sqrt{2(c-1)}$.

The basis elements $\mathbf{u}$, $\mathbf{e}$ satisfy Eq. (1.2).

It is easily verified that for $c > 1$ algebra A_c has the following idempotents

$$i_1 = \frac{k_1}{k_1 + k_2}\mathbf{u} - \frac{\mathbf{f}\sqrt{2}}{k_1 + k_2}\mathbf{e},$$

$$i_2 = \frac{k_2}{k_1 + k_2}\mathbf{u} + \frac{\mathbf{f}\sqrt{2}}{k_1 + k_2}\mathbf{e}, \tag{2.1}$$

where $k_1 = \sqrt{c+1} - \sqrt{c-1}$, $k_2 = \sqrt{c+1} + \sqrt{c-1}$.

Therefore, we have

$$i_1 + i_2 = \mathbf{u}$$

and

$$i_1\, i_2 = \frac{k_1 k_2}{(k_1 + k_2)^2}\mathbf{u} - \frac{\sqrt{2}k_2}{(k_1 + k_2)^2}\mathbf{fe} + \frac{\sqrt{2}k_1}{(k_1 + k_2)^2}\mathbf{fe} - \frac{2}{(k_1 + k_2)^2}\mathbf{u} + \frac{2m}{(k_1 + k_2)^2}\mathbf{fe} = 0.$$

It is easily seen that

$$\mathbf{e} = \mathbf{f}\frac{k_1}{\sqrt{2}} i_2 - \mathbf{f}\frac{k_2}{\sqrt{2}} i_1. \tag{2.2}$$

Consider a subspace B_c of algebra A_c of the following form

$$B_c = \{x\mathbf{u} + y\mathbf{e} \mid x, y \in \mathbb{R}\}.$$

Definition 2.1. *A function $g : B_c \to A_c$ is called differentiable (or monogenic) on B_c if for any $B_c \ni w = x\mathbf{u} + y\mathbf{e}$ there exists a unique element $g'(w)$ such that for any $h \in B_c$*

$$\lim_{\mathbb{R}\ni\varepsilon\to 0} \frac{g(w + \varepsilon h) - g(w)}{\varepsilon} = h g'(w),$$

where $hg'(w)$ is the product of h and $g'(w)$ as elements of A_c.

It follows from [7] that a function $g(w) = \mathbf{u}\, u_1(x, y) + \mathbf{f}\, u_2(x, y) + \mathbf{e}\, u_3(x, y) + \mathbf{f}\,\mathbf{e}\, u_4(x, y)$ is monogenic if and only if there exist continuous partial derivatives $\frac{\partial u_i(x,y)}{\partial x}$, $\frac{\partial u_i(x,y)}{\partial y}$, $i = 1, 2, 3, 4$ and it satisfies the following Cauchy-Riemann type conditions

$$\mathbf{e}\frac{\partial}{\partial x} g(w) = \mathbf{u}\frac{\partial}{\partial y} g(w), \ \forall w \in B_c,$$

or

$$\frac{\partial u_1(x, y)}{\partial y} = \frac{\partial u_3(x, y)}{\partial x},$$

$$\frac{\partial u_2(x, y)}{\partial y} = \frac{\partial u_4(x, y)}{\partial x},$$

$$\frac{\partial u_3(x,y)}{\partial y} = \frac{\partial u_1(x,y)}{\partial x} - m\frac{\partial u_4(x,y)}{\partial x},$$
$$\frac{\partial u_4(x,y)}{\partial y} = \frac{\partial u_2(x,y)}{\partial x} - m\frac{\partial u_3(x,y)}{\partial x}.$$

It is also proved in [7] that if g is monogenic then its components $u_i(x,y)$ satisfies Eq. (1.1).

By passing in B_c from the basis $\mathbf{u}$, $\mathbf{e}$ to the basis i_1, i_2, we have

$$w = x\,\mathbf{u} + y\,\mathbf{e} = \left(x - \mathbf{f}\frac{k_2}{\sqrt{2}}y\right) i_1 + \left(x + \mathbf{f}\frac{k_1}{\sqrt{2}}y\right) i_2.$$

Lemma 2.1. *A function $g : B_c \to A_c$, where $c > 1$, is differentiable if and only if it can be represented as follows*

$$g(w) = \alpha(w_1)\, i_1 + \beta(w_2)\, i_2, \tag{2.3}$$

where $w_1 = x - \mathbf{f}\frac{k_2}{\sqrt{2}}y$, $w_2 = x + \mathbf{f}\frac{k_1}{\sqrt{2}}y$ and $\alpha(w_1)$, $\beta(w_2)$ have continuous partial derivatives $\frac{\partial}{\partial x}\alpha(w_1)$, $\frac{\partial}{\partial y}\alpha(w_1)$, $\frac{\partial}{\partial x}\beta(w_2)$, $\frac{\partial}{\partial y}\beta(w_2)$ satisfying

$$\frac{\partial}{\partial y}\alpha(w_1) = -\mathbf{f}\frac{k_2}{\sqrt{2}}\frac{\partial}{\partial x}\alpha(w_1),$$
$$\frac{\partial}{\partial y}\beta(w_2) = \mathbf{f}\frac{k_1}{\sqrt{2}}\frac{\partial}{\partial x}\beta(w_2).$$

Proof. Sufficiency can be verified directly. Indeed,

$$\frac{\partial}{\partial y}g(w) = \frac{\partial}{\partial y}\alpha(w_1)\, i_1 + \frac{\partial}{\partial y}\beta(w_2)\, i_2$$
$$= -\mathbf{f}\frac{k_2}{\sqrt{2}}\frac{\partial}{\partial x}\alpha(w_1)\, i_1 + \mathbf{f}\frac{k_1}{\sqrt{2}}\frac{\partial}{\partial x}\beta(w_2)\, i_2$$

On the other hand, taking into account Eqs. (2.1), (2.2), we have

$$\mathbf{e}\frac{\partial}{\partial x}g(w) = \left(\mathbf{f}\frac{k_1}{\sqrt{2}}i_2 - \mathbf{f}\frac{k_2}{\sqrt{2}}i_1\right)\left(\frac{\partial}{\partial x}\alpha(w_1)\, i_1 + \frac{\partial}{\partial x}\beta(w_2)\, i_2\right)$$
$$= -\mathbf{f}\frac{k_2}{\sqrt{2}}\frac{\partial}{\partial x}\alpha(w_1)\, i_1 + \mathbf{f}\frac{k_1}{\sqrt{2}}\frac{\partial}{\partial x}\beta(w_2)\, i_2.$$

Hence,

$$\mathbf{e}\frac{\partial}{\partial x}g(w) = \mathbf{u}\frac{\partial}{\partial y}g(w).$$

Now let us prove necessity. Suppose that a function

$$g(w) = \mathbf{u}u_1(x,y) + \mathbf{f}\, u_2(x,y) + \mathbf{e}\, u_3(x,y) + \mathbf{f}\,\mathbf{e}\, u_4(x,y)$$

is monogenic on B_c. Let us define

$$\alpha(w_1) = \mathbf{u}\left(u_1(x,y) - \frac{k_2}{\sqrt{2}}u_4(x,y)\right) + \mathbf{f}\left(u_2(x,y) - \frac{k_2}{\sqrt{2}}u_3(x,y)\right),$$
$$\beta(w_2) = \mathbf{u}\left(u_1(x,y) + \frac{k_1}{\sqrt{2}}u_4(x,y)\right) + \mathbf{f}\left(u_2(x,y) + \frac{k_1}{\sqrt{2}}u_3(x,y)\right).$$

Thus, we have

$$\frac{\partial}{\partial y}\alpha(w_1) = \mathbf{u}\left(\frac{\partial u_3(x,y)}{\partial x} - \frac{k_2}{\sqrt{2}}\left(\frac{\partial u_2(x,y)}{\partial x} - m\frac{\partial u_3(x,y)}{\partial x}\right)\right)$$
$$+\mathbf{f}\left(\frac{\partial u_4(x,y)}{\partial x} - \frac{k_2}{\sqrt{2}}\left(\frac{\partial u_1(x,y)}{\partial x} - m\frac{\partial u_4(x,y)}{\partial x}\right)\right)$$

$$= -\mathbf{f}\frac{k_2}{\sqrt{2}}\frac{\partial u_1(x,y)}{\partial x} - \mathbf{u}\frac{k_2}{\sqrt{2}}\frac{\partial u_2(x,y)}{\partial x} + \mathbf{u}\left(\frac{k_2}{\sqrt{2}}\,m+1\right)\frac{\partial u_3(x,y)}{\partial x}$$
$$+\mathbf{f}\left(\frac{k_2}{\sqrt{2}}\,m+1\right)\frac{\partial u_4(x,y)}{\partial x}.$$

Taking into account that

$$\frac{k_2}{\sqrt{2}}m+1 = \sqrt{c^2-1}+c = \frac{k_2^2}{2},$$

we have $\frac{\partial}{\partial y}\alpha(w_1) = -\mathbf{f}\frac{k_2}{\sqrt{2}}\frac{\partial}{\partial x}\alpha(w_1)$.

Much in the same manner, it can be shown that $\frac{\partial}{\partial y}\beta(w_2) = \mathbf{f}\frac{k_1}{\sqrt{2}}\frac{\partial}{\partial x}\beta(w_2)$. □

Remark 2.1. *Considering variables* x, $y_1 = -\frac{k_2}{\sqrt{2}}y$ *and* x, $y_2 = \frac{k_1}{\sqrt{2}}y$, *we have*

$$\frac{\partial}{\partial y_1}\alpha = \mathbf{f}\frac{\partial}{\partial x}\alpha, \qquad (2.4)$$
$$\frac{\partial}{\partial y_2}\beta = \mathbf{f}\frac{\partial}{\partial x}\beta.$$

Hence, it is easily verified that if the components α_1, α_2 *of* $\alpha(w_1) = \alpha_1(w_1) + \mathbf{f}\alpha_2(w_1)$ *have continuous partial derivatives* $\frac{\partial^2}{\partial x^2}\alpha_k(w_1)$ *and* $\frac{\partial^2}{\partial y_1^2}\alpha_k(w_1)$, $k = 1, 2$ *then they satisfy the wave equation*

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y_1^2}\right)u(x,y_1) = 0.$$

Similarly, the components β_1, β_2 *of* $\beta(w_2) = \beta_1(w_2) + \mathbf{f}\beta_2(w_2)$ *satisfy the wave equation*

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y_2^2}\right)u(x,y_2) = 0.$$

Theorem 2.2. $u(x,y)$ *is a solution of Eq.* (1.1) *for* $c > 1$ *if and only if for some* $i, j \in \{1, 2\}$ *it can be represented as follows*

$$u(x,y) = \alpha_i(\omega_1) + \beta_j(\omega_2),$$

where $\alpha_i(\omega_1), \beta_j(\omega_2)$ *are four times continuous differentiable components of* $\alpha(\omega_1)$ *and* $\beta(\omega_2)$ *of monogenic function* $g(\omega)$ *in the decomposition* (2.3) *i.e.,*

$$g(\omega) = \alpha(\omega_1)\,i_1 + \beta(\omega_2)\,i_2,$$

where $\alpha(\omega_1) = \alpha_1(\omega_1) + \mathbf{f}\alpha_2(\omega_1)$, $\beta(\omega_2) = \beta_1(\omega_2) + \mathbf{f}\beta_2(\omega_2)$ *satisfy Eq.* (2.4).

Proof. As it was mentioned above $u(x,y) = \alpha_i(\omega_1) + \beta_j(\omega_2)$ is a solution for $c > 1$ of Eq. (1.1).

Now suppose that $u(x,y)$ is a solution of Eq. (1.1). It is easily verified that

$$\left(\frac{\partial^4}{\partial x^4} - 2c\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}\right)u(x,y) = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y_1^2}\right)\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y_2^2}\right)u(x,y) = 0. \qquad (2.5)$$

It is easily seen that Eq. (2.5) is equivalent to the set of the following systems

$$\begin{cases} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y_2^2}\right)u(x,y) = v_1(x,y), \\ \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y_1^2}\right)v_1(x,y) = 0 \end{cases}$$

or

$$\begin{cases} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y_1^2}\right) u\,(x,y) = v_2\,(x,y)\,, \\ \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y_2^2}\right) v_2\,(x,y) = 0. \end{cases}$$

Let us consider the first system. Since any solution of $\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y_k^2}\right) v_k\,(x,y) = 0$ is of the form $v_k\,(x,y) = f_1\,(x+y_k) + f_2\,(x-y_k)$, where f_i, $i=1,2$ are arbitrary twice differentiable functions it follows from the second equation of the system that

$$v_1\,(x,y) = f_1\,(x+y_1) + f_2\,(x-y_1)\,.$$

Thus, the first equation of the system is

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y_2^2}\right) u\,(x,y) = f_1\,(x+y_1) + f_2\,(x-y_1)\,. \tag{2.6}$$

It is easily seen that a partial solution of Eq. (2.6) is

$$\begin{aligned} U\,(x,y) &= \frac{k_1^2}{k_1^2-k_2^2}\left(F_1\left(x - \frac{k_2}{\sqrt{2}}y\right) + F_2\left(x + \frac{k_2}{\sqrt{2}}y\right)\right) \\ &= \frac{k_1^2}{k_1^2-k_2^2}\left(F_1\,(x+y_1) + F_2\,(x-y_1)\right), \end{aligned}$$

where $F_k^{''} = f_k$, $k=1,2$.

Thus, the general solution of the system is as follows

$$u\,(x,y) = g_1\,(x+y_2) + g_2\,(x-y_2) + \frac{k_1^2}{k_1^2-k_2^2}\left(F_1\,(x+y_1) + F_2\,(x-y_1)\right)$$

Let us put $\alpha_1\,(\omega_1) = \frac{k_1^2}{k_1^2-k_2^2}\left(F_1\,(x+y_1) + F_2\,(x-y_1)\right)$ and $\beta_2\,(\omega_2) = g_1\,(x+y_2) + g_2\,(x-y_2)$.

Taking into account that $\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y_1^2}\right)\left[\frac{k_1^2}{k_1^2-k_2^2}\left(F_1\,(x+y_1) + F_2\,(x-y_1)\right)\right] = 0$ and $\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y_2^2}\right)\left[g_1\,(x+y_2) + g_2\,(x-y_2)\right] = 0$ we conclude the proof for the first system.

The case of the second system can be proved similarly. □

3. Elliptic case

Now we consider an associative commutative algebra A_c, where $0 < c < 1$, over the complex field $\mathbb{C}$ with a basis $\mathbf{u}$, $\mathbf{e}$ and the following Cayley table $\mathbf{u}\,\mathbf{e} = \mathbf{e}\,\mathbf{u} = \mathbf{e}$, $\mathbf{e}^2 = \mathbf{u} + \mathbf{i}\mu\mathbf{e}$, where $\mu = \sqrt{2(1-c)}$. The matrix representations of $\mathbf{u}$ and $\mathbf{e}$ are

$$\mathbf{u} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} 0 & 1 \\ 1 & \mathbf{i}\mu \end{pmatrix}.$$

Hence, we have the following traces of these representations

$$\mathrm{tr}\,(\mathbf{uu}) = 2, \quad \mathrm{tr}\,(\mathbf{ue}) = \mathbf{i}\mu, \quad \mathrm{tr}\,(\mathbf{ee}) = 2-\mu^2.$$

Since

$$\det\begin{pmatrix} \mathrm{tr}\,(\mathbf{uu}) & \mathrm{tr}\,(\mathbf{ue}) \\ \mathrm{tr}\,(\mathbf{ue}) & \mathrm{tr}\,(\mathbf{ee}) \end{pmatrix} = 2(1+c) \neq 0,$$

then, A_c is a semi-simple algebra [8].

By following similar steps as in Eq. (2.1) we can show that for $0 < c < 1$ algebra A_c has the following idempotents

$$I_- = \frac{k_1}{k_1+k_2}\mathbf{u} + \frac{\sqrt{2}}{k_1+k_2}\mathbf{e}, \quad I_+ = \frac{k_2}{k_1+k_2}\mathbf{u} - \frac{\sqrt{2}}{k_1+k_2}\mathbf{e}, \tag{3.1}$$

where $k_1 = \sqrt{c+1} - \mathbf{i}\sqrt{1-c}$, $k_2 = \sqrt{c+1} + \mathbf{i}\sqrt{1-c}$.

It is also easily verified that these idempotents also satisfy

$$I_- + I_+ = \mathbf{u}$$

and

$$I_- \, I_+ = 0.$$

It is straightforward to see that

$$\mathbf{e} = \frac{k_2}{\sqrt{2}} I_- - \frac{k_1}{\sqrt{2}} I_+. \tag{3.2}$$

Lemma 3.1. *All non-zero elements of subspace $B_c = \{x\,\mathbf{u} + y\,\mathbf{e} \mid x, y \in \mathbb{R}\}$ of algebra A_c are invertible, that is, if $0 \neq w \in B_c$ then there exists $w^{-1} \in B_c$.*

Proof. Suppose $w = s\,\mathbf{u} + t\,\mathbf{e} \in B_c$. Let us show that there exists $w^{-1} = x\,\mathbf{u} + y\,\mathbf{e}$, $x, y \in \mathbb{R}$ such that $w\,w^{-1} = 1$. Indeed, the equation

$$(s\,\mathbf{u} + t\,\mathbf{e})\,(x\,\mathbf{u} + y\,\mathbf{e}) = \mathbf{u}$$

has a unique solution since the determinant of the system

$$\begin{gathered} sx + ty = 1, \\ tx + (s + \mathbf{i}\mu t)\,y = 0, \end{gathered}$$

where x, y are unknown, is $\Delta = s^2 - t^2 + \mathbf{i}\mu ts$ and $\Delta = 0$ if and only if $s = t = 0$. A function $f(w)$, $w \in B_c$ is said to be differentiable if it is differentiable in the common sense, i.e., for all $w \in B_c$ there exists the following limit

$$\lim_{B_c \ni \Delta w \to 0} \frac{f(w + \Delta w) - f(w)}{\Delta w} = f'(w).$$

It is easily seen that if f is differentiable then it is monogenic and hence, it satisfies the following Cauchy-Riemann type of conditions [7]

$$\mathbf{e}\frac{\partial}{\partial x} f(w) = \mathbf{u}\frac{\partial}{\partial y} f(w)$$

or in this case we have

$$\frac{\partial u_1(x,y)}{\partial y} = \frac{\partial u_3(x,y)}{\partial x}, \quad \frac{\partial u_2(x,y)}{\partial y} = \frac{\partial u_4(x,y)}{\partial x},$$

$$\begin{aligned} \frac{\partial u_3(x,y)}{\partial y} &= \frac{\partial u_1(x,y)}{\partial x} - \mu\frac{\partial u_4(x,y)}{\partial x}, \\ \frac{\partial u_4(x,y)}{\partial y} &= \frac{\partial u_2(x,y)}{\partial x} + \mu\frac{\partial u_3(x,y)}{\partial x}. \end{aligned}$$

In [7] it is also proved that if a function $f(x,y) = \mathbf{u}u_1(x,y) + \mathbf{i}\,u_2(x,y) + \mathbf{e}\,u_3(x,y) + \mathbf{i}\,\mathbf{e}\,u_4(x,y)$ is monogenic then $u_i(x,y)$ satisfies Eq. (1.1). We should mention that a constructive description of monogenic functions in a three-dimensional harmonic algebra was studied in [5, 6].

By passing from the basis $\mathbf{u}$, $\mathbf{e}$ to the basis I_-, I_+ we have

$$w = x\,\mathbf{u} + y\,\mathbf{e} = \left(x + \frac{k_2}{\sqrt{2}}y\right) I_- + \left(x - \frac{k_1}{\sqrt{2}}y\right) I_+.$$

□

Lemma 3.2. *A function $f : B_c \to A_c$, $0 < c < 1$, is differentiable if and only if it can be represented as follows*

$$f(w) = \alpha(w_1) I_- + \beta(w_2) I_+,$$

where $w_1 = x_1 + \mathbf{i}\, y_1$, $x_1 = x$, $y_1 = -\mathbf{i}\,\frac{k_2}{\sqrt{2}} y$, $w_2 = x_2 + \mathbf{i}\, y_2$, $x_2 = x$, $y_2 = \mathbf{i}\,\frac{k_1}{\sqrt{2}} y$ *and* $\alpha(w_1)$, $\beta(w_2)$ *are analytical functions of variables* w_1, w_2, *respectively, as follows*

$$\frac{\partial}{\partial y_1}\alpha(w_1) = \mathbf{i}\frac{\partial}{\partial x}\alpha(w_1), \quad \frac{\partial}{\partial y_2}\beta(w_2) = \mathbf{i}\frac{\partial}{\partial x}\beta(w_2).$$

Proof. The sufficiency can be verified directly. Indeed,

$$\begin{aligned}\frac{\partial}{\partial y} f(w) &= \frac{\partial}{\partial y_1}\alpha(w_1)\frac{\partial y_1}{\partial y} I_- + \frac{\partial}{\partial y_2}\beta(w_2)\frac{\partial y_2}{\partial y} I_+ \\ &= -\mathbf{i}\,\frac{k_2}{\sqrt{2}}\frac{\partial}{\partial y_1}\alpha(w_1) I_- + \mathbf{i}\frac{k_1}{\sqrt{2}}\frac{\partial}{\partial y_2}\beta(w_2) I_+ \\ &= \frac{k_2}{\sqrt{2}}\frac{\partial}{\partial x}\alpha(w_1) I_- - \frac{k_1}{\sqrt{2}}\frac{\partial}{\partial x}\beta(w_2) I_+.\end{aligned}$$

On the other hand, taking into account Eqs. (2.4), (3.1), we have

$$\begin{aligned}\mathbf{e}\frac{\partial}{\partial x} f(w) &= \left(\frac{k_2}{\sqrt{2}} I_- - \frac{k_1}{\sqrt{2}} I_+\right)\left(\frac{\partial}{\partial x}\alpha(w_1) I_- + \frac{\partial}{\partial x}\beta(w_2) I_+\right) \\ &= \frac{k_2}{\sqrt{2}}\frac{\partial}{\partial x}\alpha(w_1) I_- - \frac{k_1}{\sqrt{2}}\frac{\partial}{\partial x}\beta(w_2) I_+.\end{aligned}$$

Hence,

$$\mathbf{e}\,\frac{\partial}{\partial x} f(w) = \frac{\partial}{\partial y} f(w).$$

Now let us prove necessity. Suppose that a function

$$f(w) = u_1(x,y) + \mathbf{i}\, u_2(x,y) + \mathbf{e}\, u_3(x,y) + \mathbf{i}\,\mathbf{e}\, u_4(x,y)$$

is monogenic on B_c, i.e., $\mathbf{e}\,\frac{\partial}{\partial x} f(w) = \frac{\partial}{\partial y} f(w)$.

By using Eq. (3.1) we can represent $f(w)$ in the following manner

$$f(w) = \alpha(w_1) I_- + \beta(w_2) I_+,$$

where

$$\begin{aligned}\alpha(w_1) &= u_1(x,y) + \frac{k_2}{\sqrt{2}} u_3(x,y) + \mathbf{i}\left(u_2(x,y) + \frac{k_2}{\sqrt{2}} u_4(x,y)\right), \\ \beta(w_2) &= u_1(x,y) - \frac{k_1}{\sqrt{2}} u_3(x,y) + \mathbf{i}\left(u_2(x,y) - \frac{k_1}{\sqrt{2}} u_4(x,y)\right).\end{aligned}$$

Consider

$$\begin{aligned}\mathbf{u}\frac{\partial}{\partial y} f &= \frac{\partial}{\partial y_1}\alpha(w_1)\frac{\partial y_1}{\partial y} I_- + \frac{\partial}{\partial y_2}\beta(w_2)\frac{\partial y_2}{\partial y} I_+ \\ &= -\frac{\mathbf{i}\, k_2}{\sqrt{2}}\frac{\partial \alpha}{\partial y_1} I_- + \frac{\mathbf{i}\, k_1}{\sqrt{2}}\frac{\partial \beta}{\partial y_2} I_+.\end{aligned}$$

Then, taking into account Eq. (3.1), we have

$$\begin{aligned}\mathbf{e}\,\frac{\partial}{\partial x} f(w) &= \left(\frac{k_2}{\sqrt{2}} I_- - \frac{k_1}{\sqrt{2}} I_+\right)\left(\frac{\partial \alpha}{\partial x} I_- + \frac{\partial \beta}{\partial x} I_+\right) \\ &= \frac{k_2}{\sqrt{2}}\frac{\partial \alpha}{\partial x} I_- - \frac{k_1}{\sqrt{2}}\frac{\partial \beta}{\partial x} I_+.\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial y_1}\alpha(w_1) = \mathbf{i}\,\frac{\partial}{\partial x}\alpha(w_1),$$
$$\frac{\partial}{\partial y_2}\beta(w_2) = \mathbf{i}\,\frac{\partial}{\partial x}\beta(w_2).$$

Suppose $\alpha(\omega_1) = \alpha_1(\omega_1) + \mathbf{i}\,\alpha_2(\omega_1)$ and $\beta(\omega_2) = \beta_1(\omega_2) + \mathbf{i}\,\beta_2(\omega_2)$. It follows from the proof of Lemma 3.2 that $\alpha_i(\omega_1) + \beta_j(\omega_2)$, $i, j \in \{1, 2\}$ are solutions of Eq. (1.1) for $0 < c < 1$. □

Theorem 3.3. *$u(x, y)$ is a solution of Eq. (1.1) for $0 < c < 1$ if and only if for some $i, j \in \{1, 2\}$ it can be represented as follows*

$$u(x, y) = \alpha_i(\omega_1) + \beta_j(\omega_2),$$

where $\alpha_i(\omega_1), \beta_j(\omega_2)$ are components of $\alpha(\omega_1)$ and $\beta(\omega_2)$ of monogenic function $g(\omega)$ in the decomposition (2.3) i.e.,

$$f(\omega) = \alpha(\omega_1) I_- + \beta(\omega_2) I_+,$$

where $\alpha(\omega_1)$, $\beta(\omega_2)$ are complex analytical functions of respective variables.

Proof. As mentioned above $\alpha_i(\omega_1) + \beta_j(\omega_2)$, $i, j \in \{1, 2\}$ are solutions of Eq. (1.1) for $0 < c < 1$.

If $u(x, y)$ is a solution of Eq. (1.1) for $0 < c < 1$ much in the same way as in proving Theorem 2.2 we can show that $u(x, y) = \alpha_i(\omega_1) + \beta_j(\omega_2)$. □

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