LIMIT-ILL-POSED EQUATIONS IN A HILBERT SPACE

A. A. Pogorui

UDC 517.983:517.968

The behavior of the solution of a limit-ill-posed problem on fixed compacta is investigated for integral operators, acting in a Hilbert space.

In the investigation of the problem of the time of reaching the "receding" boundary of the domain of the state space of a Markov process [1, 2] there often appear equations, which are called limit-ill-posed equations in [3]. There, a method for the investigation of these equations is described and an analysis of the limit-ill-posed equations for the spaces, in which the operators under consideration admit matrix representation, is carried out. In the sequel we consider limit-ill-posed equations in general setting.

Let us consider an invertibly reducible bounded operator A_0 [3] of the following form on the space $L_2[0, \infty)$:

$$A_0 g = g(x) - \int_0^\infty k(x, y) g(y)\, dy.$$

Let $N(A_0)$ be the kernel of the operator and $\dim N(A_0) = r \geq 1$. Let $f_i(x)$, $\varphi_i(x)$, $i = \overline{1,r}$, denote bases of the spaces $N(A_0)$ and $N(A_0^*)$ respectively. Since the oeprator A_0 is invertibly reducible without loss of generality we can assume that

$$\int_0^\infty f_i(x) \varphi_j(x)\, dx = \delta_{ij}, \quad i,j = \overline{1, r}.$$

We introduce the operator

$$\Pi_T = \begin{cases} f(x), & x \in [0,T], \\ 0, \ x > T, & f(x) \in L_2[0, \infty). \end{cases}$$

Let $A_T = \Pi_T A_0 \Pi_T$, and $\hat{A}_T$ be the restriction of A_T to $L_2[0, T]$ and suppose that $\exists\, T_1 > 0$ such that $\forall T \in (T_1, \infty)$ there exists in the space $L_2[0, T]$ a unique solution of the equation

$$\hat{A}_T g(x) = h_T(x), \tag{1}$$

where $h_T(x) = h(x)$, $x \in [0, T]$.

To this end, e.g., it is sufficient to demand that $\|k(x, y)\|_{L_2[0,T]} < 1 \ \ \forall T$, $T \in (T_1, \infty)$, although this condition is not necessary.

Let us consider an $h(x) \in L_2[0, \infty)$, for which $\exists\, i \in \{1, \ldots, \tau\}$ such that $(\varphi_i, h) \neq 0$. Then the equation $A_0 g = h$ is unsolvable.

Using the method, set forth in [1-3], we investigate the behavior of the solution of Eq. (1) as $T \to \infty$.

We fix a certain positive $T_0 < T$. Then $L_2[0, \infty) = L_2[0, T_0[\oplus L_2(T_0, T] \oplus L_2(T, \infty)$.

We introduce the operators

$$A_{00} : L_2[0, T_0] \to L_2[0, T_0],$$

$$A_{00} g = g(x) - \int_0^{T_0} k(x, y) g(y)\, dy, \quad x \in [0, T_0],$$

Mathematics Institute, Academy of Sciences of the Ukrainian, SSR, Kiev. Translated from Ukrainskii Matematicheskii Zhurnal, vol. 43, No. 2, pp. 241-247, February, 1991. Original article submitted July 26, 1989.

$$A_{01}: L_2(T_0, T] \to L_2[0, T_0],$$

$$A_{01}g = -\int_{T_0}^{T} k(x, y) g(y) dy, \quad x \in [0, T_0],$$

$$A_{10}: L_2[0, T_0] \to L_2(T_0 T],$$

$$A_{10}g = -\int_{0}^{T_0} k(x, y) g(y) dy, \quad x \in (T_0, T],$$

$$A_{11}: L_2(T_0, T] \to L_2(T_0, T],$$

$$A_{11}g = g(x) - \int_{T_0}^{T} k(x, y) g(y) dy, \quad x \in (T_0, T].$$

We set $g_T^0(x) = g(x)$, $x \in [0, T_0]$, $h_T^0(x) = h(x)$, $x \in [0, T_0]$; $g_T^1(x) = g(x)$, $h_T^1(x) = h(x)$, $x \in (T_0, T]$.

We write Eq. (1) in the form

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \begin{pmatrix} g_T^0 \\ g_T^1 \end{pmatrix} = \begin{pmatrix} h_T^0 \\ h_T^1 \end{pmatrix}. \tag{2}$$

We will suppose that the operator A_{11} is invertible $\forall T \in (T_0, \infty)$. Then, from (2) we get the equation

$$(A_{00} - A_{01}A_{11}^{-1}A_{10}) g_T^0 = z_T^0, \tag{3}$$

where $z_T^0 = h_T^0 - A_{01}A_{11}^{-1}h_T^1$.

To investigate (3) we consider the functions

$$a_{0T}^{(i)}(x) = -\int_{T}^{\infty} k(x, y) f_i(y) dy, \quad x \in [0, T_0], \ i = \overline{1, r},$$

$$a_{1T}^{(i)}(x) = -\int_{T}^{\infty} k(x, y) f_i(y) dy, \ x \in (T_0, T], \quad i = \overline{1, r},$$

$$a_{T0}^{(i)}(y) = -\int_{T}^{\infty} k(x, y) \varphi_i(x) dx, \quad y \in [0, T_0], \quad i = \overline{1, r},$$

$$a_{T1}^{(i)}(y) = -\int_{T}^{\infty} k(x, y) \varphi_i(x) dx, \quad y \in (T_0, T], \quad i = \overline{1, r},$$

and the quantities

$$a_T^{(i,j)} = \int_{T}^{\infty} \varphi_i(x) f_j(x) dx - \int_{T}^{\infty}\int_{T}^{\infty} k(x, y) \varphi_i(x) f_j(y) dxdy.$$

We introduce the vector-valued functions

$$a_{0T} = \{a_{0T}^{(i)}(x), \quad i = \overline{1, r}\}, \quad a_{1T} = \{a_{1T}^{(i)}(x), \quad i = \overline{1, r}\},$$

$$a_{T1} = \{a_{T1}^{(i)}(y), \ i = \overline{1, r}\}, \quad a_{T0} = \{a_{T0}^{(i)}(y), \ i = \overline{1, r}\}$$

and the matrix

$$a_T = \{a_T^{(i,j)}, \ i, j = \overline{1, r}\}.$$

Let g_i, $i = \overline{1,r}$, be the solution of the equation $A_{11}g_i = a_{1T}^{(i)}$. Let us consider the matrix $a_T - a_{T1}A_{11}^{-1}a_{1T} = \{a_T^{(i,j)} - a_{T1}^{(i)} g_j, \ i,j = \overline{1,r}\}$, where

$$a_{T1}^{(i)} g_j = \int_{T}^{\infty}\int_{0}^{T} k(x, y) g_j(y) \varphi_i(x) dxdy.$$

We suppose that $(a_T - a_{T1}A_{11}^{-1}a_{1T})^{-1}$ exists $\forall T > 0$ and consider the following operator on $L_2[0, T_0]$

$$\mathfrak{U}_{00}^T = A_{00} - A_{01}A_{11}^{-1}A_{10} - (a_{0T} - A_{01}A_{11}^{-1}a_{1T})\times$$
$$\times (a_T - a_{T1}A_{11}^{-1}a_{1T})^{-1}(a_{T0} - a_{T1}A_{11}^{-1}A_{10}).$$

While proving the fact that $f_i^0(x) = f_i(x)$, $x \in [0, T_0]$, $i = 1,4$, belong to the space $N(\mathfrak{U}_{00}^T)$, we will show how the operator $\mathfrak{U}_{00}^T$ acts.

Let us set $f_i^1(x) = f(x)$, $x \in (T_0, T]$. Since $A_0 f_i = 0$, $i = \overline{1,r}$, we have

$$A_{10}f_i^0 + A_{11}f_i^1 + a_{1T}^{(i)} = 0, \quad i = \overline{1,r}, \tag{4}$$

whence it follows that

$$(A_{00} - A_{01}A_{11}^{-1}A_{10})f_i^0 = A_{00}f_i^0 + A_{01}f_i^1 + A_{01}A_{11}a_{1T}^{(i)}. \tag{5}$$

By virtue of (4) we have

$$(a_{T0} - a_{T1}A_{11}^{-1}A_{10})f_i^0 = a_{T0}f_i^0 + a_{T1}f_i^1 + a_{T1}A_{11}^{-1}a_{1T}^{(i)}. \tag{6}$$

Since $a_{T0}f_i^0 + a_{T1}f_i^1 + a_T^{(i)} = 0$, where $a_T^{(i)}$ is the i-th column of the matrix a_T, we have

$$(a_{T0} - a_{T1}A_{11}^{-1}A_{10})f_i^0 = -(a_T - a_{T1}A_{11}^{-1}a_{1T})\,1_i, \tag{7}$$

where 1_i is the column vector of order r whose i-th component is 1 and all the remaining components are zero.

By virtue of (4)-(7), we have $\mathfrak{U}_{00}^T f_i^0 = 0$, $i = \overline{1,r}$. It is analogously verified that $\mathfrak{U}_{00}^{T*}\varphi_i^0 = 0$, $i = \overline{1,r}$.

Adding and subtracting the operator $\mathfrak{U}_{00}^T$, in the expression within brackets in (3), we get the equation

$$(\mathfrak{U}_{00}^T - B_T)g_T^0 = z_T^0, \tag{8}$$

where

$$B_T = -(a_{0T} - A_{01}A_{11}^{-1}a_{1T})(a_T - a_{T1}A_{11}^{-1}a_{1T})^{-1}(a_{T0} - a_{T1}A_{11}^{-1}A_{10}). \tag{9}$$

The eigenprojection P_0 of the operator $\mathfrak{U}_{00}$ has the form

$$P_0 = \sum_{k=1}^{r} c_k f_k^0 \otimes \varphi_k^0,$$

where $c_k = (\varphi_k^0, f_k^0)^{-1}$; whence

$$P_0 B_T P_0 = \sum_{k,l=1}^{r} c_l \gamma_{kl}^T f_k^0 \otimes \varphi_l^0,$$

where $\gamma_{kl}^T = c_k(a_T^{(k,l)} - a_{T1}^{(k)}A_{11}^{-1}a_{1T}^{(l)})$, $k, l = \overline{1,r}$.

We know that the operators A_{01}, A_{11}, and A_{10} depend on T. In the sequel, let the following condition be fulfilled:

A_1) The limit $\lim\limits_{T\to\infty} A_{01}A_{11}^{-1}A_{10} = \bar{A}_{01}\bar{A}_{11}^{-1}\bar{A}_{10}$, where $\bar{A}_{01}$, $\bar{A}_{11}$, $\bar{A}_{10}$ are A_{01}, A_{11}, and A_{10} respectively for $T = \infty$, exists.

<u>THEOREM 1.</u> Suppose that the following conditons, besides condtion A_1, are fulfilled:

A_2) $\sup\limits_{T>T_0} \|A_{11}^{-1}\|_{L_2(T_0,T]} < \infty$;

A_3) There exist i, j $\in \{1, \ldots, r\}$ such that $\varepsilon_{ij}(T) = \left(\int_T^\infty f_i^2(x)\,dx \times \int_T^\infty \varphi_j^2(x)dx\right)^{1/2} > 0$ for $\forall T > 0$.

A_4) The limit $c_{kl}^{(i,j)} = \lim\limits_{T\to\infty} \varepsilon_{ij}^{-1}(T)\gamma_{kl}^T$, k, $l = \overline{1,r}$, exists and the matrix $c^{(i,j)} = (c_{kl}^{(i,j)}$, k, $l = \overline{1,r})$ has the inverse $(c^{(i,j)})^{-1} = (c_{kl}^{(i,j)(-1)}$, k, $l = \overline{1,r})$.

Then

$$\lim_{T\to\infty} \varepsilon_{ij}(T) g_T^0(x) = \sum_{k,l=1}^{r} c_l c_{kl}^{(i,j)(-1)} (\varphi_l, h) f_k^0(x), \quad x \in [0, T_0].$$

Proof. It follows from conditions A_2) and A_4) that

$$\sup_{T>T_0} \varepsilon_{ij}^{-1}(T) \| B_T \| = S_1^{(i,j)} < \infty \tag{10}$$

and, therefore, $\|B_T\| \to 0$ as $T \to \infty$. Using this fact and condition A_1), we have $\mathfrak{U}_{00}{}^T \to \mathfrak{U}_{00} = A_{00} - \bar{A}_{01}\bar{A}_{11}{}^{-1}\bar{A}_{10}$ as $T \to \infty$.

It is immediately verified that $\mathfrak{U}_{00} f_i^0 = 0$ and $\mathfrak{U}_{00}^* \varphi_i^0 = 0$, $i = \overline{1,4}$, and $f_i{}^0$ and φ_i^0, $i = \overline{1,r}$, form bases in $N(\mathfrak{U}_{00})$ and $N(\mathfrak{U}_{00}{}^*)$ respectively, i.e., ρ_0 is the eigenprojection of the oeprator $\mathfrak{U}_{00}$. Hence $R = (\mathfrak{U}_{00} + P_0)^{-1}$ exists. Since $R^T = (\mathfrak{U}_{00}{}^T + P_0)^{-1}$ exists $\forall T \in (T_0, \infty)$, it follows by virtue of a well-known result from functional analysis [4] that $\exists T_1 > T_0$:

$$\sup_{T>T_1} \| R^T \| < \infty, \tag{11}$$

whence

$$\sup_{T>T_1} \| R_0^T \| < \infty, \tag{12}$$

where $R_0{}^T = (\mathfrak{U}_{00}{}^T + P_0)^{-1} - P_0$.

Let $\tilde{B}_{ij} = \varepsilon_{ij}{}^{-1}(T) B_T$, $\Pi_{\tilde{B}_{ij}}$ be the generalized inverse operator of the operator $P_0\tilde{B}_{ij}P_0$ [3],

$$T_H^{(i,j)} = (I - \Pi_{\tilde{B}_{ij}} \tilde{B}_{ij}) R_0^T (I - \tilde{B}_{ij} \Pi_{\tilde{B}_{ij}}).$$

We show that

$$\sup_{T>T_1} \| T_H^{(i,j)} \| = S_2^{(i,j)} < \infty. \tag{13}$$

By virtue of (12) and (13), to this end it is sufficient to show that $\sup \|\Pi_{\tilde{B}_{ij}}\| < \infty$, but this follows from condition A_4). Therefore, $\exists T_2 > T_1$: $\varepsilon_{ij}(T) < (S_1{}^{(i,j)} S_2{}^{(i,j)})^{-1}$ $\forall T > T_2$. Applying [3, Lemma 3.1], for $T > T_2$ we can write

$$(\mathfrak{U}_{00}^T - \varepsilon_{ij}(T) \tilde{B}_{ij}) = - \varepsilon_{ij}^{-1}(T) \Pi_{\tilde{B}_{ij}} + T_H (I - \varepsilon_{ij}(T) B_{ij} T_H)^{-1}.$$

Hence, by virtue of (12) and (13) we have

$$(\mathfrak{U}_{00}^T - \varepsilon_{ij}(T) \tilde{B}_{ij})^{-1} = - \varepsilon_{ij}^{-1}(T) \Pi_{\tilde{B}_{ij}} + o(\varepsilon_{ij}^{-1}(T)). \tag{14}$$

Taking (14) into account, from (8) we get

$$g_T^0(x) = \sum c_l \gamma_{kl}^{(-1),T} (\varphi_l^0, z_T^0) f_k^0(x) + o(\varepsilon_{ij}^{-1}(T)), \quad x \in [0, T_0]. \tag{15}$$

Since $A_{01}^* \varphi_l^0 = - A_{11}^* \varphi_l^1 - a_T^{(l)}$, $l = \overline{1, r}$,

$$(\varphi_l^0, z_T^0) = (\varphi_l^0, h_T^0) + (\varphi_l^1, h_T^1) + a_T^{(l)} A_{11}^{-1} h_T^1.$$

Since $a_T{}^{(l)} \to 0$ as $T \to \infty$, $l = \overline{1, r}$, from condition A_2) we have

$$\lim_{T\to\infty} (\varphi_l^0, z_T^0) = (\varphi, h). \tag{16}$$

Multiplying both sides of (15) by $\varepsilon_{ij}(T)$ and passing to limit as $T \to \infty$, by virtue of A_4) and (16) we get the assertion of the theorem.

Let us consider a generalization of the above result. Let $A_{T,\alpha} g = A_T g - \alpha H_T g$, where $A_T = \Pi_T A_0 \Pi_T$, $T > 0$, and A_0 being an invertibly reducible bounded operator on $L_2[0, \infty)$:

$A_0 g = g(x) - \int_0^\infty k(x, y)g(y) \times dy$, $\dim N(A_0) = r \geq 1$, $f_i\ \varphi_i$, $i = \overline{1,r}$ are bases in $N(A_0)$, $N(A_0^*)$ respectively and $(f_i, \varphi_j) = \delta_{ij}$, $i,j = 1,r$, $h = L_2[0, \infty)$, $H_T = \Pi_T H_0 \Pi_T$, $T > 0$, where H_0 is a bounded operator on $L_2[0, \infty)$, $H_0 g = g(x) - \int_0^\infty h(x, y)g(y)dy$, α is a small parameter, $\exists\ i \in \{1, \ldots, r\}: (\varphi_i, h) \neq 0$.

Let $\hat{A}_{T,\alpha}$ be the restriction of the operator $A_{T,\alpha}$ to $L_2[0, T]$. Suppose that there exists a T_1 such that $\forall T \in (T_1, \infty)$ the equation

$$\hat{A}_{T,\alpha} g = h_T, \quad h_T(x) = h(x), \quad x \in [0, T] \tag{17}$$

has a unique solution in $L_2[0, T]$.

In the same way as from (1) to (3), we pass from (17) to

$$(A_{00}^{(\alpha)} - A_{01}^{(\alpha)} A_{11}^{(\alpha)-1} A_{10}^{(\alpha)})\, g_T^0 = \tilde{z}_T^0, \tag{18}$$

where $\tilde{z}_T^0 = h_T^0 - A_{10}(\alpha)A_{11}(\alpha)^{-1}h_T^1$, $A_{ij}(\alpha) = A_{ij} - \alpha H_{ij}$, $i, j = \overline{0, 1}$. Adding and subtracting the operator $A_{00} - A_{01}A_{11}^{-1}A_{10}$, obtained from A_0, in the expression within the brackets in (18), we have

$$(A_{00} - A_{01}A_{11}^{-1}A_{10} - L_{T,\alpha}) = \tilde{z}_T^0, \tag{19}$$

where $L_{T,\alpha} = A_{00} - A_{01}A_{11}^{-1}A_{10} - (A_{00}(\alpha) - A_{11}(\alpha)A_{11}(\alpha)^{-1}A_{10}(\alpha))$. Let $\mathfrak{U}_{00}^T$, B_T, $\gamma_{k\ell}^T$, k, $l = \overline{1, r}$, be defined from the operator A_0 in the same manner as earlier [see (9)].

Adding and subtracting the operator $\mathfrak{U}_{00}^T$, in the expression within the brackets in (19), we get

$$(\mathfrak{U}_{00}^T - (B_T + L_{T,\alpha}))\, g_T^0 = \tilde{z}_T^0. \tag{20}$$

<u>THEOREM 2.</u> Let conditions A_1)-A_3) of Theorem 1 and the following condition A_4') be fulfilled: $T \to \infty$, $\alpha \to 0$ such that the limits

$$l_{ij} = \lim_{\substack{T\to\infty \\ \alpha\to 0}} \varepsilon_{ij}^{-1}(T)\alpha, \quad c_{kl}^{(i,j)} = \lim_{T\to\infty} \varepsilon_{ij}^{-1}(T)\gamma_{kl}^T, \quad k, l = \overline{1, r},$$

exist and the matrix $\tilde{C}^{(i,j)} = (\tilde{c}_{k\ell}^{(i,j)} + l_{ij}(\varphi_k, H_0 f\ ),\ k,\ l = \overline{1,r})$ has the inverse $\tilde{C}^{(i,j)-1} = (\tilde{c}_{k\ell}^{(i,j)(-1)},\ k,\ \ell\ \ l = \overline{1, r})$. Then

$$\lim_{\substack{T\to\infty \\ \alpha\to 0}} \varepsilon_{ij}(T)\, g_T^0(x) = \sum_{k,l=1}^{r} c_l \tilde{c}_{kl}^{(i,j)(-1)} (\varphi_k, h) f_l^0(x), \quad x \in [0, T_0].$$

<u>Proof.</u> We show that

$$\lim_{\substack{\alpha\to 0 \\ T\to\infty}} \alpha^{-1}(\varphi_k^0, L_{T,\alpha} f_l^0) = (\varphi_k, H_0 f_l), \quad k, l = \overline{1, r}. \tag{21}$$

Indeed,

$$(\varphi_k^0, L_{T,\alpha} f_l^0) = (\varphi_k^0, (A_{00} - A_{00}^{(\alpha)}) f_l^0) + (\varphi_k^0, (A_{01}^{(\alpha)} - A_{01}) A_{11}^{(\alpha)-1} A_{10} f_l^0) +$$
$$+ (\varphi_k^0, A_{01} A_{11}^{(\alpha)-1} (A_{10}^{(\alpha)} - A_{10}) f_l^0) + (\varphi_k^0, A_{01}(A_{11}^{(\alpha)-1} - A_{11}^{-1}) A_{10} f_l^0).$$

By virtue of the boundedness of the operator H_0 and condition A_2), we have

$$A_{11}^{(\alpha)-1} - A_{11}^{-1} = (A_{11} - \alpha H_{11})^{-1} - A_{11}^{-1} = \alpha A_{11}^{-1} H_{11} A_{11}^{-1} + o(\alpha).$$

Hence

$$(\varphi_k^0, L_{T,\alpha} f_l^0) = \alpha[(\varphi_k^0, H_{00} f_l^0) - (\varphi_k^0, H_{01} A_{11}^{-1} A_{10} f_l^0 - (\varphi_k^0, A_{01} A_{11}^{-1} H_{10} f_l^0) +$$
$$+ (\varphi_k^0, A_{01} A_{11}^{-1} H_{11} A_{11}^{-1} A_{10} f_l^0)] + o(\alpha). \tag{22}$$

With regard for condition A_2) and the fact that $f_\ell \in N(A_0)$, $\varphi_l \in N(A_0^*)$, $l = \overline{1, r}$, (21) follows from (22).

The further proof of Theorem 2 is analogous to the proof of Theorem 1.

Example. Let

$$A_0 g = g(x) - \int_0^\infty e^{-(x+y)/2} g(y)\,dy,$$

$$H_0 g = \int_0^\infty e^{-(x^2+y^2)+\frac{1}{2}(x+y)-1} d(y)\,dy.$$

Let us set $\alpha = e^{-T}$. It is easily verified that $\exists T_1 > 0$: the equation

$$g(x) - \int_0^T e^{-(x+y)/2} g(y)\,dy - e^{-T} \int_0^T e^{-(x^2+y^2)+\frac{1}{2}(x+y)-4} g(y)\,dy = h_T \tag{23}$$

has a unique solution for $\forall T \in (T_1, \infty)$ and $h(x) \in L_2[0, \infty)$. We can immediately verify that $f(x) = e^{-x/2} \in N(A_0)$, and $\dim N(A_0) = 1$. Since A_0 is self-adjoint, it follows that $\varphi(x) = e^{-x/2} \in N(A_0^*)$.

In order to apply Theorem 2, we compute

$$(\varphi, H_0 f) = e^{-4} \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dxdy = \frac{\pi}{4e^4},$$

$$\varepsilon(T) = \int_T^\infty f^2(x)\,dx = \int_T^\infty e^{-x} dx = e^{-T},$$

$$c = \lim_{T\to\infty} \varepsilon^{-1}(T)\,\gamma_{11}^T = \lim_{T\to\infty} \frac{1}{1-e^{-T_0}} \left(1 - \frac{e^{-T}}{1-e^{-T_0}+e^{-T}}\right) = \frac{1}{1-e^{-T_0}}.$$

Hence, applying Theorem 2, we see that the solution $g_T^0(x)$ of Eq. (23), considered on the segment $[0, T_0]$, behaves in the following manner as $T \to \infty$:

$$\lim_{T\to\infty} e^{-T} g_T^0(x) = \frac{4e^{-x/2+4}}{(4e^4+\pi)} \int_0^\infty e^{-y/2} h(y)\,dy, \quad h(y) \in L_2[0, \infty), \quad x \in [0, T_0].$$

LITERATURE CITED

1. V. S. Korolyuk, A. A. Tomusyak, and A. F. Turbin, "Sojourn time of a semi-Markov process in an expanding state set," in: Analytic Methods in Probability Theory [in Russian], Naukova Dumka, Kiev (1979), pp. 69-80.
2. A. A. Borovkov, "Boundary-value problems for random observations and large deviations in functional spaces," Teor. Veroyatn. Ee Primen., 12, No. 4, 635-654 (1967).
3. V. S. Korolyuk and A. F. Turbin, Mathematical Foundations of the Phase Extension of Complex Systems [in Russian], Naukova Dumka, Kiev (1978).
4. K. Yoshida, Functional Analysis, Springer Verlag, Berlin—New York (1965).