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Analytic functions of a vector argument and partially conformal mappings in continuum complex spaces

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Abstract. A vector generalization of the main concepts in the theory of functions of a complex variable—the concepts of the modulus and the argument of the complex number—is proposed. The authors introduce a certain generalization of the concept of holomorphic functions and mappings in the case of continuum complex spaces.

1. Introduction

In works [1–3], the linear vector space $\mathbb{C}^\infty$, i.e., the space of ordered countable sequences of complex numbers was considered. Thus, $\mathbb{C}^\infty$ is the Cartesian product of a countable number of instances of the complex plane $\mathbb{C}$: $\mathbb{C}^\infty = \mathbb{C} \times \mathbb{C} \times \ldots \times \mathbb{C} \times \ldots$.

In this work, the results published in the original sources [1–3] are transferred onto the linear vector space $\mathbb{C}_{\text{cont}}$ whose basis has a power of continuum.

Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of natural, real, and complex numbers, respectively, $R_+ = [0, +\infty)$, and $\overline{\mathbb{C}}$ the extended complex plane.

2. Theory in space $\mathbb{C}_{\text{cont}}$

Let $\mathbb{C}_{\text{cont}}$ be the Cartesian product of a continuum number of $\mathbb{C}$ instances: $\mathbb{C}_{\text{cont}} = \mathbb{C} \times \mathbb{C} \times \ldots \times \mathbb{C} \times \ldots$. Similarly, $\mathbb{R}_{\text{cont}} = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} \times \ldots$, $\mathbb{R}_{\text{cont}} \subset \mathbb{C}_{\text{cont}}$.

The basis of the space $\mathbb{C}_{\text{cont}}$ is a continuum set of vectors

$$(1, 0, 0, ...), (0, 1, 0, ...), ...$$

with a continuum number of coordinates.

Let $F(\alpha)$ be a mapping of the segment $[0, 1]$ on the basis of the space $\mathbb{C}_{\text{cont}}$; i.e., every coordinate plane in the space $\mathbb{C}_{\text{cont}}$ is related to a number α, $\alpha \in [0, 1]$.

The elements of the space $\mathbb{C}_{\text{cont}}$ are vectors with a continuum number of complex coordinates $\mathbb{Z} = \{z_\alpha\} \in \mathbb{C}_{\text{cont}}$, $\alpha \in [0, 1]$.

By analogy with finite-dimensional and countable cases, $\mathbb{C}_{\text{cont}}$ can be represented as a direct sum of the continuum number of instances of the algebra of complex numbers $\mathbb{C}$.

Let us transfer some concept of works [1–3] onto the case of the space $\mathbb{C}_{\text{cont}}$.

1. Algebra $\mathbb{C}_{\text{cont}}$

Definition. A binary operation acting from $\mathbb{C}_{\text{cont}} \times \mathbb{C}_{\text{cont}}$ into $\mathbb{C}_{\text{cont}}$ by the rule

$$\mathbb{Z} \cdot \mathbb{W} = \{z_\alpha \cdot w_\alpha\}, \alpha \in [0, 1]\,, \tag{2.1}$$

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where $\mathbb{Z} = \{z_\alpha\} \in \mathbb{C}_{\text{cont}}$, $\mathbb{W} = \{w_\alpha\} \in \mathbb{C}_{\text{cont}}$, will be called *the vector product of elements* $\mathbb{C}_{\text{cont}}$. Note that this operation converts $\mathbb{C}_{\text{cont}}$ into a commutative, associative algebra with the unity $\mathbb{1} = (1, 1, ..., 1, \ldots) \in \mathbb{C}_{\text{cont}}$.

Invertible to the product operation introduced in this way are those and only those elements $\mathbb{Z} = \{z_\alpha\} \in \mathbb{C}_{\text{cont}}$ for which $z_\alpha \neq 0$ for an arbitrary $\alpha \in [0, 1]$.

Inverse to such elements $\mathbb{Z} \in \mathbb{C}_{\text{cont}}$ are elements $\mathbb{Z}^{-1} = \{z_\alpha^{-1}\} \in \mathbb{C}_{\text{cont}}$ because $\mathbb{Z} \cdot \mathbb{Z}^{-1} = \mathbb{Z}^{-1} \cdot \mathbb{Z} = \mathbb{1}$.

Therefore, the set Θ of all elements $A = \{a_\alpha\} \in \mathbb{C}_{\text{cont}}$ that have at least one coordinate $a_k = 0$ is a set of elements with no invertible ones.

2. Conjugation

Definition. Let us put each element $\mathbb{W} = \{w_\alpha\} \in \mathbb{C}_{\text{cont}}$, $\alpha \in [0, 1]$, in correspondence with the vector-conjugate element $\overline{\mathbb{W}} = \{\overline{w}_\alpha\} \in \mathbb{C}_{\text{cont}}$, where $\overline{w}_\alpha$ is a number that is complex conjugate to w_α in the usual sense. The correspondence defined in such a way gives an automorphism $\mathbb{C}_{\text{cont}}$ with a fixed subspace $\mathbb{R}_{\text{cont}}$.

3. (Vector) module

In works [1–3], a vector generalization of the concept of the module of a number was proposed. Let us extend it onto the space $\mathbb{C}_{\text{cont}}$. Let $\mathbb{R}_{+,\text{cont}} = R_+ \times R_+ \times \ldots \times R_+ \ldots$, where the quantity R_+ is continuum.

Definition. The vector module of an arbitrary element $\mathbb{Z} = \{z_\alpha\} \in \mathbb{C}_{\text{cont}}$ is called a vector $|\mathbb{Z}| := \{|z_\alpha|\} \in \mathbb{R}_{+,\text{cont}}$, $\alpha \in [0, 1]$.

Note that for an arbitrary $\mathbb{Z} = \{z_\alpha\} \in \mathbb{C}_{\text{cont}}$, the equality

$$\mathbb{Z} \cdot \overline{\mathbb{Z}} = |\overline{\mathbb{Z}}|^2 = |\mathbb{Z}|^2 \tag{2.2}$$

holds.

4. Vector norm

Definition. A vector $\mathbb{X} = \{x_\alpha\} \in \mathbb{R}_{\text{cont}}$ is called non-negative (strictly positive) and denoted as $\mathbb{X} \geq \mathbb{O}$ ($\mathbb{X} > \mathbb{O}$) if $x_\alpha \geq 0$ for all $\alpha \in [0, 1]$ ($x_\alpha > 0$ at least for one $\alpha \in [0, 1]$), $\mathbb{O} = (0, 0, \ldots, 0, \ldots)$.

Definition. We say that a vector $\mathbb{X} = \{x_\alpha\} \in \mathbb{R}_{\text{cont}}$, $\alpha \in [0, 1]$, is greater than or equal to (strictly greater than) a vector $\mathbb{Y} = \{y_\alpha\} \in \mathbb{R}_{\text{cont}}$, $\alpha \in [0, 1]$, if $\mathbb{X} - \mathbb{Y} \geq \mathbb{O}$ ($\mathbb{X} - \mathbb{Y} > \mathbb{O}$).

Definition. A vector space $\mathbb{Y}$ is called vector-normalized if each $y \in \mathbb{Y}$ is in correspondence with a non-negative vector $\|y\| \in \mathbb{R}_{+,\text{cont}}$ that satisfies the following conditions:

1) $\|y\| \geq \mathbb{O}$, with $\|y\| = \mathbb{O} \Longleftrightarrow y = 0_{\mathbb{Y}}$ ($0_{\mathbb{Y}}$ is the zero of the space $\mathbb{Y}$);
2) $\|\gamma y\| = |\gamma| \|y\|$, $\forall y \in \mathbb{Y}$, $\forall \gamma \in \mathbb{C}$;
3) $\|y_1 + y_2\| \leq \|y_1\| + \|y_2\|$, $\forall y_1, y_2 \in \mathbb{Y}$.

The above definition of the module satisfies the definition of the norm. Hence, the vector module is a vector norm in the algebra $\mathbb{C}_{\text{cont}} : \|\cdot\| = |\cdot|$. Then a unit open polycircle $\|z\| < \mathbb{1}$ ($\mathbb{1} = (1, 1, \ldots, 1, \ldots)$) is a unit ball, and $\mathbb{T}_{\text{cont}} = \{\mathbb{Z} \in \mathbb{C}_{\text{cont}} : \|Z\| = \mathbb{1}\}$ a unit sphere in the algebra $\mathbb{C}_{\text{cont}}$. Note that

a) $|Z_1 \cdot Z_2| = \|Z_1 \cdot Z_2\| = \|Z_1\| \|Z_2\| = |Z_1| |Z_2|$, $\forall Z_1, Z_2 \in \mathbb{C}_{\text{cont}}$;
b) $|\mathbb{1}| = \|\mathbb{1}\| = \mathbb{1}$, ($\mathbb{1} = (1, 1, \ldots, 1, \ldots)$).

5. Vector argument $a \in \mathbb{C}_{\text{cont}}$

Definition. The vector argument of a vector $\mathbb{A} = \{a_\alpha\} \in \mathbb{C}_{\text{cont}} \backslash \mathbb{O}$, $\alpha \in [0, 1]$, is an infinite-dimensional real vector defined by the formula

$$\arg \mathbb{A} = \{\arg a_\alpha\},$$

where $\arg a_\alpha$, $\alpha \in [0,1]$, is either the principal argument value or a value following from the specific content of the problem where the vector $\mathbb{A} \in \mathbb{C}_{\rm cont}$ appears.

6. Compactification of $\mathbb{C}_{\rm cont}$

As a compactification of $\mathbb{C}_{\rm cont} = \mathbb{C}\times\mathbb{C}\times\ldots\times\mathbb{C}\times\ldots$, we take the space $\overline{\mathbb{C}}_{\rm cont} = \overline{\mathbb{C}}\times\overline{\mathbb{C}}\times\ldots\times\overline{\mathbb{C}}\times\ldots$, which will be called the infinite-dimensional space of the theory of functions. Infinite are those points of $\overline{\mathbb{C}}_{\rm cont}$ that have at least one infinite coordinate.

The convergence in $\overline{\mathbb{C}}_{\rm cont}$ is defined as a coordinate-wise convergence uniform over the coordinate numbers.

In this case, the introduced convergence generates a topology in the space $\overline{\mathbb{C}}_{\rm cont}$.

The Cartesian product $\mathbb{B} = \prod\limits_{\alpha} B_\alpha$, where B_α, $\alpha \in [0,1]$, are domains in $\overline{\mathbb{C}}$, will be called the domain in $\overline{\mathbb{C}}_{\rm cont}$.

7. Differentiability

Let a domain $\mathbb{D} \subset \mathbb{C}_{\rm cont}$ and a mapping $\mathbb{F} : \mathbb{D} \to \mathbb{C}_{\rm cont}$ be given, where $\mathbb{F} = \{f_\alpha(\mathbb{Z})\} = \{f_\alpha(\mathbb{X}+i\mathbb{Y})\}$, $\alpha \in [0,1]$, $f_\alpha(\mathbb{X}+i\mathbb{Y}) = U_\alpha(\mathbb{X},\mathbb{Y}) + iV_\alpha(\mathbb{X},\mathbb{Y}) = U_\alpha(\{x_\beta\},\{y_\beta\}) + iV_\alpha(\{x_\beta\},\{y_\beta\})$, $\beta \in [0,1]$. $\mathbb{F} = \mathbb{U} + i\mathbb{V}$, $\mathbb{U} = \mathbb{U}(\mathbb{X},\mathbb{Y}) = \{U_\alpha(\mathbb{X},\mathbb{Y})\}$, $\alpha \in [0,1]$, $\mathbb{V} = \mathbb{V}(\mathbb{X},\mathbb{Y}) = \{V_\alpha(\mathbb{X},\mathbb{Y})\}$, $\alpha \in [0,1]$, $\mathbb{Z} = \mathbb{X} + i\mathbb{Y} = \{x_\alpha\} + i\{y_\alpha\} \in \mathbb{D}$, $\alpha \in [0,1]$.

Let the functions $U_k(\{x_\beta\},\{y_\beta\})$, $V_k(\{x_\beta\},\{y_\beta\})$, $\beta \in [0,1]$, have continuous partial derivatives over all variables x_β, y_β, $\beta \in [0,1]$, everywhere in $\mathbb{D}$. Then the Jacobi matrix looks like

$$\begin{pmatrix} \mathbb{U}_\mathbb{X} & \mathbb{U}_\mathbb{Y} \\ \mathbb{V}_\mathbb{X} & \mathbb{V}_\mathbb{Y} \end{pmatrix}, \tag{2.3}$$

where $\mathbb{U}_\mathbb{X}, \mathbb{U}_\mathbb{Y}, \mathbb{V}_\mathbb{X}$, and $\mathbb{V}_\mathbb{Y}$ are infinite matrices of the following forms: $\mathbb{U}_\mathbb{X} = \left[\{U^{(\alpha)}_{x_\beta}\}\right]$, $\mathbb{U}_\mathbb{Y} = \left[\{U^{(\alpha)}_{y_\beta}\}\right]$, $\mathbb{V}_\mathbb{X} = \left[\{V^{(\alpha)}_{x_\beta}\}\right]$, $\mathbb{V}_\mathbb{Y} = \left[\{V^{(\alpha)}_{y_\beta}\}\right]$, $V^{(\alpha)}_{x_p} = \frac{\partial}{\partial x_\beta} V_\alpha$, $V^{(\alpha)}_{y_\beta} = \frac{\partial}{\partial y_\beta} V_\alpha$, $U^{(\alpha)}_{x_\beta} = \frac{\partial}{\partial x_\beta} U_\alpha$, $U^{(\alpha)}_{y_\beta} = \frac{\partial}{\partial x_\beta} U_\alpha$, $\alpha, \beta \in [0,1]$.

In our case, the symbol $[\cdot]$ means an infinite matrix.

Then the Cauchy–Riemann equation acquires the following form:

$$\begin{cases} \mathbb{U}_\mathbb{X} = \mathbb{V}_\mathbb{Y}, \\ \mathbb{U}_\mathbb{Y} = -\mathbb{V}_\mathbb{X}. \end{cases} \tag{2.4}$$

Definition. Let $\mathbb{D}$ be a domain in the space $\mathbb{C}_{\rm cont}$. A mapping $\mathbb{F} : \mathbb{D} \to \mathbb{C}_{\rm cont}$ that is continuously differentiable in $\mathbb{D}$ and satisfies the matrix equation (2.4) in $\mathbb{D}$ will be called the holomorphic mapping of the domain $\mathbb{D}$.

We assume that a holomorphic mapping $\mathbb{F} : \mathbb{D} \to \mathbb{C}_{\rm cont}$, $\mathbb{D} \subset \mathbb{C}_{\rm cont}$, is biholomorphic if $\mathbb{F}$ has an inverse mapping that is holomorphic in $\mathbb{F}(\mathbb{D})$.

Consider the definition of uniform convergence inside the unit polycircle of a certain sequence of mappings.

Let $\mathbb{U}^{\rm cont}_r = U_r \times U_r \times \ldots \times U_r \times \ldots$, where $U_r = \{z : z \in \mathbb{C}, |z| < r\}$, $\mathbb{U}^{\rm cont}_1 := \mathbb{U}^{\rm cont}$. $\overline{\mathbb{U}}^{\rm cont}_r = \overline{U}_r \times \overline{U}_r \times \ldots \times \overline{U}_r \times \ldots$, and $\mathbb{F}_p : \mathbb{U}^{\rm cont} \to \mathbb{C}_{\rm cont}$ is a sequence of mappings.

Definition. We assume that a sequence $\mathbb{F}_p$, $p = \overline{1,\infty}$, uniformly converges to a certain mapping $\mathbb{F}_0 : \mathbb{U}^{\rm cont} \to \mathbb{C}_{\rm cont}$ inside $\mathbb{U}^{\rm cont}$ if, for an arbitrary $\varepsilon > 0$ and $0 < r < 1$, there exists a number $n_0 = n_0(\varepsilon, r)$, $n_0 \in \mathbb{N}$, such that

$$\|\mathbb{F}_p(\mathbb{Z}) - \mathbb{F}_0(\mathbb{Z})\| \le \varepsilon \cdot \mathbf{1}$$

for all $\mathbb{Z} \in \overline{\mathbb{U}}^{\rm cont}_r$ and all $p > n_0$.

Definition. A holomorphic mapping

$$\mathbb{F} : \mathbb{U}^{\text{cont}} \to \mathbb{C}_{\text{cont}}, \quad \mathbb{F}(\mathbb{Z}) = \{f_\alpha(z_\alpha)\}, \quad f_\alpha = \sum_{p=1}^{\infty} a_p^{(\alpha)} z_\alpha^p,$$

will be called the analytic function of the vector argument if the series

$$\mathbb{F}(\mathbb{Z}) = \sum_{p=1}^{\infty} \mathbb{A}_p \mathbb{Z}^p, \quad \mathbb{A}_p = \{a_p^{(\alpha)}\}, \quad \mathbb{Z} \in \mathbb{U}^{\text{cont}}, \quad p = \overline{1, \infty}, \quad \alpha \in [0,1]$$

uniformly converges inside the polycircle $\mathbb{U}^{\text{cont}}$.

Definition. Let $\delta \in (0,1)$ be a fixed number. Then a mapping

$$\mathbb{F}(\mathbb{Z}) = \{f_\alpha(z_\alpha)\}, \quad \mathbb{Z} \in \mathbb{U}^{\text{cont}},$$

where each $f_\alpha(z_\alpha)$, $\alpha \in [0,1]$, is a one-sheeted function in the unit circle such that $\delta < |f'_\alpha(0)| < \frac{1}{\delta}$, $\alpha \in [0,1]$, will be called the partially conformal mapping of the unit polycircle.

In this case, $\delta = \delta(\mathbb{F})$.

Note that the narrowing of partially conformal mappings onto the coordinate plane is a conformal mapping.

8. Presentation in the vector-Cartesian form

Let $\mathbb{Z} = \{z_\alpha\} \in \mathbb{C}_{\text{cont}}$. Then

$$\mathbb{Z} = \{z_\alpha\} = \{Rez_\alpha + iImz_\alpha\} = \{Rez_\alpha\} + \{iImz_\alpha\} =$$

$$= \{Rez_\alpha\} + i\{Imz_\alpha\} = Re\mathbb{Z} + iIm\mathbb{Z} = X + iY =$$

$$= \{x_\alpha\} + i\{y_\alpha\} \in \mathbb{R}_{\text{cont}} + i\mathbb{R}_{\text{cont}},$$

where $X = Re\mathbb{Z} = \{Rez_\alpha\} = \{x_\alpha\}$, $Y = Im\mathbb{Z} = \{Imz_\alpha\} = \{y_\alpha\}$, $\alpha \in [0,1]$. That is, $\mathbb{C}_{\text{cont}} = \mathbb{R}_{\text{cont}} + i\mathbb{R}_{\text{cont}}$.

9. Representation in the vector-polar form

Using the above definitions, we obtain the following chain of equalities:

$$\mathbb{Z} = \{z_\alpha\} = \{|z_\alpha| e^{i\alpha_\alpha}\} = \{|z_\alpha|\}\{e^{i\alpha_\alpha}\} =$$

$$= |\mathbb{Z}| \left[\cos\arg\mathbb{Z} + i\sin\arg\mathbb{Z}\right] = |\mathbb{Z}| e^{i\arg\mathbb{Z}},$$

where

$$\cos\beta = \{\cos\beta_\alpha\}, \quad \sin\beta = \{\sin\beta_\alpha\},$$

$$\exp i\beta = \{\exp i\beta_\alpha\}, \quad \beta = \{\beta_\alpha\} \in \mathbb{R}_{\text{cont}}, \quad \mathbb{Z} = \{z_\alpha\} \in \mathbb{C}_{\text{cont}},$$

$\alpha \in [0,1]$.

In a similar way, we determine $\ln\mathbb{Z}$, where $\mathbb{Z} = \{z_\alpha\} \in \mathbb{C}_{\text{cont}} \setminus \Theta$:

$$\ln\mathbb{Z} = \ln|\mathbb{Z}| + i\arg\mathbb{Z} = \{\ln|z_\alpha| + i\arg z_\alpha\}.$$

Here are examples of partially conformal mappings that are given by elementary functions:

1) the fractional-linear function

$$T = \frac{\mathbb{A}_1\mathbb{Z} + \mathbb{A}_2}{\mathbb{A}_3\mathbb{Z} + \mathbb{A}_4}, \mathbb{Z} \neq -\frac{\mathbb{A}_4}{\mathbb{A}_3}, \mathbb{A}_1\mathbb{A}_4 - \mathbb{A}_2\mathbb{A}_3 \neq \mathbb{O},$$

where $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$, and $\mathbb{A}_4$ are fixed complex numbers, and $\mathbb{Z} = \{z_\alpha\}$, $\alpha \in [0,1]$, is a complex variable;

2) the power function $W = \mathbb{Z}^n = \{z_\alpha^n\}$, where n is a natural number, which is holomorphic over the whole plane $\overline{\mathbb{C}}_{\text{cont}}$, $\alpha \in [0,1]$;

3) the Zhukovskii function $W = \frac{1}{2}\left(\mathbb{Z} + \frac{1}{\mathbb{Z}}\right)$, which is holomorphic in $\overline{\mathbb{C}}_{\text{cont}} \setminus \Theta$;

4) the polynomial $\mathbb{P}_n(\mathbb{Z}) = \sum\limits_{k=0}^{n} \mathbb{A}_k\mathbb{Z}^k$, $\mathbb{Z} \in \mathbb{C}_{\text{cont}}$;

5) $\frac{1}{\mathbb{Z}-\mathbb{Z}_0}$, $\mathbb{Z} - \mathbb{Z}_0 \in \mathbb{C}_{\text{cont}} \setminus \Theta$;

6) $\exp\mathbb{Z} = \{e^{z_\alpha}\} = \mathbf{1} + \mathbb{Z} + \frac{1}{2}\mathbb{Z}^2 + \ldots + \frac{1}{k!}\mathbb{Z}^k + \ldots$, $\mathbb{Z} \in \mathbb{C}_{\text{cont}}$, $\alpha \in [0,1]$;

7) $(1-\mathbb{Z})^{\frac{1}{2}} = \mathbf{1} - \frac{1}{2}\mathbb{Z} + \frac{1}{8}\mathbb{Z}^2 - \ldots + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\ldots\left(\frac{1}{2}-k+1\right)}{k!}\mathbb{Z}^k - \ldots, \quad \mathbb{Z} \in \mathbb{U}^\infty = \{\mathbb{Z} : \|z\| < 1\}$.

10. Polycylindrical Riemann theorem

As is known, a domain $B \subset \overline{\mathbb{C}}$ is of hyperbolic type if its boundary is a connected set that contains more than one point.

Let $0 < \delta < 1$ and $\mathbb{A} = \{a_\alpha\} \in \overline{\mathbb{C}}_{\text{cont}}$. Then $\mathbb{B} = \mathbb{B}(\delta) = \mathbb{B}_\delta(\mathbb{A}) = \prod\limits_\alpha B_\alpha \subset \overline{\mathbb{C}}_{\text{cont}}$, $\mathbb{A} \in \mathbb{B}_\delta(\mathbb{A})$, where each domain B_α is of hyperbolic type, $\delta < r(B_\alpha, a_\alpha) < \frac{1}{\delta}$, $\alpha \in [0,1]$. For an arbitrary $0 < \delta < 1$, the domain $\mathbb{B}(\delta) = \mathbb{B}_\delta(\mathbb{A})$ is called a finite with respect to $\mathbb{A}$ polycylindrical domain of hyperbolic type.

Riemann theorem. *Let $\mathbb{A} \in \overline{\mathbb{C}}_{\text{cont}}$ and $0 < \delta < 1$. Then an arbitrary finite with respect to $\mathbb{A}$ polycylindrical domain $\mathbb{B} = \mathbb{B}_\delta(\mathbb{A}) \subset \overline{\mathbb{C}}_{\text{cont}}$ of hyperbolic type is biholomorphically equivalent to the unit polycircle $\mathbb{U}^{\text{cont}} = \{\mathbb{W} \in \mathbb{C}_{\text{cont}} : \|\mathbb{W}\| < 1\}$.*

Proof. Let $\mathbb{B} = \mathbb{B}(\delta) = \prod\limits_\alpha B_\alpha$ be a domain indicated in the Riemann theorem, $\mathbb{A} = \{a_\alpha\} \in \mathbb{B}$, $a_\alpha \in B_\alpha$, $\alpha \in [0,1]$, and $w_\alpha = f_\alpha(z_\alpha)$ is a function that is holomorphic in B_α and univalently and conformally maps the domain B_α, $\alpha \in [0,1]$, into the unit circle $|w_\alpha| < 1$ so that $f(a_\alpha) = 0$, $f'(a_\alpha) > 0$.

Then the biholomorphic mapping $\mathbb{F}_\mathbb{B}(\mathbb{Z}) = \{f_\alpha(z_\alpha)\}$, $\mathbb{F}'_\mathbb{B}(\mathbb{Z}) = \{f'_\alpha\}$, $\alpha \in [0,1]$, satisfies the normalizing conditions

$$\mathbb{F}_\mathbb{B}(\mathbb{A}) = \mathbb{O}, \quad \mathbb{F}'_\mathbb{B}(\mathbb{A}) = \{f'_m(a_\alpha)\} > \mathbb{O},$$

and is the only such mapping into the unit polycircle. Then the mapping inverse to the mapping $\mathbb{F}_\mathbb{B}(\mathbb{A})$ is a partially conformal mapping of the unit polycircle. The theorem is proved. □

Thus, in the algebra $\mathbb{C}_{\text{cont}}$, the norm is defined by the equality $\|\mathbb{Z}\| := |\mathbb{Z}|$. The vector metrics in $\mathbb{C}_{\text{cont}}$: $\rho(\mathbb{Z}_1, \mathbb{Z}_2) = \|\mathbb{Z}_1 - \mathbb{Z}_2\|$. We will call the so-defined vector norm and metrics polycylindrical. The convergence by the polycylindrical norm uniformly over the numbers is given by the relationship $\mathbb{Z}_p \underset{p\to\infty}{\longrightarrow} \mathbb{Z}_0 \Longleftrightarrow \|\mathbb{Z}_p - \mathbb{Z}_0\| \underset{p\to\infty}{\longrightarrow} \mathbb{O} = (0,0,\ldots 0,\ldots) \Longleftrightarrow |z_p^{(\alpha)} - z_0^{(\alpha)}| \underset{p\to\infty}{\rightrightarrows} 0$, $\forall\alpha \in [0,1]$, where the symbol "$\rightrightarrows$" denotes the uniform convergence over $\alpha \in [0,1]$.

REFERENCES

1. A. K. Bakhtin, "Generalization of some results of the theory of univalent functions onto multidimensional complex spaces," *Dop. NANU,* **3**, 7–11 (2011).

2. A. K. Bakhtin, *Analytic functions of vector argument and partially-conformal mappings in multidimensional complex spaces*, 8th International ISAAC Congress: Abstracts, 1–9 (2011).

3. A. K. Bakhtin, “Analytic functions of vector argument and partially conformal mappings in multidimensional complex spaces,” *Dop. NANU,* **2**, 13–18 (2012).

4. A. K. Bakhtin, “Inequalities for the inner radii of nonoverlapping domains and open sets,” *Ukr. Math. J.,* **61**(5), 716–733 (2009).

5. A. K. Bakhtin and A. L. Targonskii, “Extremal problems and quadratic differential,” *Nonlin. Oscillations,* **8**(3), 296–301 (2005).

6. A. L. Targonskii, “Extremal problems of partially nonoverlapping domains on a Riemann sphere”, *Dop. NAN Ukr.,* **9**, 31–36 (2008).

7. A. Targonskii, “Extremal problems on the generalized (n; d)-equiangular system of points,” *An. St. Univ. Ovidius Constanta,* **22**(2), 239–251 (2014).

8. A. Targonskii, “Extremal problem (2n; 2m-1)-system points on the rays,” *An. St. Univ. Ovidius Constanta,* **24**(2), 283–299 (2016).

9. A. L. Targonskii, “Extremal problems for partially non-overlapping domains on equiangular systems of points,” *Bull. Soc. Sci. Lett. Lodz,* **63**(1), 57–63 (2013).

10. A. Targonskii and I. Targonskaya, “On the One Extremal Problem on the Riemann Sphere,” *International Journal of Advanced Research in Mathematics,* **4**, 1–7 (2016).

11. A. K. Bakhtin and A. L. Targonskii, “Generalized (n, d)-ray systems of points and inequalities for nonoverlapping domains and open sets,” *Ukr. Math. J.,* **63**(7), 999–1012 .(2011).

12. A. K. Bakhtin and A. L. Targonskii, “Some extremal problems in the theory of nonoverlapping domains with free poles on rays,” *Ukr. Math. J.,* **58**(12), 1950–1954 (2006).

13. V. N. Dubinin, “Asymptotic representation of the modulus of a degenerating condenser and some its applications,” *Zap. Nauchn. Sem. Peterburg. Otdel. Mat. Inst.,* **237**, 56–73 (1997).

14. V. N. Dubinin, *Capacities of condensers and symmetrization in geometric function theory of complex variables*, Dal’nauka, Vladivostok, 2009.

15. A. Targonskii, “An extremal problem for the nonoverlapping domains,” *Journal of Mathematical Sciences,* **227**(1), 98–104 (2017); transl. from *Ukr. Mat. Bull.,* **14**(1), 126–134 (2017).

16. A. Targonskii and I. Targonskaya, “Extreme problem for partially nonoverlapping domains on a Riemann sphere,” *Journal of Mathematical Sciences,* **235**(1), 74–80 (2018); transl. from *Ukr. Mat. Bull.,* **15**(1), 94–102 (2018).

17. A. L. Targonskii, “About one extremal problem for projections of the points on unit circle,” *Ukrainian Mathematical Bulletin,* **15**(3), 418–430 (2018).

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