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Extreme problem for a mosaic system of points on the open sets and non-overlapping domains

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Dedicated to the memory of Professor Oleksandr Bakhtin

Abstract. In the geometric theory of functions of a complex variable, the well-known direction is related to the estimates of the products of the inner radii of pairwise non-overlapping domains. This direction is called extreme problems in classes of pairwise non-overlapping domains. One of the problems of this type is considered in the present work.

Анотація. У геометричній теорії функцій комплексної змінної добре відомий напрям пов'язаний з оцінками добутків внутрішніх радіусів взаємно неперетинних областей. Цей напрям отримав назву екстремальних задач на класах попарно неперетинних областей. Одна з задач такого типу і розглянута у цій роботі.

1. Introduction

This article belongs to the theory of extreme problems on classes of pairwise non-overlapping domains, which is a separate trend in the geometric theory of functions of a complex variable. The start of this direction is related with the work by M. Lavrent'ev [12]. He found the maximum of a functional including the product of the conformal radii of two non-overlapping domains relatively to fixed points of the complex plane. In 1947, G. Goluzin solved a similar problem for three fixed points of the complex plane [10]. After that, this field of research began to develop rapidly. In this connection, we mention the results of many authors, in particular, Yu. Alenitsyn, M. Lebedev, J. Jenkins, P. Tamrazov, P. Kufarev,

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Ключові слова: функція комплексної змінної, внутрішній радіус, попарно неперетинні області

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G. Kuz'min, and others. In 1974, P. Tamrazov advanced the idea of the consideration of extreme problems, where the poles of quadratic differentials have a certain freedom (see [13]). In the frame of this idea, G. Bakhtina formulated a number of problems with the so-called "free poles" on the unit circle (see, e.g., [6]).

The works of V. Dubinin was an important step on this way. He proposed several methods, including the method of piecewise separating transformation, which allowed him to solve a number of extreme problems for any multiconnected domains (see, e.g., [2, 7, 8]). Now, these results are used even in studies of the holomorphic dynamics.

In the last decade, the method of "controlling functionals" was developed and used in the solution of many extreme problems for the so-called "ray systems of points" (see, e.g., [1–5, 14–18]).

Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the plane of complex numbers, $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be its one-point compactification, and $\mathbb{R}^+ = (0,\infty)$.

Let us also fix some numbers n, m, $d \in \mathbb{N}$.

The system of points $A_{n,m} = \{a_{k,p} \in \mathbb{C} : k = \overline{1,n},\ p = \overline{1,m}\}$ is called an (n,m)*-ray system of points*, if for all $k = \overline{1,n}$, the following relations hold:

$$0 < |a_{k,1}| < \ldots < |a_{k,m}| < \infty;$$
$$\arg a_{k,1} = \arg a_{k,2} = \ldots = \arg a_{k,m} =: \theta_k;$$
$$0 = \theta_1 < \theta_2 < \ldots < \theta_n < \theta_{n+1} := 2\pi.$$

For such systems of points consider the following quantities:

$$\alpha_k = \frac{1}{\pi}\left(\theta_{k+1} - \theta_k\right), \quad k = \overline{1,n},$$

$$\alpha_{n+1} := \alpha_1, \qquad \alpha_{n+1} := \alpha_1, \qquad \sum_{k=1}^{n} \alpha_k = 2.$$

For $m = 1$ and $k = \overline{1,n}$ we get an n-ray system of points (see [1–5, 14–18]). If the conditions $\alpha_k = \frac{2}{n}$, $k = \overline{1,n}$ are satisfied, then the system of points $A_{n,m}$ is called *equiangular*.

For any (n,m)-equiangular ray system of points $A_{n,m} = \{a_{k,p}\}$, we define the following "controlling" functional

$$M\left(A_{n,m}\right) = \prod_{k=1}^{n}\prod_{p=1}^{m} \chi\left(|a_{k,p}|^{\frac{n}{2}}\right) \cdot |a_{k,p}|,$$

where $\chi(t) = \frac{1}{2}\cdot\left(t + t^{-1}\right)$.

Define also the following system of angular domains:

$$P_k\left(A_{n,m}\right) = \Big\{w \in \mathbb{C} : \frac{2\pi}{n}(k-1) < \arg w < \frac{2\pi}{n}k\Big\}, \qquad k = \overline{1,n}.$$

For a fixed number $R \in \mathbb{R}^+$, consider the unique branch of the multi-branch analytic function

$$z_k(w) = -i\left(\frac{e^{-i\theta_k}w}{R}\right)^{\frac{n}{2}}. \tag{1.1}$$

For each $k = \overline{1,n}$ it realizes a one-sheet conformal mapping of the domain P_k onto the right half-plane $\mathrm{Re}z > 0$.

Let $\{b_k\}_{k=1}^n \subset \mathbb{C}$ be the following set of points:

$$b_k = R\cdot e^{i\frac{\pi}{n}(2k-1)}, \qquad k = \overline{1,n}.$$

It easily follows from relations (1.1) that

$$z_k(b_k) = 1, \qquad z_k(a_{k+1,p}) = is_{m-p+1}, \qquad z_k(a_{k,p}) = is_{m+p},$$

$$s_j > 0,\ j = \overline{1,m}, \qquad s_j < 0,\ j = \overline{m+1,\,2m},$$

$$s_1 > s_2 > \ldots > s_{2m}, \qquad a_{n+1,p} := a_{1,p}, \qquad k = \overline{1,n},\ p = \overline{1,m}.$$

For each $k = \overline{1,n}$ and for a collection of points $\{is_j\}_{j=1}^{2m}$ on the imaginary axis satisfying the inequalities

$$s_j > 0,\ j = \overline{1,m}, \qquad s_j < 0,\ j = \overline{m+1,2m}, \qquad s_1 > s_2 > \ldots > s_m,$$

we now consider the set of circles Γ_j^k such that the points $-1, 1, is_j \in \Gamma_j$.

For each $k = \overline{1,n}$ denote

$$\Omega_j^{(k)} := \begin{cases} \{z : z \in \Gamma_j^k,\ 0 \leqslant \arg z \leqslant \pi\}, & j = \overline{1,m}, \\ \{z : z \in \Gamma_j^k,\ -\pi \leqslant \arg z \leqslant 0\}, & j = \overline{m+1,2m}. \end{cases}$$

Let also $\{L_j^{(k)}\}_{j=1}^{2m}$, $k = \overline{1,n}$, be a collection of curves such that

$$L_j^{(k)} \subset \overline{P_k}, \qquad b_k \in L_j^{(k)},\ j = \overline{1,2m},$$

$$a_{k,p} \in L_{m-p+1}^{(k-1)}, \qquad a_{k,p} \in L_{m+p}^{(k)},\ p = \overline{1,m},$$

$$z_k\colon L_j^{(k)} \to \{z : z \in \Omega_j^{(k)},\ 0 \leqslant |\arg z| \leqslant \tfrac{\pi}{2}\},\ j = \overline{1,2m}.$$

Consider the following mapping

$$\zeta(z) = \frac{z-1}{z+1}. \tag{1.2}$$

For each $k = \overline{1,n}$ this mapping is one-sheet and conformal and maps the domains $\Omega_j^{(k)}$ onto the system of rays

$$\{\zeta : \arg\zeta = \beta_j\}, \quad j = \overline{1,\,2m},$$
$$0 \leqslant \beta_1 < \beta_2 < \ldots < \beta_{2m},$$

respectively.

Denote it by $\mathbb{A}_{2m,\,2d+1}$, where

$$\nu_{j,t} \in \mathbb{A}_{2m,\,2d+1}, \qquad j = \overline{1,\, 2m}, \qquad t = \overline{1,\, 2d+1},$$

so that

$$\begin{aligned} \zeta\big(z_k(a_{k,p})\big) &= \nu_{m+p,d+1}, \\ \zeta\big(z_{k-1}(a_{k,p})\big) &= \nu_{p,d+1}, \\ z_0 &:= z_n, \\ p &= \overline{1,m}. \end{aligned}$$

Rotating (if necessary) the obtained ray system by some angle one can assume that $\beta_1 = 0$. On each ray of the obtained ray system, we choose $2d+1$ points and get the $(2m,\, 2d+1)$-ray system of points.

For each $k = \overline{1,n}$ consider the systems of preimages of the composition $\zeta \circ z_k$ of mappings (1.1) and (1.2) and denote the corresponding systems of points by

$$D^{(k)}_{2m,d} = \left\{ c^{(k)}_{j,s} \in L^{(k)}_j \, : \, j = \overline{1,2m},\ s = \overline{1,d} \right\}.$$

The system of points

$$AD_{n,m,d} = \bigcup_{k=1}^{n} D^{(k)}_{2m,d} \cup A_{n,m}$$

is called *mosaic*.

For any mosaic system of points $AD_{n,m,d}$, consider the following "controlling" functional

$$\mu\left(AD_{n,m,d}\right) := \prod_{k=1}^{n} \left(\prod_{p=1}^{m} \left| a_{k,p} \right| \cdot \prod_{j=1}^{2m} \prod_{s=1}^{d} \left| c^{(k)}_{j,s} \right| \right)^{1-\frac{n}{2}}.$$

For the images of any mosaic system $AD_{n,m,d}$ under mapping (1.1), consider the system of points $\{\omega^*_{j,t}\}_{j=1,\,t=1}^{2m,\,2d+1}$, that are the second-order poles of the following quadratic differential.

$$\begin{aligned} Q(z)dz^2 = &-\frac{(z-1)^{2m-2}}{(z+1)^{6m+2}} \times \\ &\times \frac{\left((z+1)^{2m} + (z-1)^{2m}\right)^{4d}}{\left(\left((z+1)^m - i(z-1)^m\right)^{4d-2} + \left((z+1)^m + i(z-1)^m\right)^{4d-2}\right)^2} dz^2. \end{aligned} \tag{1.3}$$

We note that the main results of the theory of quadratic differentials can be found in [11].

In this case, we introduce a $(2m,\,2d+1)$-ray system of points $\mathbb{A}_{2m,\,2d+1}$, whose points are the poles of the quadratic differential

$$Q(\zeta)d\zeta^2 = -\frac{\zeta^{2m-2}\cdot\left(1+\zeta^{2m}\right)^{4d}}{\left(\left(1-i\zeta^m\right)^{4d+2}+\left(1+i\zeta^m\right)^{4d+2}\right)^2}\cdot d\zeta^2. \tag{1.4}$$

Let $\left\{B_{j,s}^{(k)}\right\}$ be any collection of pairwise non-overlapping domains such that

$$c_{j,s}^{(k)} \in B_{j,s}^{(k)}, \qquad\qquad B_{j,s}^{(k)} \subset P_k, \tag{1.5}$$

where $k=\overline{1,n}$, $j=\overline{1,2m}$, $s=\overline{1,d}$.

Let $D\subset\overline{\mathbb{C}}$ be any open set and $a\in D$ a point. By $D(a)$ we denote a connected component of D containing the point a. For an arbitrary system of points

$$A_{n,m}=\{a_{k,p}\in\mathbb{C}: k=\overline{1,n},\ p=\overline{1,m}\}\subset D$$

denote by $D_k\left(a_{s,p}\right)$ the connected component of the set $D\left(a_{s,p}\right)\cap\overline{P_k}$ containing the point $a_{s,p}$, for $k=\overline{1,n}$, $s=k,k+1$ and $a_{n+1,\,p}:=a_{1,p}$.

We say that the open set D, $A_{n,m}\subset D$ satisfies the condition of disjointness relatively to the system of points $A_{n,m}$, if the following relation holds on all angles $\overline{P_k}$.

$$\begin{gathered}[D_k(a_{k,t})\cap D_k(a_{k,u})]\cup[D_k(a_{k+1,t})\cap D_k(a_{k+1,u})]\cup\\ \cup[D_k(a_{k,t})\cap D_k(a_{k+1,p})]\cup\\ \cup\left[B_{j,s}^{(k)}\cap D_k(a_{k,p})\right]\cup\left[B_{j,s}^{(k)}\cap D_k(a_{k+1,p})\right]=\varnothing,\end{gathered} \tag{1.6}$$

$k=\overline{1,n}$, $p,\,t,\,u=\overline{1,m}$, $t\neq u$, $j=\overline{1,2m}$, $s=\overline{1,d}$ and $a_{n+1,p}:=a_{1,p}$.

The object of studies in the present paper is the following problem.

Problem 1.1. Let $n,\,m,\,d\in\mathbb{N}$, $R\in\mathbb{R}^+$, $\iota,\,\kappa\geqslant 0$, $n\geqslant 2$. Determine the maximum of the quantity

$$\prod_{k=1}^{n}\left(\prod_{p=1}^{m} r^{\iota}\left(B_{k,p},a_{k,p}\right)\cdot\prod_{j=1}^{2m}\prod_{s=1}^{d} r^{\kappa}\left(B_{j,s}^{(k)},c_{j,s}^{(k)}\right)\right)$$

for any mosaic system of points $AD_{n,m,d}$, where $\left\{B_{j,s}^{(k)}\right\}$ is any collection of pairwise non-overlapping domains that satisfies (1.5), and D is an open set satisfing (1.6).

A similar problem for non-overlapping domains was solved in [18].

It is clear that for $\kappa=0$ these problems are generalizations of the corresponding problems considered in [1,4].

2. Auxiliary result

Lemma 2.1. *Let $k, m,\ q \in \mathbb{N}$,*

$$\Omega_j^k = \begin{cases} \left\{ z:\ z \in \Gamma_j^k,\ 0 \leqslant \arg z \leqslant \pi \right\}, & j = \overline{1, m}, \\ \left\{ z:\ z \in \Gamma_j^k,\ -\pi \leqslant \arg z \leqslant 0 \right\}, & j = \overline{m+1,\ 2m}. \end{cases}$$

Then for any system of points $\{\omega_{j,t}\}_{j=1,t=1}^{2m,q}$ such that

$$\begin{aligned} &\omega_{j,t} \in \Omega_j^k, && j = \overline{1,\ 2m}, \\ &0 < |\arg \omega_{j,1}| < |\arg \omega_{j,2}| < |\arg \omega_{j,q}| < \frac{\pi}{2}, && t = \overline{1, q}, \end{aligned} \tag{2.1}$$

and any collection of pairwise non-overlapping domains $\{G_{j,t}\}_{j=1,t=1}^{2m,q}$ for $\omega_{j,t} \in G_{j,t} \subset \overline{\mathbb{C}}$ the following identity holds:

$$\prod_{j=1}^{2m} \prod_{t=1}^{q} r\big(G_{j,t}, \omega_{j,t}\big) = \frac{1}{2^{2mq}} \cdot \prod_{j=1}^{2m} \prod_{t=1}^{q} \big(|\omega_{j,t} + 1|^2 \cdot r\left(\Lambda_{j,t}, \nu_{j,t}\right)\big),$$

where

$$\begin{aligned} &\nu_{j,t} := \zeta\left(\omega_{j,t}\right), && j = \overline{1,\ 2m}, \\ &\zeta\colon G_{j,t} \to \Lambda_{j,t}, && t = \overline{1, q}, \end{aligned}$$

and the mapping $\zeta(z)$ is given by the relation (1.2).

Proof. The function (1.2) realizes an automorphism of the complex plane being one-sheet and conformal and maps the system of points $\{\omega_{j,t}\}_{j=1,\,t=1}^{2m,q}$ satisfying (2.1) onto the $(2m, q)$-ray system of points $A_{2m,q} = \{\nu_{j,t}\}_{j=1,\,t=1}^{2m,q}$.

It is clear that $\zeta\left(1\right) = 0$ and $\zeta\left(-1\right) = \infty$. Therefore

$$r\left(G_{j,t}, \omega_{j,t}\right) = \frac{|\omega_{j,t} + 1|^2}{2} \cdot r\left(\Lambda_{j,t}, \nu_{j,t}\right), \qquad j = \overline{1,\ 2m}, \qquad t = \overline{1, q},$$

whence we get the following relations

$$\prod_{j=1}^{2m} \prod_{t=1}^{q} r\left(G_{j,t}, \omega_{j,t}\right) = \frac{1}{2^{2mq}} \cdot \prod_{j=1}^{2m} \prod_{t=1}^{q} \big(|\omega_{j,t} + 1|^2 \cdot r\left(\Lambda_{j,t}, \nu_{j,t}\right)\big). \qquad \square$$

3. Main result

Theorem 3.1. *Let $n,\ m,\ d \in \mathbb{N}$, $R \in \mathbb{R}^+$, $n \geqslant 2$. Then for any mosaic system of points $AD_{n,m,d}$, any collection of pairwise non-overlapping domains $\left\{B_{j,s}^{(k)}\right\}$ satisfying condition* (1.5), *and any open set D satisfying condition* (1.6), *the following inequality holds:*

$$\prod_{k=1}^{n}\Big(\prod_{p=1}^{m} r\left(B_{k,p}, a_{k,p}\right)\cdot\prod_{j=1}^{2m}\prod_{s=1}^{d} r\big(B_{j,s}^{(k)}, c_{j,s}^{(k)}\big)\Big) \leqslant \Big(\frac{2\sqrt{R^{n}}}{mn\,(2d+1)}\Big)^{mn(1+2d)}\times$$
$$\times\,\mu\left(AD_{n,m,d}\right)\cdot\Big(\prod_{j=1}^{2m}\prod_{t=1}^{2d+1}\left|\omega_{j,t}^{*}+1\right|\Big)^{n}\cdot M^{\frac{n}{2}}\left(\mathbb{A}_{2m,2d+1}\right),$$

where the points $\{\omega_{j,t}^{*}\}_{j=1,\,t=1}^{2m,2d+1}$ *are the second-order poles of the quadratic differential* (1.3)*, and the points of the* $(2m,\,2d+1)$*-ray system* $\mathbb{A}_{2m,\,2d+1}$ *are the poles of the quadratic differential* (1.4)*.*

The equality is achieved if the points of the mosaic system $AD_{n,m,d}$ *and the domains of the system of pairwise non-overlapping domains* $\big\{B_{k,p},\,B_{j,s}^{(k)}\big\}$*,* $D=\bigcup_{k=1}^{n}\bigcup_{p=1}^{m}B_{k,p}$ *are, respectively, the second-order poles and the circular domains of the quadratic differential*

$$\begin{aligned}Q(w)dw^2 &= \frac{w^{n-2}}{(\Upsilon_1(w))^{6m+2}}\\ &\times\frac{\Upsilon_2^{2m-2}(w)\cdot\big(\Upsilon_1^{2m}(w)+\Upsilon_2^{2m}(w)\big)^{4d}}{\Big(\big(\Upsilon_1^{m}(w)-i\Upsilon_2^{m}(w)\big)^{4d+2}+\big(\Upsilon_1^{m}(w)+i\Upsilon_2^{m}(w)\big)^{4d+2}\Big)^{2}}dw^2,\end{aligned}\tag{3.1}$$

where $\Upsilon_1(w)=\frac{-iw^{\frac{n}{2}}}{R^{\frac{n}{2}}}+1,\ \Upsilon_2(w)=\frac{-iw^{\frac{n}{2}}}{R^{\frac{n}{2}}}-1.$

Proof. Firstly, the condition of disjointness implies that $\operatorname{cap}\overline{\mathbb{C}}\backslash D>0$, the set D has the generalized Green function

$$g_D(z,a)=\begin{cases} g_{D(a)}(z,a), & z\in D(a),\\ 0, & z\in\overline{\mathbb{C}}\backslash\overline{D(a)},\\ \lim\limits_{\zeta\to z} g_{D(a)}(\zeta,a), & \zeta\in D(a),\ z\in\partial D(a),\end{cases}$$

relatively to the point $a\in D$, and $g_{D(a)}(z,a)$ is the Green function of the domain $D(a)$ relatively to the point $a\in D(a)$.

Further, we will use methods from works [1, 2, 8]. Let

$$E_0=\overline{\mathbb{C}}\backslash D,\qquad E(a_{k,p},t)=\left\{w\in\mathbb{C}:\ |w-a_{k,p}|\leqslant t\right\},$$

$k=\overline{1,n}$, $p=\overline{1,m}$, $n,\ m\geqslant 2$, $t\in\mathbb{R}_{+}$. For sufficiently small $t>0$ consider a condenser

$$C\left(t,\,D,\,A_{n,m}\right)=\left\{E_0,\,E_1\right\},$$

where

$$E_1=\bigcup_{k=1}^{n}\bigcup_{p=1}^{m}E(a_{k,p},t).$$

The capacity of the condenser $C\,(t,\ D,\ A_{n,m})$ is introduced in [2, 8] as the quantity

$$\operatorname{cap} C\,(t,\ D,\ A_{n,m}) = \inf \iint \left[(G'_x)^2 + (G'_y)^2\right]\ dxdy,$$

where the infimum is taken over all continuous functions $G = G(z)$ having the Lipschitz property in $\overline{\mathbb{C}}$ and $G|_{E_0} = 0$, $G|_{E_1} = 1$.

As the condenser modulus we consider the quantity

$$|C| = [\operatorname{cap} C]^{-1}\,.$$

From [9, Theorem 1] we get

$$|C\,(t, D, A_{n,m})\,| = \frac{1}{2\pi}\cdot\frac{1}{mn}\cdot\log\frac{1}{t} + M(D, A_{n,m}) + o(1), \quad t \to 0, \qquad (3.2)$$

where

$$M(D, A_{n,m}) = \\ = \frac{1}{2\pi}\cdot\frac{1}{m^2n^2}\cdot\Big[\sum_{k=1}^{n}\sum_{p=1}^{m}\log r(D, a_{k,p}) + \sum_{\substack{k\neq q,\\ p\neq h}} g_D(a_{k,p}, a_{q,h})\Big]. \qquad (3.3)$$

Let $z_0 := z_n$ and

$$\begin{aligned} \omega^{(k)}_{m+p,\,d+1} &:= \ z_k\,(a_{k,p})\,,\ k = \overline{1,n}, \\ \omega^{(k-1)}_{m-p+1,\,d+1} &:= \ z_{k-1}\,(a_{k,p})\,,\ p = \overline{1,m}, \\ \omega^{(k)}_{j,s} &:= \ z_k\big(c^{(k)}_{j,s}\big),\ j = \overline{1,2m}, \\ \omega^{(0)}_{m-p+1,\,d+1} &:= \ z_n\,(a_{n,p})\,,\ s = \overline{1,d}. \end{aligned}$$

For an arbitrary domain $\Delta \in \mathbb{C}$ we put

$$(\Delta)^* := \left\{w \in \overline{\mathbb{C}} :\ \overline{w} \in \Delta\right\}.$$

Let also $\Omega^{(k)}_{m+p,d+1}$ denotes the connected component of the set

$$z_k\left(D \cap \overline{P}_k\right) \cup \left(z_k\left(D \cap \overline{P}_k\right)\right)^*$$

containing the point $\omega^{(k)}_{m+p,\,d+1}$, and let $\Omega^{(k-1)}_{m-p+1,d+1}$ be the connected component of the set

$$z_{k-1}\left(D \cap \overline{P}_{k-1}\right) \cup \left(z_{k-1}\left(D \cap \overline{P}_{k-1}\right)\right)^*$$

containing the point $\omega^{(k-1)}_{m-p+1,\,d+1}$ with $k = \overline{1,n}$, $p = \overline{1,m}$, $\overline{P}_0 := \overline{P}_n$.

The pair of domains $\Omega^{(k)}_{m+p,\,d+1}$ and $\Omega^{(k-1)}_{m-p+1,\,d+1}$ is a result of the piecewise separating transformation of the open set D relatively to the family of angles $\{P_{k-1}, P_k\}$, $\{z_{k-1}, z_k\}$ at the point $a_{k,p}$, $k = \overline{1,n}$, $p = \overline{1,m}$.

Simlarly, for each $k = \overline{1,n}$, $j = \overline{1,\,2m}$, $s = \overline{1,d}$ denote by $\Omega^{(k)}_{j,s}$ and $\Omega^{(k)}_{j,\,2d-s+2}$ the result of the piecewise separating transformation $B^{(k)}_{j,s}$ relatively to the families P_k, z_k at the point $c^{(k)}_{j,s}$, which contains, respectively, the points $\omega^{(k)}_{j,s}$, $\omega^{(k)}_{j,\,2d-s+2}$. It is clear that $\Omega^{(k)}_{j,t}$ are, in general, multiconnected domains, $k = \overline{1,n}$, $j = \overline{1,\,2m}$, $t = \overline{1,\,2d+1}$.

Now (1.1) implies the following asymptotic expressions:

$$\begin{gathered}\Big|z_k(t) - z_t(a_{k,p})\Big| \sim \frac{1}{R^{\frac{n}{2}}} \cdot \frac{n}{2} \cdot \Big|a_{k,p}\Big|^{\frac{n}{2}-1} \cdot |w - a_{k,p}|,\\ w \to a_{k,p}, \quad w \in \overline{P}_t, \quad k = \overline{1,n}, \quad p = \overline{1,m}, \quad t = k-1, k.\end{gathered} \tag{3.4}$$

Consider the condensers

$$C_k\,(t,\,D,\,A_{n,m}) = \big(E^{(k)}_0,\,E^{(k)}_1\big),$$

where

$$E^{(k)}_s = \zeta_k\,\big(E_s \cap \overline{P}_k\big) \cup \big[\zeta_k\,\big(E_s \cap \overline{P}_k\big)\big]^*\,,$$

$k = \overline{1,n}$, $s = 0,1$, and $\{P_k\}^n_{k=1}$ is the system of angles corresponding to the system of points $A_{n,m}$. It follows that the condenser $C\,(t,\,D,\,A_{n,m})$ corresponds to a set of condensers $\big\{C_k\,(t,\,D,\,A_{n,m})\big\}^n_{k=1}$ under a piecewise separating transformation relatively to the angles $\{P_k\}^n_{k=1}$ and the functions $\{z_k\}^n_{k=1}$. Then, due to the results from [2, 9], we obtain that

$$\operatorname{cap} C\,(t, D, A_{n,m}) \geqslant \frac{1}{2}\sum_{k=1}^{n} \operatorname{cap} C_k\,(t, D, A_{n,m})\,. \tag{3.5}$$

Hence

$$|C\,(t, D, A_{n,m})\,| \leqslant 2\Big(\sum_{k=1}^{n} |C_k\,(t, D, A_{n,m})\,|^{-1}\Big)^{-1}. \tag{3.6}$$

Formula (3.2), as $t \to 0$, gives the asymptotics of the absolute values of the condenser $C\,(t,\,D,\,A_{n,m})$ and $M\,(D, A_{n,m})$ is called the *modulus* of the set D relatively to the system of points $A_{n,m}$.

Using formulae (3.4) and the fact that the set D satisfies the condition of disjointness relatively to the system of points $A_{n,m}$, we get a similar asymptotic representations $C_k\,(t, D, A_{n,m})$, $k = \overline{1,n}$, for condensers:

$$|C_k\,(t, D, A_{n,m})\,| = \frac{1}{4\pi m}\log\frac{1}{t} + M_k\,(D, A_{n,m}) + o(1), \qquad t \to 0, \tag{3.7}$$

where

$$M_k\,(D, A_{n,m}) = \frac{1}{8\pi m^2} \cdot \left[\sum_{p=1}^{m} \log \frac{r\left(\Omega^{(k)}_{m+p,d+1}, \omega^{(k)}_{m+p,d+1}\right)}{\frac{1}{R^{\frac{n}{2}}} \cdot \frac{n}{2} \cdot \Big|a_{k,p}\Big|^{\frac{n}{2}-1}} + \right.$$

$$+\sum_{p=1}^{m}\log\frac{r\left(\Omega_{m-p+1,d+1}^{(k-1)},\omega_{m-p+1,d+1}^{(k-1)}\right)}{\frac{1}{R^{\frac{n}{2}}}\cdot\frac{n}{2}\cdot\left|a_{k,p}\right|^{\frac{n}{2}-1}}\Bigg].$$

By means of (3.7), we obtain

$$\begin{aligned}|C_k(t,D,A_{n,m})|^{-1}&=\frac{4\pi m}{\log\frac{1}{t}}\cdot\left(1+\frac{4\pi m}{\log\frac{1}{t}}M_k(D,A_{n,m})+o\Big(\frac{1}{\log\frac{1}{t}}\Big)\right)^{-1}\\&=\frac{4\pi m}{\log\frac{1}{t}}-\Big(\frac{4\pi m}{\log\frac{1}{t}}\Big)^2M_k(D,A_{n,m})+o\left(\Big(\frac{1}{\log\frac{1}{t}}\Big)^2\right),\quad t\to0.\end{aligned}\tag{3.8}$$

Further, relation (3.8) implies that

$$\begin{aligned}&\sum_{k=1}^{n}|C_k(t,D,A_{n,m})|^{-1}=\\&=\frac{4\pi nm}{\log\frac{1}{t}}-\Big(\frac{4\pi m}{\log\frac{1}{t}}\Big)^2\cdot\sum_{k=1}^{n}M_k(D,A_{n,m})+o\left(\Big(\frac{1}{\log\frac{1}{t}}\Big)^2\right),\quad t\to0.\end{aligned}\tag{3.9}$$

In turn, relation (3.9) leads to the following asymptotic representation:

$$\begin{aligned}&\left(\sum_{k=1}^{n}|C_k(t,D,A_{n,m})|^{-1}\right)^{-1}=\\&=\frac{\log\frac{1}{t}}{4\pi nm}\cdot\left(1-\frac{4\pi m}{n\log\frac{1}{t}}\cdot\sum_{k=1}^{n}M_k(D,A_n)+o\Big(\frac{1}{\log\frac{1}{t}}\Big)\right)^{-1}\\&=\frac{\log\frac{1}{t}}{4\pi nm}+\frac{1}{n^2}\cdot\sum_{k=1}^{n}M_k(D,A_{n,m})+o(1),\quad t\to0.\end{aligned}\tag{3.10}$$

Inequalities (3.5) and (3.6) and relations (3.2) and (3.10) imply that

$$\begin{aligned}&\frac{1}{2\pi}\cdot\frac{1}{mn}\cdot\log\frac{1}{t}+M(D,A_{n,m})+o(1)\leqslant\\&\leqslant\frac{\log\frac{1}{t}}{2\pi nm}+\frac{2}{n^2}\cdot\sum_{k=1}^{n}M_k(D,A_{n,m})+o(1).\end{aligned}\tag{3.11}$$

It follows from relation (3.11) that

$$M(D,A_{n,m})\leqslant\frac{2}{n^2}\cdot\sum_{k=1}^{n}M_k(D,A_{n,m})\ \text{ as } t\to0.\tag{3.12}$$

Then formulae (3.3), (3.7) and (3.12) lead to the relation

$$\frac{1}{2\pi}\cdot\frac{1}{m^2n^2}\cdot\Big[\sum_{k=1}^{n}\sum_{p=1}^{m}\log r(D,a_{k,p})+\sum_{(k,p)\neq(q,h)}g_D(a_{k,p},a_{q,h})\Big]\leqslant$$
$$\leqslant\frac{1}{4\pi m^2n^2}\cdot\left[\sum_{k=1}^{n}\sum_{p=1}^{m}\log\frac{r\left(\Omega^{(k)}_{m+p,d+1},\omega^{(k)}_{m+p,d+1}\right)}{\frac{1}{R^{\frac{n}{2}}}\cdot\frac{n}{2}\cdot\left|a_{k,p}\right|^{\frac{n}{2}-1}}+\right.$$
$$\left.+\sum_{k=1}^{n}\sum_{p=1}^{m}\log\frac{r\left(\Omega^{(k-1)}_{m-p+1,d+1},\omega^{(k-1)}_{m-p+1,d+1}\right)}{\frac{1}{R^{\frac{n}{2}}}\cdot\frac{n}{2}\cdot\left|a_{k,p}\right|^{\frac{n}{2}-1}}\right],$$

and we finally get

$$\prod_{k=1}^{n}\prod_{p=1}^{m}r\left(D,a_{k,p}\right)\leqslant$$
$$\leqslant\left(\frac{2}{n}\right)^{mn}\cdot R^{\frac{mn^2}{2}}\cdot\prod_{k=1}^{n}\prod_{p=1}^{m}\left|a_{k,p}\right|^{1-\frac{n}{2}}\times \tag{3.13}$$
$$\times\prod_{k=1}^{n}\prod_{p=1}^{m}\left(r\big(\Omega^{(k)}_{m+p,d+1},\omega^{(k)}_{m+p,d+1}\big)\cdot r\big(\Omega^{(k-1)}_{m-p+1,d+1},\omega^{(k-1)}_{m-p+1,d+1}\big)\right)^{\frac{1}{2}}.$$

As a result of the separating transformation of the domains $B^{(k)}_{j,s}$ relatively to the families P_k, z_k at the point $c^{(k)}_{j,s}$, we get the relations

$$r\Big(B^{(k)}_{j,s},c^{(k)}_{j,s}\Big)=$$
$$=\frac{2\cdot R^{\frac{n}{2}}}{n\left|c^{(k)}_{j,s}\right|^{\frac{n}{2}-1}}\cdot\left[r\left(\Omega^{(k)}_{j,s},\omega^{(k)}_{j,s}\right)\cdot r\Big(\Omega^{(k)}_{j,\,2d-s+2},\omega^{(k)}_{j,\,2d-s+2}\Big)\right]^{\frac{1}{2}}, \tag{3.14}$$

where $k=\overline{1,n}$, $j=\overline{1,\,2m}$, $s=\overline{1,d}$.

Then relations (3.13) and (3.14) yield

$$\prod_{k=1}^{n}\Big(\prod_{p=1}^{m}r\left(D,a_{k,p}\right)\cdot\prod_{j=1}^{2m}\prod_{s=1}^{d}r\left(B^{(k)}_{j,s},c^{(k)}_{j,s}\right)\Big)\leqslant$$
$$\leqslant\mu(AD_{n,m,d})\cdot\left(\frac{2}{n}\right)^{mn(1+2d)}\cdot R^{\frac{mn^2}{2}(1+2d)}\cdot\prod_{k=1}^{n}\Big(\prod_{j=1}^{2m}\prod_{t=1}^{2d+1}r\big(\Omega^{(k)}_{j,t},\omega^{(k)}_{j,t}\big)\Big)^{\frac{1}{2}}.$$

Using Lemma 2.1 and the previous inequality, we get

$$\prod_{k=1}^{n}\left(\prod_{p=1}^{m}r\left(D,a_{k,p}\right)\cdot\prod_{j=1}^{2m}\prod_{s=1}^{d}r\left(B^{(k)}_{j,s},c^{(k)}_{j,s}\right)\right)\leqslant\mu\left(AD_{n,m,d}\right)\cdot\left(\frac{2}{n}\right)^{mn(1+2d)}\times$$

$$\times R^{\frac{mn^2}{2}(1+2d)} \cdot \prod_{k=1}^{n} \left(\frac{1}{2^{2m(2d+1)}} \cdot \prod_{j=1}^{2m} \prod_{t=1}^{2d+1} \left(\left| \omega_{j,t}^{(k)} + 1 \right|^2 \cdot r\left(\Lambda_{j,t}^{(k)}, \nu_{j,t}^{(k)} \right) \right) \right)^{\frac{1}{2}} =$$

$$= \left(\frac{1}{n} \right)^{mn(1+2d)} \cdot R^{\frac{mn^2}{2}(1+2d)} \cdot \mu\left(AD_{n,m,d} \right) \cdot \prod_{k=1}^{n} \left(\prod_{j=1}^{2m} \prod_{t=1}^{2d+1} \left| \omega_{j,t}^{(k)} + 1 \right| \right) \times$$

$$\times \prod_{k=1}^{n} \left(\prod_{j=1}^{2m} \prod_{t=1}^{2d+1} r\left(\Lambda_{j,t}^{(k)}, \nu_{j,t}^{(k)} \right) \right)^{\frac{1}{2}}, \tag{3.15}$$

where at each $k = \overline{1, n}$

$$\begin{aligned} &\nu_{j,t}^{(k)} := \zeta\big(\omega_{j,t}^{(k)}\big), && j = \overline{1,\, 2m}, \\ &\zeta \colon G_{j,t}^{(k)} \to \Lambda_{j,t}^{(k)}, && t = \overline{1,\, 2d+1}. \end{aligned}$$

For each $k = \overline{1, n}$, the system of points $\left\{ \nu_{j,t}^{(k)} \right\}_{j=1,\, t=1}^{2m,\, 2d+1}$ is a $(2m,\ 2d{+}1)$-ray system of points.

Now by [2, Corollary 3.1.5] for each $k = \overline{1, n}$ we have that

$$\prod_{j=1}^{2m} \prod_{t=1}^{2d+1} r\left(\Lambda_{j,t}^{(k)},\, \nu_{j,t}^{(k)} \right) \leqslant \left(\frac{2}{m\,(2d+1)} \right)^{2m(2d+1)} \cdot M\left(\mathbb{A}_{2m,2d+1} \right). \tag{3.16}$$

The equality is achieved if the points $\big\{ \nu_{j,t}^{(k)} \big\}_{j=1,\, t=1}^{2m,\, 2d+1}$ and the domains $\big\{ \Lambda_{j,t}^{(k)} \big\}_{j=1,\, t=1}^{2m,\, 2d+1}$ are, respectively, the poles and circular domains of the quadratic differential (1.4).

From relation (3.15) with regard to (3.16), we obtain the relations

$$\prod_{k=1}^{n} \Big(\prod_{p=1}^{m} r\left(B_{k,p}, a_{k,p} \right) \cdot \prod_{j=1}^{2m} \prod_{s=1}^{d} r\big(B_{j,s}^{(k)}, c_{j,s}^{(k)} \big) \Big) \leqslant \Big(\frac{2\sqrt{R^n}}{mn\,(2d+1)} \Big)^{mn(1+2d)} \times$$

$$\times\ \mu\left(AD_{n,m,d} \right) \cdot \prod_{k=1}^{n} \Big(\prod_{j=1}^{2m} \prod_{t=1}^{2d+1} \left| \omega_{j,t}^{(k)} + 1 \right| \Big) \cdot \big(M\left(\mathbb{A}_{2m,2d+1} \right) \big)^{\frac{n}{2}}.$$

The quadratic differential (1.3) can be obtained from the differential (1.4) with the help of substitution (1.2). The quadratic differential (3.1) follows from differential (1.3) with the help of the substitution

$$z(w) = \frac{-iw^{\frac{n}{2}}}{R^{\frac{n}{2}}}. \qquad \square$$

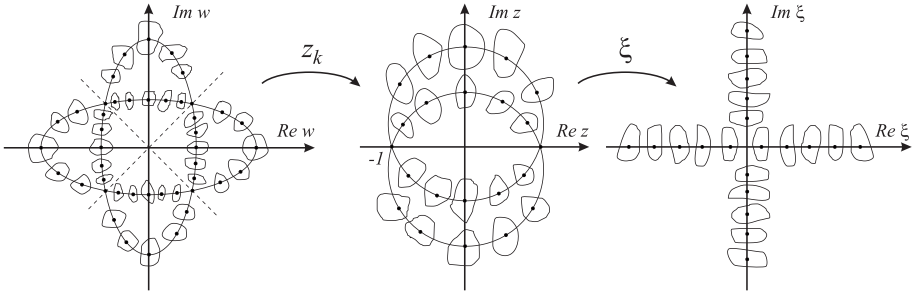

FIGURE 3.1. The figures show the transformations used during the proof of the theorem

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