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On the inverse K_I-inequality for one class of mappings

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Abstract. We study mappings differentiable almost everywhere, possessing the N-Luzin property, the N^{-1}-property on the spheres with respect to the $(n-1)$-dimensional Hausdorff measure and such that the image of the set where its Jacobian equals to zero has a zero Lebesgue measure. It is proved that such mappings satisfy the lower bound for the Poletsky-type distortion in their definition domain.

1. Introduction

The present article is devoted to the establishment of a distortion inequality for the modulus of families of paths under mappings. As is well known, such inequalities are often assumed to be the definition of quasiconformal mappings and some more general classes, and they may also be useful in describing various classes of mappings, cf. [1], [2], [3], [4], [5], [6], [7]–[8], [9] and [10]. In particular, the definition of K-quasiconformal mappings may be given by the relation

$$\frac{M(\Gamma)}{K} \leqslant M(f(\Gamma)) \leqslant K \cdot M(\Gamma)\,,$$

where $M(\Gamma)$ denotes the conformal modulus of a family Γ and f is assumed to be a homeomorphism (see [9, Definition 13.1]). Moreover, a part of a definition of K-quasiregular mappings contains the inequality

$$M(\Gamma) \leqslant N(f,D)K_O(f)M(f(\Gamma))\,,$$

where $1 \leqslant K_O(f) < \infty$ is some number, and $N(f,D)$ denotes the multiplicity function (see below), see [11, Theorem 3.2].

In this occasion, we also note our recent publication [12], where an inequality of the form

$$M_\alpha(\Gamma_f(y_0,r_1,r_2)) \leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} N^\alpha(f,D)\cdot K_{O,p}^{\frac{n-1}{p-n+1}}(y,f^{-1})(y)\cdot \eta^\alpha(|y-y_0|)\,dm(y) \tag{1}$$

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was established, where M_α denotes the α-modulus of families of paths, $\alpha > 1$, $p = \frac{\alpha(n-1)}{\alpha-1}$, $K_{O,p}(y, f^{-1}) = \sum\limits_{x\in f^{-1}(y)} K_{I,p}(x, f)$,

$$K_{I,p}(x,f) = \begin{cases} \frac{|J(x,f)|}{l(f'(x))^p}, & J(x,f) \neq 0, \\ 1, & f'(x) = 0, \\ \infty, & \text{otherwise} \end{cases}, \tag{2}$$

$$K_{O,p}(x,f) = \begin{cases} \frac{\|f'(x)\|^p}{|J(x,f)|}, & J(x,f) \neq 0, \\ 1, & f'(x) = 0, \\ \infty, & \text{otherwise} \end{cases},$$

respectively, besides that,

$$l(f'(x)) = \min_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f'(x)h|}{|h|}, \quad \|f'(x)\| = \max_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f'(x)h|}{|h|} \tag{3}$$

are the *operator minimum* and *operator maximum (norm)* of the derivative $f'(x)$, $J(x,f) = \det f'(x)$ is a *Jacobian of the mapping f at the point x*,

$$N(f,D) = \sup_{y\in\mathbb{R}^n} N(y,f,D), \quad N(y,f,D) = \text{card}\,\{x \in D : f(x) = y\}$$

are *multiplicity functions* of the mapping f,

$$A = A(y_0, r_1, r_2) = \{y \in \mathbb{R}^n : r_1 < |y - y_0| < r_2\} \tag{4}$$

is a *spherical ring* centered at y_0 of the radii r_1 and r_2, $\Gamma_f(y_0, r_1, r_2)$ denotes the family of all paths γ in D such that $f(\gamma)$ join $S(y_0, r_1)$ and $S(y_0, r_2)$ in $A(y_0, r_1, r_2)$), $S(y_0, r) := \{y \in \mathbb{R}^n : |y - y_0| = r\}$, and $\eta : (r_1, r_2) \to [0, \infty]$ may be chosen as arbitrary Lebesgue measurable function in (1) such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\,. \tag{5}$$

This article is devoted to a significant improvement of this result, namely, under similar conditions, we will establish a similar inequality, but with a more appropriate function Q_*, see below. In addition, we will show this result without one rather cumbersome and difficult-to-verify condition

$$\overline{f^{-1}(S(y_0, r_1))} \cap \overline{f^{-1}(S(y_0, r_2))} = \varnothing\,, \tag{6}$$

present in [12]. It should be noted that the research methodology here is approximately the same as in [12], however, the main results of the article require significant efforts to establish them and cannot be obtained from [12] as direct consequences. With all this, the "uncomfortable condition" (6), as we shall see, is rather easily overcome.

Let us turn now to definitions. Let X and Y be two spaces with measures μ and μ', respectively. We say that a mapping $f : X \to Y$ has *N-property of Luzin*, if from the condition $\mu(E) = 0$ it follows that $\mu'(f(E)) = 0$. Similarly, we say that a mapping $f : X \to Y$ has *N'-Luzin property*, if from the condition $\mu'(E) = 0$ it follows that $\mu(f^{-1}(E)) = 0$. Let A be a set where f does not have a total differential, and let $y \notin f(A)$. If $N(f, D) < \infty$, then we set

$$Q(y) := K_{I,\alpha}(y, f^{-1}) = \sum_{x\in f^{-1}(y)} K_{O,\alpha}(x, f)\,. \tag{7}$$

Observe that, $N(f, D) < \infty$ for open, discrete and closed mappings of D, see [13, Lemma 3.3].

Given sets $E, F \subset \overline{\mathbb{R}^n}$ and a domain $D \subset \mathbb{R}^n$ we denote by $\Gamma(E, F, D)$ a family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in [a, b]$. Let $\Gamma_f(y_0, r_1, r_2)$ a family of all paths γ in D such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$. Let $Q_* : \mathbb{R}^n \to [0, \infty]$ be a Lebesgue measurable function, and $M_\alpha(\Gamma)$ denotes the α-modulus of a family Γ (see, e.g., [9, section 6]). We say that f *satisfies the inverse Poletsky inequality at a point $y_0 \in \overline{f(D)} \setminus \{\infty\}$ with respect to α-modulus* if there is $r_0 > 0$ such that, the relation

$$M_\alpha(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} Q_*(y) \cdot \eta^\alpha(|y - y_0|)\, dm(y) \tag{8}$$

holds for any $0 < r_1 < r_2 < r_0$ and any Lebesgue measurable function η with (5). The following statement holds.

Theorem 1.1. *Let $n - 1 < \alpha \leqslant n$, let $y_0 \in \overline{f(D)} \setminus \{\infty\}$, $r_0 = \sup\limits_{y\in f(D)} |y - y_0| > 0$, and let $f : D \to \mathbb{R}^n$ be an open, discrete and closed mapping that is differentiable almost everywhere and has N-Luzin property with respect to the Lebesgue measure in $\mathbb{R}^n$. Suppose that $\overline{D}$ is a compact set in $\mathbb{R}^n$, and, in addition,*

$$m(f(\{x \in D : J(x, f) = 0\})) = 0\,. \tag{9}$$

Suppose that f has N^{-1}-property on $S(y_0, r) \cap f(D)$ for almost all $r \in (\varepsilon, r_0)$ relative to the Hausdorff measure $\mathcal{H}^{n-1}$ on $S(y_0, r)$. If the function Q which is defined in (7), belongs to the class $L^1(f(D))$, then the mapping f satisfies the inverse Poletsky inequality with respect to α-modulus with $Q_*(y) := N^\alpha(f, D) \cdot Q(y)$, more precisely,

$$M_\alpha(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} N^\alpha(f, D) K_{I,\alpha}(y, f^{-1}) \cdot \eta^\alpha(|y - y_0|)\, dm(y)\,. \tag{10}$$

Corollary 1.2. *The assertion of Theorem 1.1 holds if instead of the condition (9) a stronger condition is required: $J(x, f) \neq 0$ almost everywhere.*

Remark 1.3. *Observe that, the inequality (10) implies the relation (1). Indeed, let us to show that, at the points of non-degenerate differentiability of f,*

$$K_{O,\alpha}(x, f) \leqslant K_{I,p}^{\frac{n-1}{p-n+1}}(x, f)\,, \quad \alpha = \frac{p}{p - n + 1}\,, \quad p > n - 1\,. \tag{11}$$

To prove the inequality (11), we use the relations

$$|J(x, f)| = \lambda_1(x) \cdots \lambda_n(x), \qquad \|f'(x)\| = \lambda_n(x)\,, \tag{12}$$

$$l(f'(x)) = \lambda_1(x)\,, \tag{13}$$

$$K_{O,\alpha}(x, f) = \frac{\lambda_n^\alpha(x)}{\lambda_1(x) \cdots \lambda_n(x)}\,, \qquad K_{I,p}(x, f) = \frac{\lambda_1(x) \cdots \lambda_n(x)}{\lambda_1^p(x)}\,, \tag{14}$$

where

$$\lambda_1(x), \ldots, \lambda_n(x), \quad \lambda_1(x) \leqslant \ldots \leqslant \lambda_n(x)$$

are the so-called main stretchings see ([14, Lemma 4.2.I]). Using the relations (12)–(14), the inequality (11) may be rewritten as

$$\frac{\lambda_n^{\frac{p}{p-n+1}}(x)}{\lambda_1(x) \cdots \lambda_n(x)} \leqslant \left(\frac{\lambda_1(x) \cdots \lambda_n(x)}{\lambda_1^p(x)}\right)^{\frac{n-1}{p-n+1}}\,,$$

or, equivalently,

$$\lambda_n(x) \leqslant \frac{\lambda_1(x) \cdots \lambda_n(x)}{\lambda_1^{n-1}(x)}$$

and

$$\lambda_n(x)\cdot\lambda_1^{n-1}(x)\leqslant\lambda_1(x)\cdots\lambda_n(x)\,.$$

But the latter is obvious, because $\lambda_1(x)\leqslant\ldots\leqslant\lambda_n(x)$.

The inequality (11) yields

$$K_{I,\alpha}(y,f^{-1})=\sum\limits_{x\in f^{-1}(y)}K_{O,\alpha}(x,f)\leqslant\sum\limits_{x\in f^{-1}(y)}K^{\frac{n-1}{p-n+1}}_{I,p}(x,f)\,. \tag{15}$$

Using the inequality $(1+t)^{\gamma}\geqslant 1+t^{\gamma}$, $t\geqslant 0$, $\gamma>0$, *which may be established, for example, using the differentiation apparatus, it may be shown that* $x^{\gamma}+y^{\gamma}\leqslant(x+y)^{\gamma}$ *for any* $x,y>0$. *Then, putting* $\gamma:=\frac{n-1}{p-n+1}$, *we obtain that*

$$\sum\limits_{x\in f^{-1}(y)}K^{\frac{n-1}{p-n+1}}_{I,p}(x,f)\leqslant\left(\sum\limits_{x\in f^{-1}(y)}K_{I,p}(x,f)\right)^{\frac{n-1}{p-n+1}}=K^{\frac{n-1}{p-n+1}}_{O,p}(y,f^{-1})\,. \tag{16}$$

Due to (15) and (16), we obtain that

$$K_{I,\alpha}(y,f^{-1})\leqslant K^{\frac{n-1}{p-n+1}}_{O,p}(y,f^{-1})\,,\quad \alpha=\frac{p}{p-n+1}\,.$$

Thus, if the relation (10), then the inequality (1) holds, as well.

2. Distortion of families of sets under mappings

In what follows, we will need basic facts about the relationship between families of paths and separating sets, see [15]. Let G be a bounded domain in $\mathbb{R}^n$, and C_0, C_1 are disjoint compact sets in $\overline{G}$. Put $R=G\setminus(C_0\cup C_1)$ and $R^*=R\cup C_0\cup C_1$. For a number $p>1$ we define a p *-capacity of the pair* C_0, C_1 *relative to the closure* G by the equality

$$C_p[G,C_0,C_1]=\inf\int\limits_R|\nabla u|^p\,dm(x),$$

where the infimum is taken for all functions u, continuous in R^*, $u\in ACL(R)$, such that $u=1$ on C_1 and $u=0$ on C_0. These functions are called *admissible* for $C_p[G,C_0,C_1]$. We say that a set $\sigma\subset\mathbb{R}^n$ *separates* C_0 and C_1 in R^*, if $\sigma\cap R$ is closed in R and there are disjoint sets A and B, open relative $R^*\setminus\sigma$, such that $R^*\setminus\sigma=A\cup B$, $C_0\subset A$ and $C_1\subset B$. Let Σ denotes the class of all sets that separate C_0 and C_1 in R^*. For the number $p'=p/(p-1)$ we define the quantity

$$\widetilde{M_{p'}}(\Sigma)=\inf\limits_{\rho\in\widetilde{\mathrm{adm}}\,\Sigma}\int\limits_{\mathbb{R}^n}\rho^{p'}dm(x) \tag{17}$$

where the notation $\rho\in\widetilde{\mathrm{adm}}\,\Sigma$ denotes that ρ is nonnegative Borel function in $\mathbb{R}^n$ such that

$$\int\limits_{\sigma\cap R}\rho\,d\mathcal{H}^{n-1}\geqslant 1\quad\forall\,\sigma\in\Sigma\,. \tag{18}$$

Note that,

$$\widetilde{M_{p'}}(\Sigma)=C_p[G,C_0,C_1]^{-1/(p-1)}\,, \tag{19}$$

see [15, Theorem 3.13] for $p=n$ and [16, p. 50] for $1<p<\infty$, in addition, by the Hesse result

$$M_p(\Gamma(E,F,D))=C_p[D,E,F]\,, \tag{20}$$

where $(E \cup F) \cap \partial D = \varnothing$ (see [17, Theorem 5.5]). Shlyk has proved that the requirement $(E \cup F) \cap \partial D = \varnothing$ can be omitted, in other words, the equality (20) holds for any disjoint non-empty sets $E, F \subset \overline{D}$ (see [18, Theorem 1]).

Let S be a surface, in other words, $S : D_s \to \mathbb{R}^n$ be a continuous mapping of an open set $D_s \subset \mathbb{R}^{n-1}$. We put $N(y, S) = \operatorname{card} S^{-1}(y) = \operatorname{card}\{x \in D_s : S(x) = y\}$ and recall this function the *multiplicity function* of the surface S with respect to a point $y \in \mathbb{R}^n$. Given a Borel set $B \subset \mathbb{R}^n$, its $(n-1)$-measured Hausdorff area associated with the surface S is determined by the formula $\mathcal{A}_S(B) = \mathcal{A}_S^{n-1}(B) = \int\limits_B N(y, S)\, d\mathcal{H}^{n-1}y$, see [19, item 3.2.1]. For a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ its integral over the surface S is determined by the formula $\int\limits_S \rho\, d\mathcal{A} = \int\limits_{\mathbb{R}^n} \rho(y) N(y, S)\, d\mathcal{H}^{n-1}y$. In what follows, $J_k f(x)$ denotes the *k-dimensional Jacobian* of the mapping f at a point x (see [19, § 3.2, Ch. 3]).

Let $n \geqslant 2$, and let Γ be a family of surfaces S. A Borel function $\rho\colon \mathbb{R}^n \to \overline{\mathbb{R}^+}$ is called *an admissible* for Γ, abbr. $\rho \in \operatorname{adm} \Gamma$, if

$$\int\limits_S \rho^{n-1}\, d\mathcal{A} \geqslant 1 \tag{21}$$

for any $S \in \Gamma$. Given $p \in (1, \infty)$, a *p-modulus* of Γ is called the quantity

$$M_p(\Gamma) = \inf_{\rho \in \operatorname{adm} \Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x)\, dm(x)\,.$$

We also set $M(\Gamma) := M_n(\Gamma)$. Let us say that some property P holds for *p-almost all surfaces* of the domain D, if this property holds for all surfaces in D, except, maybe be, some of their subfamily, p -modulus of which is zero. If we are talking about the conformal modulus $M(\Gamma) := M_n(\Gamma)$, the prefix "n" in the expression "n-almost all", as a rule, is omitted. We say that a Lebesgue measurable function $\rho\colon \mathbb{R}^n \to \overline{\mathbb{R}^+}$ is *p-extensively admissible* for the family Γ of surfaces S in $\mathbb{R}^n$, abbr. $\rho \in \operatorname{ext}_p \operatorname{adm} \Gamma$, if the relation (21) is satisfied for p-almost all surfaces S of the family Γ.

Below we give a statement about the distortion of the modulus of families of sets (surfaces) under the pre-image of a mapping, related to the "inner" dilatation $K_{I,\alpha}(y, f^{-1})$ (see (2)). Similar $K_{O,p}$-version is published in [12, Lemma 2.1], cf. [5, Theorem 5] and [20, Theorem 4].

Lemma 2.1. *Let $p > n-1$, $f : D \to \mathbb{R}^n$ be a sense-preserving mapping that is differentiable almost everywhere and has N-Luzin property with respect to the Lebesgue measure in $\mathbb{R}^n$, let $N(f, D) < \infty$ and let $y_0 \in \overline{f(D)} \setminus \{\infty\}$, $r_0 = \sup\limits_{y \in f(D)} |y - y_0|$, $0 < \varepsilon_0 < r_0$, $0 < \varepsilon < \varepsilon_0$. Suppose that the condition (9) is also satisfied. Fix $\varepsilon > 0$, and denote by Σ_ε the family of all sets of the form*

$$\{f^{-1}(S(y_0, r)) \cap f(D)\}, \quad r \in (\varepsilon, r_0)\,. \tag{22}$$

Suppose, in addition, that f has N^{-1}-property on $S(y_0, r) \cap f(D)$ for almost all $r \in (\varepsilon, r_0)$ relative to the Hausdorff measure $\mathcal{H}^{n-1}$ on $S(y_0, r)$. Then

$$\widetilde{M}_{\frac{p}{n-1}}(\Sigma_\varepsilon) \geqslant \frac{1}{N^{\frac{p}{n-1}}(f, D)} \inf_{\rho \in \operatorname{ext}_p \operatorname{adm} f(\Sigma_\varepsilon)} \int\limits_{f(D) \cap A(y_0, \varepsilon, r_0)} \frac{\rho^p(y)}{Q^{\frac{p-n+1}{n-1}}(y)}\, dm(y)\,, \tag{23}$$

where

$$Q(y) := K_{I,\alpha}(y, f^{-1}) = \sum_{x \in f^{-1}(y)} K_{O,\alpha}(x, f)\,, \tag{24}$$

and $\alpha = \frac{p}{p-n+1}$.

Proof. Without loss of generality, we may assume that $r_0 > 0$. We will generally follow the methodology set forth in proving [5, Theorem 5] (see also [6, Theorem 8.6]).

Denote by B a Borel set of all points $x \in D$, where the mapping f has a total differential $f'(x)$ and $J(x, f) \neq 0$. By Kirsbraun's theorem and by the unity of the approximate differential (see, for example, [19, 2.10.43 and Theorem 3.1.2]) it follows that the set B is a countable union of Borel sets B_k, $k = 1, 2, \dots$, such that the mappings $f_k = f|_{B_k}$ are Bilipschitz homeomorphisms (see [19, Lemma 3.2.2 and Theorems 3.1.4 and 3.1.8]). Without loss of generality, we may assume that the sets B_k are disjoint. We also denote by B_* the set of all points $x \in D$, where f has a total differential, but $J(x, f) = 0$.

Since the set $B_0 := D \setminus (B \cup B_*)$ has a Lebesgue measure zero, and the mapping f has N-Luzin property, then $m(f(B_0)) = 0$. By [6, Theorem 9.3] $\mathcal{A}_{S_r}(f(B_0)) = 0$ for p-almost all spheres $S_r := S(y_0, r) \cap f(D)$ centered at a point y_0, where "almost all" is understood in the sense of p-modulus of families of surfaces. Note that, the function $\psi(r) := \mathcal{H}^{n-1}(f(B_0) \cap S_r)$ is Lebesgue due to the Fubini theorem ([21, Section 8.1, Ch. III]). Thus, the set $E \subset \mathbb{R}$ of all $r \in \mathbb{R}$ such that $\mathcal{H}^{n-1}(f(B_0) \cap S_r) = 0$ is Lebesgue measurable. Then by [2, Lemma 4.1] $\mathcal{A}_{S_r}(f(B_0)) = 0$ for almost all spheres $S_r := S(y_0, r)$ centered at the point y_0, where "almost all" is understood in the sense of a one-dimensional Lebesgue measure with respect to the parameter $r \in (\varepsilon, r_0)$. Now, by the assumption of Lemma,

$$\mathcal{H}^{n-1}(f^{-1}(S_r) \cap B_0) = 0 \tag{25}$$

for almost all $r \in (\varepsilon, \varepsilon_0)$. Arguing similarly, we obtain that

$$\mathcal{H}^{n-1}(f^{-1}(S_r) \cap B_*) = 0 \tag{26}$$

for almost all $r \in (\varepsilon, \varepsilon_0)$.

Let $\rho^{n-1} \in \widetilde{\mathrm{adm}}\, \Sigma_\varepsilon$ and let

$$\tilde{\rho}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y) \cap D \setminus B_0} \rho_*(x)\,, & y \in f(D) \setminus f(B \cap B_*) \\ 0\,, & y \in f(B \cap B_*) \end{cases}, \tag{27}$$

where

$$\rho_*(x) = \begin{cases} \rho(x) \cdot \left(\frac{\|f'(x)\|}{J(x,f)}\right)^{1/(n-1)}, & x \in D \setminus B_0, \\ 0, & \text{otherwise}. \end{cases} \tag{28}$$

Observe that $\tilde{\rho} = \sup \rho_k$, where

$$\rho_k(y) = \begin{cases} \rho_*(f_k^{-1}(y)), & y \in f(B_k), \\ 0, & \text{otherwise} \end{cases} \tag{29}$$

and, moreover, each mapping $f_k = f|_{B_k}$, $k = 1, 2, \dots$, is injective. Thus, a function $\tilde{\rho}$ is Borel (see, e.g., [21, Theorem I (8.5)]).

Let $f^{-1}(S_r) := S_r^*$. Then

$$\int\limits_{S_r \cap f(D)} \widetilde{\rho}^{n-1}(y)\, d\mathcal{A}_* = \int\limits_{\mathbb{R}^n} \widetilde{\rho}^{n-1}(y) \chi_{S_r \cap f(D)}(y)\ d\mathcal{H}^{n-1}y \geqslant$$

$$\geqslant \int\limits_{\mathbb{R}^n} \frac{1}{N(f,D)} \cdot \sum_{k=1}^{\infty} \widetilde{\rho}^{n-1}(y) \chi_{S_r \cap f(D)}(y) N(y, f, B_k \cap S_r^*)\, d\mathcal{H}^{n-1}y =$$

$$= \frac{1}{N(f,D)} \sum_{k=1}^{\infty} \int\limits_{\mathbb{R}^n} {\rho_*}^{n-1}(f_k^{-1}(y)) N(y, f, B_k \cap S_r^*)\, d\mathcal{H}^{n-1}y = \tag{30}$$

$$= \frac{1}{N(f,D)} \sum_{k=1}^{\infty} \int\limits_{f(B_k \cap S_r^*)} {\rho_*}^{n-1}(f_k^{-1}(y))\, d\mathcal{H}^{n-1} y\,.$$

Let $\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x)$, $\lambda_1(x) \leqslant \lambda_2(x) \leqslant \ldots \leqslant \lambda_n(x)$, $\|f'(x)\| = \lambda_n(x)$ are the main stretchings of the mapping f, see e.g. [14, Lemmas 4.1.I, 4.2.I]. Due to (12), $J(x, f) = \lambda_1(x) \cdots \lambda_n(x)$ and

$$J_{n-1} f(x) = \widetilde{\lambda}_1(x) \cdots \widetilde{\lambda}_n(x)\,, \tag{31}$$

where

$$\lambda_1(x) \leqslant \widetilde{\lambda}_1(x) \leqslant \lambda_2(x) \leqslant \widetilde{\lambda}_2(x) \leqslant \lambda_3(x) \leqslant \ldots \leqslant \lambda_{n-1}(x) \leqslant \widetilde{\lambda}_{n-1}(x) \leqslant \lambda_n(x)\,. \tag{32}$$

Due to (12), (31) and (32), we obtain that

$$\left(\frac{\|f'(x)\|}{J(x,f)}\right)^{1/(n-1)} = \left(\frac{1}{\lambda_1(x) \ldots \lambda_{n-1}(x)}\right)^{\frac{1}{n-1}} \geqslant \left(\frac{1}{J_{n-1} f(x)}\right)^{\frac{1}{n-1}}\,. \tag{33}$$

Due to (25), (26) and (33), by [19, Corollary 3.2.20] for $m = n-1$, we obtain that

$$\sum_{k=1}^{\infty} \int\limits_{f(B_k \cap S_r^*)} {\rho_*}^{n-1}(f_k^{-1}(y))\, d\mathcal{H}^{n-1} y = \sum_{k=1}^{\infty} \int\limits_{B_k \cap S_r^*} {\rho_*}^{n-1}(x)\ J_{n-1} f(x)\, d\mathcal{H}^{n-1} x =$$

$$= \sum_{k=1}^{\infty} \int\limits_{B_k \cap S_r^*} \frac{\rho^{n-1}(x) \|f'(x)\|}{J(x,f)}\, J_{n-1} f(x)\, d\mathcal{H}^{n-1} x \geqslant$$

$$\geqslant \sum_{k=1}^{\infty} \int\limits_{B_k \cap S_r^*} \rho^{n-1}(x)\ d\mathcal{H}^{n-1} x = \int\limits_{f^{-1}(S_r)} \rho^{n-1}(x)\, d\mathcal{H}^{n-1} x \geqslant 1 \tag{34}$$

for almost any $S_r = f \circ S_r^* \in f(\Sigma_\varepsilon)$. It follows from (30) and (34) that $N^{\frac{1}{n-1}}(f, D)\tilde{\rho} \in \rho \in \mathrm{ext}_p \mathrm{adm}\, f(\Sigma_\varepsilon)$ (see [2, Lemma 4.1]).

Recall that $Q(y) := K_{I,\alpha}(y, f^{-1}) = \sum\limits_{x \in f^{-1}(y)} K_{O,\alpha}(x, f)$. Since $\tilde{\rho}^p(y) = \sup\limits_{k \in \mathbb{N}} \rho_k^p(y) \leqslant \sum\limits_{k=1}^{\infty} \rho_k^p(y)$ and $m(f(B_*)) = m(f(B_0)) = 0$, then

$$\int\limits_{f(D)} \frac{\tilde{\rho}^p(y)}{Q^{\frac{p-n+1}{n-1}}(y)}\, dm(y) \leqslant \sum_{k=1}^{\infty} \int\limits_{f(B_k)} \frac{\rho_k^p(y)}{Q^{\frac{p-n+1}{n-1}}(y)}\, dm(y) \leqslant \sum_{k=1}^{\infty} \int\limits_{f(B_k)} \frac{\rho_k^p(y)}{K_{O,\alpha}^{\frac{p-n+1}{n-1}}(f_k^{-1}(y), f)}\, dm(y)\,. \tag{35}$$

Using the change of variables formula on each B_k, $k = 1, 2, \ldots$, see, for example, [19, Theorem 3.2.5], we obtain that

$$\int\limits_{f(B_k)} \frac{\rho_k^p(y)}{K_{O,\alpha}^{\frac{p-n+1}{n-1}}(f_k^{-1}(y), f)}\, dm(y) =$$

$$= \int\limits_{f(B_k)} \frac{\rho^p(f_k^{-1}(y)) J^{\frac{p-n+1}{n-1}}(f_k^{-1}(y), f)}{\|f'(f_k^{-1}(y))\|^{\frac{p}{p-n+1} \cdot \frac{p-n+1}{n-1}}} \cdot \frac{\|f'(f_k^{-1}(y))\|^{\frac{p}{n-1}}}{(J(f_k^{-1}(y), f))^{\frac{p}{n-1}}}\, dm(y) = \tag{36}$$

$$= \int\limits_{f(B_k)} \rho^p(f_k^{-1}(y)) J(y, f_k^{-1})\, dm(y) = \int\limits_{B_k} \rho^p(x)\, dm(x)\,.$$

The relations (35) and (36) imply that

$$\int\limits_{f(D)} \frac{\widetilde{\rho}^p(y)}{Q^{\frac{p-n+1}{n-1}}(y)}\,dm(y) \leqslant \sum\limits_{k=1}^{\infty}\int\limits_{B_k} \rho^p(x)\,dm(x)\,. \tag{37}$$

Summing (37) over $k=1,2,\ldots$ and using the countable additivity of the Lebesgue integral (see, for example, [21, Theorem I.12.3]), we obtain that

$$\int\limits_{f(D)} \frac{1}{Q^{\frac{p-n+1}{n-1}}(y)}\cdot \widetilde{\rho}^p(y)\,dm(y) \leqslant \int\limits_{D} \rho^p(x)\,dm(x)\,. \tag{38}$$

Going in the ratio (38) to inf over all functions $\rho^{n-1}\in \widetilde{\rm adm}\,\Sigma_{\varepsilon}$, we obtain that

$$\int\limits_{f(D)} \frac{\widetilde{\rho}^p(y)}{Q^{\frac{p-n+1}{n-1}}(y)}\,dm(y) \leqslant \widetilde{M_{\frac{p}{n-1}}}(\Sigma_{\varepsilon})\,,$$

whence we obtain that

$$\int\limits_{f(D)} \frac{N^{\frac{p}{n-1}}(f,D)}{Q^{\frac{p-n+1}{n-1}}(y)}\cdot \widetilde{\rho}^p(y)\,dm(y) \leqslant N^{\frac{p}{n-1}}(f,D)\cdot \widetilde{M_{\frac{p}{n-1}}}(\Sigma_{\varepsilon})\,.$$

Put $\widetilde{\rho_1}(y):=N^{\frac{1}{n-1}}(f,D)\cdot\widetilde{\rho}(y)$. Due to the latter relation, we obtain that

$$\int\limits_{f(D)} \frac{\widetilde{\rho}_1^p(y)}{Q^{\frac{p-n+1}{n-1}}(y)}\,dm(y) \leqslant N^{\frac{p}{n-1}}(f,D)\cdot \widetilde{M_{\frac{p}{n-1}}}(\Sigma_{\varepsilon})\,. \tag{39}$$

Since by the above $\widetilde{\rho_1}(y)=N^{\frac{1}{n-1}}(f,D)\tilde{\rho}\in {\rm ext}_p{\rm adm}\,f(\Sigma_{\varepsilon})$, it follows from (39) that the relation (23) holds. Lemma is proved. □

We have the following simple consequence.

Corollary 2.2. *Let $f:D\rightarrow\mathbb{R}^n$ be a sense-preserving map which is differentiable almost everywhere, and has N and N^{-1} Luzin properties with respect to the Lebesgue measure. Let $y_0\in\overline{f(D)}\setminus\{\infty\}$, $r_0=\sup\limits_{y\in f(D)}|y-y_0|$. We fix $\varepsilon>0$, and denote by Σ_{ε} the family of all sets of the form (22). In addition, suppose that f has N^{-1}-Luzin property on $S(y_0,r)\cap f(D)$ for almost all $r\in(\varepsilon,\varepsilon_0)$ with respect to $\mathcal{H}^{n-1}$ on $S(y_0,r)$. Then the relation (23) is fulfilled, where Q is defined by the relation (7).*

Proof. Since f has N^{-1}-Luzin property, by Ponomarev's theorem we have that $J(x,f)\ne 0$ almost everywhere (see, for example, [22, Theorem 1]), we may assume that $J(x,f)\ne 0$ on any B_k, $k=1,2,\ldots$. Then, since the mapping f has N-property, the condition (9) is also fulfilled. The desired statement, in this case, follows from Lemma 2.1. □

3. Proof of the main result

Let $Q_*:D\rightarrow[0,\infty]$ be a Lebesgue measurable function. Denote by $q_{x_0}(r)$ the integral average of $Q_*(x)$ under the sphere $|x-x_0|=r$,

$$q_{x_0}(r):=\frac{1}{\omega_{n-1}r^{n-1}}\int\limits_{|x-x_0|=r} Q_*(x)\,d\mathcal{H}^{n-1}\,, \tag{40}$$

where ω_{n-1} denotes the area of the unit sphere in $\mathbb{R}^n$. Below we also assume that the following standard relations hold: $a/\infty = 0$ for $a \neq \infty$, $a/0 = \infty$ for $a > 0$ and $0 \cdot \infty = 0$ (see, e.g., [21, § 3, section I]). The following conclusion was obtained by V. Ryazanov together with the second author in the case $p = n$, see, e.g., [6, Lemma 7.4]. In the case of an arbitrary $p > 1$, see, for example, [23, Lemma 2] and [12, Remark 3.1].

Proposition 3.1. *Let $p > 1$, $n \geqslant 2$, $x_0 \in \mathbb{R}^n$, $r_1, r_2 \in \mathbb{R}$, $r_1, r_2 > 0$, and let $Q_*(x)$ be a Lebesgue measurable function, $Q_* : \mathbb{R}^n \to [0, \infty]$, $Q_* \in L^1_{\rm loc}(\mathbb{R}^n)$. We put*

$$I = I(x_0, r_1, r_2) = \int\limits_{r_1}^{r_2} \frac{dr}{r^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)}\,,$$

and let $q_{x_0}(r)$ be defined by (40). Then

$$\frac{\omega_{n-1}}{I^{p-1}} \leqslant \int\limits_{A} Q_*(x) \cdot \eta^p(|x - x_0|)\, dm(x) \qquad (41)$$

for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\,, \qquad (42)$$

where $A = A(x_0, r_1, r_2)$ is defined in (4).

Proof of Theorem 1.1. Fix $y_0 \in \overline{f(D)} \setminus \{\infty\}$, $0 < r_1 < r_2 < r_0 = \sup\limits_{y \in f(D)} |y - y_0|$, $C_1 \subset \overline{B(y_0, r_1)} \cap f(D)$ and $C_2 \subset f(D) \setminus B(y_0, r_2)$. Set

$$C_0 := f^{-1}(C_1)\,, \quad C_0^* := f^{-1}(C_2)$$

(see Figure 1).

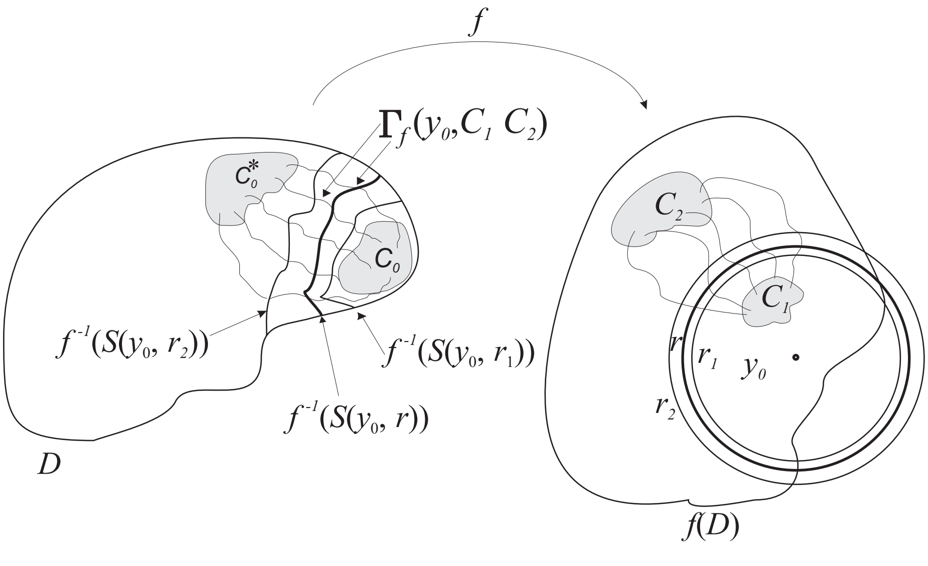

Figure 1: To the proof of Theorem 1.1

Given a mapping $f : D \to \mathbb{R}^n$, a point $y_0 \in \overline{f(D)} \setminus \{\infty\}$, and $0 < r_1 < r_2 < r_0 = \sup\limits_{y \in f(D)} |y - y_0|$, we denote by $\Gamma_f(y_0, C_1, C_2)$ a family of all paths γ in D such that $f(\gamma) \in \Gamma(C_1, C_2, A(y_0, r_1, r_2))$. Let us firstly prove that

$$M_\alpha(\Gamma_f(y_0, C_1, C_2)) \leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} Q_*(y) \cdot \eta^\alpha(|y - y_0|)\, dm(y) \tag{43}$$

for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that the relation (5) holds, where $Q_*(y) := N^\alpha(f, D) \cdot K_{I,\alpha}(y, f^{-1}) = N^\alpha(f, D) \cdot \sum\limits_{x \in f^{-1}(y)} K_{O,\alpha}(x, f)$.

Observe that C_0 and C_0^* are disjoint compact sets in D, see [10, Theorem 3.3]. Besides that, C_0 and C_0^* are non empty by the choose of r_0, r_1 and r_2.

Let us to show that a set $\sigma_r := f^{-1}(S(y_0, r))$ separates C_0 from C_0^* in D for any $r \in (r_1, r_2)$. Indeed, σ_r is closed in D as a preimage of a closed set $S(y_0, r)$ under the continuous mapping f (see, e.g., [24, Theorem 1.IV.13, Ch. 1]). In particular, σ_r is also closed with respect to $R := D \setminus (C_0 \cup C_0^*)$. We put

$$A := f^{-1}(B(y_0, r))$$

and

$$B := D \setminus \overline{f^{-1}(B(y_0, r))}\,.$$

Observe that, A and B are not empty by the choice of r_0, r_1, r_2 and r. Since f is continuous, $f^{-1}(B(y_0, r))$ and $D \setminus \overline{f^{-1}(B(y_0, r))}$ are open in D. In other words, A and B are open in

$$R^* := R \cup C_0 \cup C_0^* = D\,.$$

Note that $A \cap B = \varnothing$, and $R^* \setminus \sigma_r = A \cup B$. Let Σ_{C_0, C_0^*} be the family of all sets separating C_0 and C_0^* in R^*. In this case, by the equations of Ziemer and Hesse, see (19) and (20), respectively, we obtain that

$$M_\alpha(\Gamma(C_0, C_0^*, D)) = (\widetilde{M}_{p/(n-1)}(\Sigma_{C_0, C_0^*}))^{1-\alpha}\,, \tag{44}$$

where $\alpha = \frac{p}{p-n+1}$. Then by Lemma 2.1 and by the relation (44), we obtain that

$$M_\alpha(\Gamma_f(y_0, C_1, C_2)) \leqslant$$

$$\leqslant \left(\inf\limits_{\rho \in \mathrm{ext}_p \mathrm{adm}\, f(\Sigma_\varepsilon)} \int\limits_{f(D)\cap A(y_0,r_1,r_2)} \frac{\rho^p(y)}{N^{\frac{p}{n-1}}(f, D) \cdot Q^{\frac{p-n+1}{n-1}}(y)}\, dm(y)\right)^{-\frac{n-1}{p-n+1}}, \tag{45}$$

where Q is defined by (7). Using the second remote formula in the proof of Theorem 9.2 in [6], we obtain that

$$\inf\limits_{\rho \in \mathrm{ext}_p\, \mathrm{adm}\, f(\Sigma_\varepsilon)} \int\limits_{f(D)\cap A(y_0,r_1,r_2)} \frac{\rho^p(y)}{N^{\frac{p}{n-1}}(f, D) \cdot Q^{\frac{p-n+1}{n-1}}(y)}\, dm(y) =$$

$$= \int\limits_{r_1}^{r_2} \left(\inf\limits_{\alpha \in I(r)} \int\limits_{S(y_0,r)\cap f(D)} \frac{\alpha^q(y)}{N^{\frac{p}{n-1}}(f, D) \cdot Q^{\frac{p-n+1}{n-1}}(y)}\, \mathcal{H}^{n-1}(y)\right) dr\,, \tag{46}$$

where $q = \frac{p}{n-1}$, and $I(r)$ denotes the set of all measurable functions on $S(y_0, r) \cap f(D)$ such that

$$\int\limits_{S(y_0,r)\cap f(D)} \alpha(x)\, \mathcal{H}^{n-1} = 1\,.$$

Then, choosing in [6, Lemma 9.2] $X = S(y_0, r) \cap f(D)$, $\mu = \mathcal{H}^{n-1}$ and $\varphi = \frac{1}{Q}|_{S(y_0,r)\cap f(D)}$, we obtain that

$$\int\limits_{r_1}^{r_2}\left(\inf\limits_{\alpha\in I(r)}\int\limits_{S(y_0,r)\cap f(D)}\frac{\alpha^q(y)}{Q(y)}\,d\mathcal{H}^{n-1}\right)dr=\int\limits_{r_1}^{r_2}\frac{dr}{\|Q\|_s(r)}\,, \tag{47}$$

where $\|Q\|_s(r)=\left(\int\limits_{S(y_0,r)\cap f(D)}Q^s(x)\,d\mathcal{H}^{n-1}\right)^{1/s}$ and $s := \frac{n-1}{p-n+1}$. Thus, by (45), (46) and (47) we obtain that

$$M_\alpha(\Gamma_f(y_0, C_1, C_2)) \leqslant N^\alpha(f, D)\cdot\left(\int\limits_{r_1}^{r_2}\frac{dr}{\|Q\|_1(r)}\right)^{-\frac{n-1}{p-n+1}}=$$

$$=\frac{N^\alpha(f,D)\cdot\omega_{n-1}}{\left(\int\limits_{r_1}^{r_2}\frac{dr}{r^{\frac{n-1}{\alpha-1}}\widetilde{q}_{y_0}^{1/(\alpha-1)}(r)}\right)^{\frac{n-1}{p-n+1}}}=\frac{N^\alpha(f,D)\cdot\omega_{n-1}}{\left(\int\limits_{r_1}^{r_2}\frac{dr}{r^{\frac{n-1}{\alpha-1}}\widetilde{q}_{y_0}^{1/(\alpha-1)}(r)}\right)^{\alpha-1}}\,, \tag{48}$$

where $q_{y_0}(r)=\frac{1}{\omega_{n-1}r^{n-1}}\int\limits_{S(y_0,r)}\widetilde{Q}\,d\mathcal{H}^{n-1}$ and $\widetilde{Q}(y)=\begin{cases}Q(y)\,, & y\in f(D)\,,\\ 0\,, & y\notin f(D)\end{cases}$. Finally, it follows from (48) and Proposition 3.1 that the relation

$$M_{\frac{p}{p-n+1}}(\Gamma_f(y_0, C_1, C_2)) \leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} N^\alpha(f, D)\cdot Q(y)\cdot\eta^\alpha(|y-y_0|)\,dm(y)$$

holds for a function $Q(y) = K_{I,\alpha}(y, f^{-1}) := \sum\limits_{x\in f^{-1}(y)} K_{O,\alpha}(x, f)$. Thus, the relation (43) is proved.

Now, we take the increasing sequences of compacta C_1^m and C_2^m, $m = 1, 2, \ldots$, exhausting $S(y_0, r_1) \cap f(D)$ and $S(y_0, r_2) \cap f(D)$, respectively. By the proving above,

$$M_{\frac{p}{p-n+1}}(\Gamma_f(y_0, C_1^m, C_2^m)) \leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} N^\alpha(f, D)\cdot Q(y)\cdot\eta^\alpha(|y-y_0|)\,dm(y)\,.$$

Letting here to the limit as $m\to\infty$ and using the theorem on monotonicity of the modulus (see [6, Theorem A.7]) , we obtain that

$$M_{\frac{p}{p-n+1}}(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} N^\alpha(f, D)\cdot Q(y)\cdot\eta^\alpha(|y-y_0|)\,dm(y)\,,$$

that is desired conclusion. □

Proof of Corollary 1.2 immediately follows by Theorem 1.1 and additional arguments used under the proof of Corollary 2.2. □

Remark 3.2. *Observe that, the local and boundary behavior of mappings that satisfy condition (8) is described in sufficient detail in [25], which makes it possible to transfer these results to mappings participating in Theorem 1.1. Note also that the mappings with the inverse Poletsky inequality are part of the definition of quasiconformality in the case of a bounded function Q (see [9, Ch. 13.1]), and in the unbounded case were obtained by different authors under different conditions for Q (see, eg, [6, Theorem 8.5], [1, Lemma 3.1], [4] and [26, Theorem 1.3]). In particular, the statement below follows directly from Theorem 1.1 and [27, Theorem 4.1].*

For domains $D, D' \subset \mathbb{R}^n$, $n \geqslant 2$, a number $N \in \mathbb{N}$ and a Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty]$, $Q(y) \equiv 0$ for $y \in \mathbb{R}^n \setminus D'$, we denote by $\mathfrak{R}_{Q,N}(D, D')$ the family of all open discrete mappings $f : D \to D'$ which are differentiable almost everywhere, have N-Luzin property with respect to the Lebesgue measure in $\mathbb{R}^n$, satisfy relation (9) and have N^{-1}-property on $S(y_0, r) \cap D'$ for almost all $r \in (\varepsilon, r_0)$ relative to the Hausdorff measure $\mathcal{H}^{n-1}$ on $S(y_0, r)$ for any $y_0 \in D'$ and $r_0 = \sup\limits_{y \in D'} |y - y_0|$ such that

1) $N(f, D) \leqslant N$,

2) $K_{I,n}(y, f^{-1}) = \sum\limits_{x \in f^{-1}(y)} K_{O,n}(x, f) \leqslant Q(y)$ for any $y \in D'$.

If $Q \in L^1(D')$, D' is bounded and K is a compact set in D, then the inequality

$$|f(x) - f(y)| \leqslant \frac{C}{\log^{1/n}\left(1 + \frac{r_*}{2|x-y|}\right)} \tag{49}$$

holds for any $x, y \in K$ and all $f \in \mathfrak{R}, \mathfrak{R}_Q(D, D')$, where $C = C(n, N, K, \|Q\|_1, D, D') > 0$ is some constant depending only on n, N, K and $\|Q\|_1$, $\|Q\|_1$ denotes L^1-norm of Q in D', and $r_* = d(K, \partial D)$.

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