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On the Dirichlet problem for A-harmonic functions

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We study the Dirichlet boundary value problem with continuous boundary data for the A-harmonic equations $\mathrm{div}[A\,\mathrm{grad}\,u] = 0$ *in an arbitrary bounded domain D of the complex plane* $\mathbb{C}$ *with no boundary component degenerated to a single point. We provide integral criteria, including the BMO and FMO criteria expressed in terms of* $A(z)$*, for the existence of weak solutions to the problem. We also discuss the connections between A-harmonic functions and potential theory.*

Keywords: *A-harmonic equations, degenerate Beltrami equations, BMO, bounded mean oscillation, FMO, finite mean oscillation, Dirichlet problem, potential theory.*

Introduction. The existence theorems of normalized homeomorphic solutions for the degenerate Beltrami equation $f_{\bar{z}} = \mu(z) f_z$ in the whole complex plane $\mathbb{C}$ established in [1] have several basic consequences, including the solvability of the Dirichlet problem for this equation in simply connected domains, as shown in [2]. In this paper, we provide another example of its application to degenerate elliptic equations of the form

$$\mathrm{div}\,[A(z)\nabla u(z)] = 0\,, \tag{1}$$

which arise naturally in hydrodynamics, nonlinear elasticity, and other related fields.

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From now on we will assume that 2x2 matrix functions

$$A(z)=\begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix} \tag{2}$$

with measurable real-valued entries $a_{ij}(z)$ are symmetric, have $\det A(z)=1$ and satisfy the ellipticity condition $(1+a_{11}(z))(1+a_{22}(z))>a_{12}(z)a_{21}(z)$ almost everywhere. The set of all such matrix functions we will denoted by $\mathrm{M}^{2\times 2}$.

Let $\mu: D\to C$ be a measurable function with $|\mu(z)|<1$ a.e. in D. If D is simply connected, then by lengthy but elementary algebraic manipulation (see, for instance, Theorem 16.1.6 in [3]), it can be shown that if f is a $W^{1,1}_{\mathrm{loc}}$ solution to the Beltrami equation

$$f_{\bar z}=\mu(z)f_z \tag{3}$$

then both $u(z)=\operatorname{Re} f(z)$ and $v(z)=\operatorname{Im} f(z)$ satisfy the equation (1) with the matrix coefficient

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}:=\begin{bmatrix} \dfrac{|1-\mu|^2}{1-|\mu|^2} & \dfrac{-2\mathrm{I}m\mu}{1-|\mu|^2} \\ \dfrac{-2\mathrm{I}m\mu}{1-|\mu|^2} & \dfrac{|1+\mu|^2}{1-|\mu|^2} \end{bmatrix}. \tag{4}$$

The matrix identities (4) can be converted a.e. to express the coefficient $\mu(z)$ of the Beltrami equation (3) through the elements of the matrices $A(z)$:

$$\mu=\mu_A:=-\frac{a_{11}-a_{22}+i(a_{12}+a_{21})}{2+a_{11}+a_{22}}, \tag{5}$$

see e.g. the formula (16.20) in [3]. Vice versa, every matrix valued coefficient $A\in\mathrm{M}^{2x2}(D)$ in (2) generates by formula (5) the complex coefficient μ of the corresponding Beltrami equation (3).

A continuous function $u: D\to\mathbb{R}$ is called *the A-harmonic function*, see e.g. [4], if u satisfies (1) in the sense of distributions, i.e., if $u\in W^{1,1}_{\mathrm{loc}}(D)$ and

$$\int_D \langle A(z)\nabla u(z), \nabla\psi(z)\rangle\, dm(z)=0 \quad \forall\, \psi\in C_0^\infty(D), \tag{6}$$

where $C_0^\infty(D)$ denotes the collection of all infinitely differentiable functions $\psi: D\to\mathbb{R}$ with compact support in D, $\langle a,b\rangle$ means the scalar product of vectors a and b in $\mathbb{R}^2$, and $dm(z)$ stands for the Lebesgue measure in $\mathbb{C}$.

A continuous function $v: D\to\mathbb{R}$ is called the *A-harmonic conjugate of* u or sometimes a *stream function of the potential* u, if $v\in W^{1,1}_{\mathrm{loc}}(D)$ and

$$\nabla v(z)=\mathbb{H}A(z)\nabla u(z), \tag{7}$$

where $\mathbb{H}$ is the Hodge operator,

$$\mathbb{H}=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}:\mathbb{R}^2\to\mathbb{R}^2, \tag{8}$$

i.e., the counterclockwise rotation by the angle $\pi/2$ in $\mathbb{R}^2$.

The matrix $\mathbb{H}$ plays the role of an imaginary unit in the space of two-dimensional square matrices with real elements, because $\mathbb{H}^2 = -\mathrm{I}$. Thus, the relation (7) is equivalent to the equation

$$A(z)\nabla u(z) = -\mathbb{H}\nabla v(z). \tag{9}$$

As known, the curl of any gradient field is equal to zero in the sense of distributions and the Hodge operator $\mathbb{H}$ transforms curl-free fields into divergence-free fields, and vice versa, see e.g. 16.1.3 in [3]. Hence (9) itself implies (1).

Thus, the above considerations alow us to involve the theory of the Beltrami equations in the development of the theory of A-harmonic functions.

Recall that a Beltrami equation (3) is called *degenerate* if $\operatorname{ess\,sup} K_\mu(z) = \infty$, where

$$K_\mu(z) := \frac{1+|\mu(z)|}{1-|\mu(z)|}. \tag{10}$$

The case of degeneracy is particularly interesting from the viewpoint of applications since it allows for the study of equation (1) in strongly anisotropic and inhomogeneous media.

2. On multi-valued solutions for the Beltrami equations. In this section we present criteria for the existence of multi-valued solutions f of the Dirichlet problem to the Beltrami equations in the spirit of the theory of multi-valued analytic functions in arbitrary bounded domains D in $\mathbb{C}$ with no boundary component degenerated to a single point. These criteria are formulated both in terms of K_μ and the more refined quantity that takes into account not only the modulus of μ but also its argument

$$K_\mu^T(z, z_0) := \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1-|\mu(z)|^2} \tag{11}$$

that is called the *tangent dilatation quotient* of (3) with respect to the point $z_0 \in \mathbb{C}$. Note that

$$K_\mu^{-1}(z) \leqslant K_\mu^T(z, z_0) \leqslant K_\mu(z) \qquad \forall\, z \in D,\ z_0 \in \mathbb{C}. \tag{12}$$

Let $B(z, \varepsilon)$ be an open disk centered at a point z of radius ε. We say that a discrete open mapping $f : B(z_0, \varepsilon_0) \to \mathbb{C}$, where $B(z_0, \varepsilon_0) \subseteq D$, is a *local regular solution of the equation* (3) if $f \in W^{1,1}_{\mathrm{loc}}$, $J_f(z) \neq 0$ and f satisfies (3) a.e. in $B(z_0, \varepsilon_0)$. The local regular solutions $f_0 : B(z_0, \varepsilon_0) \to \mathbb{C}$ and $f_* : B(z_*, \varepsilon_*) \to \mathbb{C}$ of the equation (3) will be called extension of each to other if there is a finite chain of such solutions $f_i : B(z_i, \varepsilon_i) \to \mathbb{C}$, $i = 1, \ldots, m$, such that $f_1 = f_0$, $f_m = f_*$ and $f_i(z) \equiv f_{i+1}(z)$ for $z \in E_i := B(z_i, \varepsilon_i) \cap B(z_{i+1}, \varepsilon_{i+1}) \neq \varnothing$, $i = 1, \ldots, m-1$.

A collection of local regular solutions $f_j : B(z_j, \varepsilon_j) \to \mathbb{C}$, $j \in J$, will be called a *multi-valued solution* of the equation (3) in D if the disks $B(z_j, \varepsilon_j)$ cover the whole domain D and f_j are extensions of each to other through the collection, and the collection is maximal by inclusion.

A multi-valued solution of the equation (3) will be called a *multi-valued solution of the Dirichlet problem*

$$\lim_{z \to \zeta} \operatorname{Re} f(z) = \varphi(\zeta) \qquad \forall\, \zeta \in \partial D \tag{13}$$

for a prescribed continuous function $\varphi:\partial D\to\mathbb{R}$, if $u(z)=\mathrm{Re}\,f(z)=\mathrm{Re}\,f_j(z)$, $z\in B(z_j,\varepsilon_j)$, $j\in J$, is a *single-valued function* in D satisfying the condition $\lim\limits_{z\to\zeta}u(z)=\varphi(\zeta)$ for all ζ in ∂D.

From now on, we will assume that the functions $K_\mu^T(z,z_0)$ and $K_\mu(z)$ are extended by 1 outside of the domain D.

Lemma 1. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $\mu:D\to C$ be measurable, $|\mu(z)|<1$ a.e., $K_\mu\in L^1(D)$ and*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu^T(z,z_0)\cdot\psi^2_{z_0,\varepsilon}(|z-z_0|)\,dm(z)=o(I^2_{z_0}(\varepsilon)) \quad as \quad \varepsilon\to 0\ \ \forall\, z_0\in\overline{D} \tag{14}$$

for $\varepsilon_0=\varepsilon(z_0)>0$ and a family of measurable functions $\psi_{z_0,\varepsilon}:(0,\varepsilon_0)\to(0,\infty)$ with

$$I_{z_0}(\varepsilon):=\int\limits_{\varepsilon}^{\varepsilon_0}\psi_{z_0,\varepsilon}(t)\,dt<\infty \qquad \forall\varepsilon\in(0,\varepsilon_0). \tag{15}$$

Then the Beltrami equation (3) has a multi-valued solution f of the Dirichlet problem (13) in D for each continuous function $\varphi:\partial D\to\mathbb{R}$.

Moreover, such a solution f can be represented as the composition

$$f=h\circ g, \qquad g(z)=z+o(1) \quad as \quad z\to\infty, \tag{16}$$

where $g:\mathbb{C}\to\mathbb{C}$ is a regular homeomorphic solution of the Beltrami equation (3) in $\mathbb{C}$ with μ extended by zero outside of D and $h:D_*\to\mathbb{C}$, $D_*:=g(D)$, is a multi-valued analytic function with a single-valued harmonic function $\mathrm{Re}\,h$ satisfying the Dirichlet condition

$$\lim_{\xi\to\varsigma}\mathrm{Re}\,h(\xi)=\varphi_*(\zeta) \quad \forall\,\zeta\in\partial D_*, \quad where\ \varphi_*:=\varphi\circ g^{-1}. \tag{17}$$

Proof. Indeed, by Lemma 1 in [1], there is a regular homeomorphic solution with hydrodynamic normalization $g(z):=z+o(1)$ as $z\to\infty$ of the Beltrami equation (3) in $\mathbb{C}$ with μ extended by zero outside of D. It should be noted that $D_*=g(D)$ is also a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point due to homeomorphism $g:\mathbb{C}\to\mathbb{C}$. Therefore, based on Theorem 4.2.2 and Corollary 4.1.8 in [5], there is a unique harmonic function $u:D_*\to\mathbb{R}$ that satisfies the Dirichlet boundary condition

$$\lim_{\xi\to\varsigma}u(\xi):=\varphi_*(\zeta) \quad \forall\zeta\in\partial D_*, \quad \text{where } \varphi_*:=\varphi\circ g^{-1}. \tag{18}$$

Let $B_0=B(z_0,r_0)$ be a disk in the domain D. Then $D_0=g(B_0)$ is a simply connected subdomain of the domain $D_*=g(D)$, where there exists a conjugate harmonic function v determined up to an additive constant such that $h^*=u+iv$ is a single–valued analytic function. Let us denote through h_0 the holomorphic function corresponding to the choice of such a harmonic function v_0 in D_0 with normalization $v_0(g(z_0))=0$. Thus, we have determined the initial element of a multi-valued analytic function in D_0. The function h_0 can be extended along any path in D_* to, generally speaking, multi-valued analytic function h, because u is given in the whole domain D_*. Hence, $f=h\circ g$ is just a desired multi-valued function, that solves the Dirichlet problem (13) in D for the Beltrami equation (3).

3. The Dirichlet problem for A-harmonic functions. Taking into account the connection between the solutions of the A-harmonic equation (1) and the corresponding Beltrami equation (3), noted in the introduction, we arrive to the following result.

Lemma 2. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point and $A \in \mathrm{M}^{2\times 2}(D)$ with $K_{\mu_A} \in L^1(D)$. Suppose that*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_{\mu_A}(z, z_0)\cdot \psi^2_{z_0,\varepsilon}(|z-z_0|)\, dm(z) = o(I^2_{z_0}(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0 \;\; \forall\, z_0 \in \overline{D} \tag{19}$$

for some $\varepsilon_0 = \varepsilon(z_0) > 0$ and a family of measurable functions $\psi_{z_0,\varepsilon} : (0, \varepsilon_0) \to (0, \infty)$ with

$$I_{z_0}(\varepsilon) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\, dt < \infty \qquad \forall \varepsilon \in (0, \varepsilon_0). \tag{20}$$

Then there exist $A-$harmonic solutions u of the Dirichlet problem

$$\lim_{z\in\zeta} u(z) = \varphi(\zeta) \qquad \forall \zeta \in \partial D \tag{21}$$

for each continuous function $\varphi : \partial D \to \mathbb{R}$.

Moreover, such a solution u can be represented as the composition

$$u = H \circ g, \qquad g(z) = z + o(1) \quad as \quad z \to \infty, \tag{22}$$

where $g : \mathbb{C} \to \mathbb{C}$ is a regular homeomorphic solution of the Beltrami equation (7) in $\mathbb{C}$ with μ_A extended by zero outside of D and $H : D_* \to \mathbb{C}$, $D_* := g(D)$, is a unique harmonic function satisfying the Dirichlet condition

$$\lim_{\xi\to\varsigma} H(\xi) = \varphi_*(\zeta) \quad \forall\, \zeta \in \partial D_*, \quad \text{where } \varphi_* := \varphi \circ g^{-1}. \tag{23}$$

Choosing $\psi(t) = 1/(t\log(1/t))$ in Lemma 2, we obtain by Lemma 2 in [1] the following result in terms of FMO, finite mean oscillation.

Theorem 1. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point and $A \in \mathrm{M}^{2\times 2}(D)$ with $K_{\mu_A} \in L^1(D)$. Suppose that $K^T_{\mu_A}(z, z_0) \leqslant Q_{z_0}(z)$ a.e. in U_{z_0} for every point $z_0 \in \overline{D}$, a neighborhood U_{z_0} of z_0 and a function $Q_{z_0} : U_{z_0} \to [0, \infty]$ in the class $\mathrm{FMO}(z_0)$. Then there exist $A-$harmonic solutions of Dirichlet problem (21) in D with representation (22) for each continuous function $\varphi : \partial D \to \mathbb{R}$.*

By Corollary 2 in [1], we can derive the following consequence of Theorem 1.

Corollary 1. *Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point and $A \in \mathrm{M}^{2\times 2}(D)$ with $K_{\mu_A} \in L^1(D)$. If*

$$\overline{\lim_{\varepsilon\to 0}} \frac{1}{\pi\varepsilon^2} \int\limits_{B(z_0,\varepsilon)} K^T_{\mu_A}(z, z_0)\, dm(z) < \infty \qquad \forall\, z_0 \in \overline{D}, \tag{24}$$

then there exist A-harmonic solutions of Dirichlet problem (21) in D with representation (22) for each continuous function $\varphi : \partial D \to \mathbb{R}$.

By (12), we also obtain the following consequences of Theorem 1.

Corollary 2. *Let* D *be a bounded domain in* $\mathbb{C}$ *with no boundary component degenerated to a single point,* $A \in \mathrm{M}^{2\times 2}(D)$ *and* K_{μ_A} *have a dominant* $Q:\mathbb{C}\to[1,\infty)$ *in the class* $\mathrm{BMO}_{\mathrm{loc}}$. *Then there exist A-harmonic solutions of Dirichlet problem (21) in* D *with representation (22) for each continuous function* $\varphi:\partial D\to\mathbb{R}$.

Corollary 3. *Let* D *be a bounded domain in* $\mathbb{C}$ *with no boundary component degenerated to a single point,* $A \in \mathrm{M}^{2\times 2}(D)$ *and* $K_{\mu_A}(z) \leqslant Q(z)$ *a.e. in* D *with a function* Q *in the class* $\mathrm{FMO}(\overline{D})$. *Then there exist A-harmonic solutions of Dirichlet problem (21) in* D *with representation (22) for each continuous function* $\varphi:\partial D\to\mathbb{R}$.

By taking the function $\psi(t)=1/t$, in Lemma 2, we arrive to the Calderon-Zygmund type criterion.

Theorem 2. *Let* D *be a bounded domain in* $\mathbb{C}$ *with no boundary component degenerated to a single point,* $A \in \mathrm{M}^{2\times 2}(D)$ *with* $K_{\mu_A}\in L^1(D)$. *Suppose that*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_{\mu_A}(z,z_0)\,\frac{dm(z)}{|z-z_0|^2} = o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right) \qquad as\ \varepsilon\to 0 \qquad \forall\, z_0\in\overline{D} \tag{25}$$

for $\varepsilon_0=\varepsilon(z_0)>0$. *Then, there exist A-harmonic solutions of Dirichlet problem (21) with representation (22) for each continuous function* $\varphi:\partial D\to\mathbb{R}$.

Of course, we could be able to give here the whole scale of conditions in terms of iterated logarithms $\psi(t)=1/(t\log 1/t\cdot\log\log 1/t\cdot\ldots\cdot\log\ldots\log 1/t)$.

Choosing in Lemma 2 $\psi_{z_0,\varepsilon}(t)\equiv\psi_{z_0}(t):=1/[tk^T_{\mu_A}(z_0,t)]$, where $k^T_{\mu_A}(z_0,r)$ is the integral mean of $K^T_{\mu_A}(z,z_0)$ over the circle $\{z\in\mathbb{C}:|z-z_0|=r\}$, we obtain the Lehto type criterion.

Theorem 3. *Let* D *be a bounded domain in* $\mathbb{C}$ *with no boundary component degenerated to a single point,* $A \in \mathrm{M}^{2\times 2}(D)$ *with* $K_{\mu_A}\in L^1(D)$. *Suppose that*

$$\int\limits_0^{\varepsilon_0}\frac{dr}{rk^T_{\mu_A}(z_0,r)}=\infty \qquad \forall z_0\in\overline{D} \tag{26}$$

for $\varepsilon_0=\varepsilon(z_0)>0$. *Then there exist A-harmonic solutions of Dirichlet problem (21) with representation (22) for each continuous function* $\varphi:\partial D\to\mathbb{R}$.

Corollary 4. *Let* D *be a bounded domain in* $\mathbb{C}$ *with no boundary component degenerated to a single point,* $A \in \mathrm{M}^{2\times 2}(D)$ *with* $K_{\mu_A}\in L^1(D)$ *and*

$$k^T_{\mu_A}(z_0,\varepsilon)=O\left(\log\frac{1}{\varepsilon}\right) \qquad as\ \varepsilon\to 0 \qquad \forall\, z_0\in\overline{D}. \tag{27}$$

Then there exist A-harmonic solutions of Dirichlet problem (21) in D *with representation (22) for each continuous function* $\varphi:\partial D\to\mathbb{R}$.

Condition (27) can be replaced by the whole series of more weak conditions

$$k^T_{\mu_A}(z_0,\varepsilon)=O\left(\left[\log\frac{1}{\varepsilon}\cdot\log\log\frac{1}{\varepsilon}\cdot\ldots\cdot\log\ldots\log\frac{1}{\varepsilon}\right]\right) \qquad \forall\, z_0\in\overline{D}. \tag{28}$$

Combining Theorems 2.5 and 3.2 in [6] and Theorems 3, we obtain the following Orlicz type criteria.

Theorem 4. *Let* D *be a bounded domain in* $\mathbb{C}$ *with no boundary component degenerated to a single point,* $A \in \mathrm{M}^{2\times 2}(D)$ *with* $K_{\mu_A} \in L^1(D)$. *Suppose that*

$$\int\limits_{U_{z_0}} \Phi_{z_0}(K^T_{\mu_A}(z, z_0))\, dm(z) < \infty \qquad \forall\, z_0 \in \overline{D} \tag{29}$$

for a neighborhood U_{z_0} *of* z_0 *and a convex non-decreasing function* $\Phi_{z_0} : [0, \infty] \to [0, \infty]$ *with*

$$\int\limits_{\Delta(z_0)}^{\infty} \log \Phi_{z_0}(t) \frac{dt}{t^2} = +\infty \tag{30}$$

for $\Delta(z_0) > 0$. *Then there exist A-harmonic solutions of Dirichlet problem (21) in* D *with representation (22) for each continuous function* $\varphi : \partial D \to \mathbb{R}$.

Corollary 5. *Let* D *be a bounded domain in* $\mathbb{C}$ *with no boundary component degenerated to a single point,* $A \in \mathrm{M}^{2\times 2}(D)$ *with* $K_{\mu_A} \in L^1(D)$ *and*

$$\int\limits_{U_{z_0}} e^{\alpha(z_0) K^T_{\mu_A}(z, z_0)}\, dm(z) < \infty \qquad \forall\, z_0 \in \overline{D} \tag{31}$$

for some $\alpha(z_0) > 0$ *and a neighborhood* U_{z_0} *of the point* z_0. *Then there exist A-harmonic solutions of Dirichlet problem (21) in* D *with representation (22) for each continuous function* $\varphi : \partial D \to \mathbb{R}$.

By applying (12), we can deduce the following consequence of Theorem 4.

Corollary 6. *Let* D *be a bounded domain in* $\mathbb{C}$ *with no boundary component degenerated to a single point,* $A \in \mathrm{M}^{2\times 2}(D)$ *with* $K_{\mu_A} \in L^1(D)$. *Suppose that*

$$\int\limits_{D} \Phi(K_{\mu_A}(z))\, dm(z) < \infty \tag{32}$$

for a convex non-decreasing function $\Phi : [0, \infty] \to [0, \infty]$ *with*

$$\int\limits_{\delta}^{\infty} \log \Phi(t) \frac{dt}{t^2} = +\infty \tag{33}$$

for some $\delta > 0$. *Then there exist A-harmonic solutions of Dirichlet problem (21) in* D *with representation (22) for each continuous function* $\varphi : \partial D \to \mathbb{R}$.

Remark 1. By the Stoilow theorem, see e.g. [7], a multi-valued solution $f = u + iv$ of the Dirichlet problem (21) in D for the Beltrami equation (3) with $K_{\mu_A} \in L^1_{\mathrm{loc}}(D)$ can be represented in the form $f = h \circ F$ where h is a multi-valued analytic function and F is a homeomorphic solution of (3) with $\mu := \mu_A$ in the class $W^{1,1}_{\mathrm{loc}}$. Therefore, as per Theorem 5.1 in [6] (also see Theorem 16.1.6 in [3]), condition (33) is not only sufficient but also necessary to have A-harmonic solutions u of Dirichlet problem (21) in D with integral constraints (32) for all continuous functions $\varphi : \partial D \to \mathbb{R}$.

Corollary 7. *Let* D *be a bounded domain in* $\mathbb{C}$ *with no boundary component degenerated to a single point,* $A \in \mathrm{M}^{2\times 2}(D)$ *and such that, for some* $\alpha > 0$,

$$\int\limits_{D} e^{\alpha K_{\mu_A}(z)}\, dm(z) < \infty. \tag{34}$$

Then there exist A-harmonic solutions of Dirichlet problem (21) in D with representation (22) for each continuous function $\varphi : \partial D \to \mathbb{R}$.

Remark 2. The requirement for domains to have no boundary component degenerated to a single point is necessary even for harmonic functions. Consider, for instance, the punctured unit disk $\mathbb{D}_0 := \mathbb{D} \setminus \{0\}$. By setting $\varphi(\zeta) \equiv 1$ on $\partial\mathbb{D}$ and $\varphi(0) = 0$, we see that φ is continuous on $\partial\mathbb{D}_0 = \partial\mathbb{D} \cup \{0\}$. Let us assume that there is a harmonic function u satisfying (21) with such φ . Then u is bounded by the maximum principle for harmonic functions and by the classic Cauchy—Riemann theorem, see also Theorem V.4.2 in [8], the extended u is harmonic in $\mathbb{D}$. Thus, by contradiction with the Mean-Value-Property we disprove the above assumption, as stated in Theorem 0.2.4 in [9].

Finally, recall that a point $p \in \partial D$ for a domain D in $\mathbb{R}^n, n \geqslant 2,$ is called a *regular point* if each solution of the Dirichlet problem for the Laplace equation in D , whose boundary function is continuous at p , is also continuous at p . The well-known Wiener criterion for regularity of a boundary point, as formulated in terms of barrier functions in [10], has simple geometric interpretation in the complex plane. Specifically, a point $p \in \partial D$ is regular if p belongs to a component of ∂D that is not degenerated to a single point, as stated in Theorem 4.2.2 in [5]. The example given above shows that this condition is not only sufficient but also necessary for regularity of a boundary point in the plane.

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ПРО ЗАДАЧУ ДІРІХЛЕ ДЛЯ А-ГАРМОНІЧНИХ ФУНКЦІЙ

Для А-гармонічного рівняння досліджено задачу Діріхле з неперервними межовими даними в обмежених областях комплексної площини. Нами встановлені критерії існування слабких розв'язків поставленої задачі у довільній обмеженій області без вироджених межових компонент в сенсі розподілів, здійснених у термінах умов на матричний коефіцієнт рівняння типу BMO (функцій обмеженого середнього коливання) і FMO (функцій скінченного середнього коливання). Наведено також ряд інтегральних критеріїв типу Кальдерона—Зигмунда, Лехто та Орлича. Відповідні приклади показують, що умова невиродженості межових компонент області є не лише достатньою, але й необхідною умовою розв'язності задачі Діріхле навіть для гармонічних функцій. Останнє узгоджується з відомою умовою Вінера. Показано, що отримані розв'язки мають зображення у вигляді композиції гармонічних розв'язків відповідних задач Діріхле і регулярних гомеоморфних розв'язків рівнянь Бельтрамі всієї комплексної площини з відповідними комплексними коефіцієнтами, які задовольняють гідродинамічну умову нормування у нескінченно віддаленій точці.

Ключові слова: *BMO, обмежене середнє коливання, FMO, скінченне середнє коливання, задача Діріхле, теорія потенціалу.*