On boundary estimates of mappings, acting onto domains with a locally quasiconformal boundary

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The following definitions are from [1]. A path γ in $\mathbb{R}^n$ is a continuous mapping $\gamma : \Delta \to \mathbb{R}^n$ where Δ is an interval in $\mathbb{R}$. Its locus $\gamma(\Delta)$ is denoted by $|\gamma|$. Given a family Γ of paths γ in $\mathbb{R}^n$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm}\Gamma$, if $\int\limits_{\gamma} \rho(x)|dx| \geqslant 1$ for each (locally rectifiable) $\gamma \in \Gamma$. The *modulus* of Γ is defined by the relation

$$M(\Gamma) := \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^n(x)\, dm(x) \qquad (1)$$

interpreted as $+\infty$ if $\operatorname{adm}\Gamma = \varnothing$. Everywhere below, unless otherwise stated, the boundary and the closure of a set are understood in the sense of the extended Euclidean space $\overline{\mathbb{R}^n}$.

Let $Q:\mathbb{R}^n\rightarrow [0,\infty]$ be Lebesgue measurable function. We will say that f *satisfies the inverse Poletsky's inequality* if the ratio

$$M(\Gamma)\leqslant \int\limits_{f(D)} Q(y)\cdot \rho_*^n(y)\,dm(y) \tag{2}$$

holds for any family of (locally rectifiable) paths Γ in D and for any $\rho_*\in {\rm adm}\,f(\Gamma)$. Note that estimates of the type (2) are well known and holds for classes of mappings (see, e.g., [2, Theorem 6.7.II] and [3, theorem 8.5]).

Given sets E and F and a given domain D in $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$, we denote by $\Gamma(E,F,D)$ the family of all paths $\gamma:[0,1]\rightarrow \overline{\mathbb{R}^n}$ joining E and F in D, that is, $\gamma(0)\in E$, $\gamma(1)\in F$ and $\gamma(t)\in D$ for all $t\in(0,1)$. In accordance with [4], a domain D in $\mathbb{R}^n$ is called *quasiextremal distance domain* (*QED-domain for short)* if

$$M(\Gamma(E,F,\mathbb{R}^n))\leqslant A_0\cdot M(\Gamma(E,F,D)) \tag{3}$$

for some finite number $A_0\geqslant 1$ and all continua E and F in D. In the extended Euclidean space $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$ we use the *spherical (chordal) metric* $h(x,y)=|\pi(x)-\pi(y)|$, where π is a stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1},\frac{1}{2})=\{x\in\mathbb{R}^{n+1}: |x-e_{n+1}/2|=1/2\}$ in $\mathbb{R}^{n+1}$, and

$$h(x,\infty)=\frac{1}{\sqrt{1+{|x|}^2}}\,,$$

$$h(x,y)=\frac{|x-y|}{\sqrt{1+{|x|}^2}\sqrt{1+{|y|}^2}}\,,\quad x\ne\infty\ne y \tag{4}$$

(see e.g. [1, Definition 12.1]). In what follows, given $A,B\subset\overline{\mathbb{R}^n}$ we set $h(A,B)=\inf\limits_{x\in A,y\in B}h(x,y)$, where h is a chordal metric in (4). Consider the following definition that has been proposed by Näkki [5], cf. [6]. The boundary of a domain D is called *locally quasiconformal* if every point $x_0\in\partial D$ has a neighborhood U, for which there exists a quasiconformal mapping φ of U onto the unit ball $\mathbb{B}^n\subset\mathbb{R}^n$ such that $\varphi(\partial D\cap U)$ is the intersection of the unit sphere $\mathbb{B}^n$ with a coordinate hyperplane $x_n=0$, where $x=(x_1,\ldots,x_n)$. Note that, with slight differences in the definition, domains with such boundaries are also called collared domains.

Given $\delta>0$, domains $D,D^{\,\prime}\subset\mathbb{R}^n$, $n\geqslant 2$, a nondegenerate continuum $A\subset D^{\,\prime}$ and a Lebesgue-measurable function $Q:D^{\,\prime}\rightarrow[0,\infty]$ denote by $\mathfrak{S}_{\delta,A,Q}(D,D^{\,\prime})$ the family of all open discrete and closed mappings f of the domain D onto the domain $D^{\,\prime}$ satisfying the condition (2) and such that $h(f^{\,-1}(A),\partial D)\geqslant\ \delta$. The following statement is true.

Theorem 1. *Let $Q\in L^1(D^{\,\prime})$, let D be a QED-domain, and $D^{\,\prime}$ is a bounded domain with a locally quasiconformal boundary. Then any mapping $f\in\mathfrak{S}_{\delta,A,Q}(D,D^{\,\prime})$ which satisfies the relation (2) has a continuous extension $f:\overline{D}\rightarrow\overline{D^{\,\prime}}$, while, for each point $x_0\in\partial D$ there will be U neighborhoods of this point and constants $C=C(n,A,D,D^{\,\prime},x_0)>0$ and $0<\alpha=\alpha(n,A,D,D^{\,\prime},x_0)\leqslant 1$ such that*

$$|\overline{f}(x)-\overline{f}(y)|^{\frac{n}{\alpha^2}}\leqslant\frac{C\cdot\|Q\|_1}{\log\left(1+\frac{\delta}{2|x-y|}\right)} \tag{5}$$

for all $x,y\in U\cap\overline{D}$, where $\|Q\|_1$ is the norm of the function Q in $L^1(D^{\,\prime})$.

The result mentioned above is accepted for publication in [7].

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