On singularities of mappings with a Lebesgue integrable majorant

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The following definitions are from [1]. A path γ in $\mathbb{R}^n$ is a continuous mapping $\gamma:\Delta\to\mathbb{R}^n$ where Δ is an interval in $\mathbb{R}$. Its locus $\gamma(\Delta)$ is denoted by $|\gamma|$. Given a family Γ of paths γ in $\mathbb{R}^n$, a Borel function $\rho:\mathbb{R}^n\to[0,\infty]$ is called *admissible* for Γ, abbr. $\rho\in\operatorname{adm}\Gamma$, if

$$\int\limits_{\gamma}\rho(x)|dx|\geqslant 1$$

for each (locally rectifiable) $\gamma\in\Gamma$. Given $p\geqslant 1$, the *p-modulus* of Γ is defined by the relation

$$M_p(\Gamma):=\inf_{\rho\in\operatorname{adm}\Gamma}\int\limits_{\mathbb{R}^n}\rho^p(x)dm(x) \tag{1}$$

interpreted as $+\infty$ if $\operatorname{adm}\Gamma=\varnothing$.

Given sets E and F and a given domain D in $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$, we denote by $\Gamma(E,F,D)$ the family of all paths $\gamma:[0,1]\to\overline{\mathbb{R}^n}$ joining E and F in D, that is, $\gamma(0)\in E$, $\gamma(1)\in F$ and $\gamma(t)\in D$ for all $t\in(0,1)$. Everywhere below, unless otherwise stated, the boundary and the closure of a set are understood in the sense of the extended Euclidean space $\overline{\mathbb{R}^n}$. Let $x_0\in\overline{D}$, $x_0\ne\infty$,

$$S(x_0,r)=\{x\,\in\,\mathbb{R}^n:|x-x_0|=r\}\,,S_i=S(x_0,r_i)\,,\quad i=1,2\,,$$

$$A=A(x_0,r_1,r_2)=\{x\,\in\,\mathbb{R}^n:r_1<|x-x_0|<r_2\}\,.$$

Let $f:D\to\mathbb{R}^n$, $n\geqslant 2$, and let $Q:\mathbb{R}^n\to[0,\infty]$ be a Lebesgue measurable function such that $Q(y)\equiv 0$ for $y\in\mathbb{R}^n\setminus f(D)$. Let $A=A(y_0,r_1,r_2)$ and $\Gamma_f(y_0,r_1,r_2)$ denotes the family of all paths $\gamma:[a,b]\to D$ such that $f(\gamma)\in\Gamma(S(y_0,r_1),S(y_0,r_2),A(y_0,r_1,r_2))$, i.e., $f(\gamma(a))\in S(y_0,r_1)$, $f(\gamma(b))\in S(y_0,r_2)$, and $f(\gamma(t))\in A(y_0,r_1,r_2)$ for any $a<t<b$. We say that f *satisfies the inverse Poletsky inequality at* $y_0\in f(D)$ *with respect to p-modulus,* if the relation

$$M_p(\Gamma_f(y_0,r_1,r_2))\leqslant\int\limits_A Q(y)\cdot\eta^p(|y-y_0|)\,dm(y) \tag{2}$$

holds for any $0<r_1<r_2<r_0:=\sup\limits_{y\in f(D)}|y-y_0|$ and any Lebesgue measurable function $\eta:(r_1,r_2)\to[0,\infty]$ such that

$$\int\limits_{r_1}^{r_2}\eta(r)\,dr\geqslant 1\,. \tag{3}$$

Note that estimates of the type (2) are well known and hold at least for $p=n$ in many classes of mappings (see, e.g., [2, Theorem 3.2], [3, Theorem 6.7.II] and [4, Theorem 8.5]). For $p\ne n$, similar estimates may be found, e.g., in [5] and [6].

A mapping $f : D \to \mathbb{R}^n$ is called *discrete* if the image $\{f^{-1}(y)\}$ of any point $y \in \mathbb{R}^n$ consists of isolated points, and *open* if the image of any open set $U \subset D$ is an open set in $\mathbb{R}^n$.

Later, in the extended space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ we use the *spherical (chordal) metric* $h(x, y) = |\pi(x) - \pi(y)|$, where π is a stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$, namely,

$$h(x,\infty) = \frac{1}{\sqrt{1+|x|^2}}\,,$$

$$h(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}\,, \quad x \neq \infty \neq y \tag{4}$$

(see, e.g., [1, Definition 12.1]). The following statement is true.

Theorem 1. *Let $n \geqslant 2$, $p \geqslant n$, let D be a domain in $\mathbb{R}^n$, $x_0 \in D$, and let $f : D \setminus \{x_0\} \to \mathbb{R}^n$ be an open discrete mapping that satisfies the conditions (2)-(3) at any point $y_0 \in \overline{D'} \setminus \{\infty\}$, where $D' := f(D \setminus \{x_0\})$.*

If $Q \in L^1(D')$, then f has a continuous extension $\overline{f} : D \to \overline{\mathbb{R}^n}$, the continuity of which should be understood in the sense of the chordal metric h in (4). The extended mapping $\overline{f}$ is open and discrete in D. Moreover, if $p = n$ and $\overline{f}(x_0) \neq \infty$, then there is a neighborhood $U \subset D$ of the point x_0 depending only on x_0, and $C = C(n, D, x_0) > 0$ such that

$$|\overline{f}(x) - \overline{f}(x_0)| \leqslant \frac{C_n \cdot (\|Q\|_1)^{1/n}}{\log^{1/n}\left(1 + \frac{\delta}{2|x-x_0|}\right)} \tag{5}$$

for any $x, y \in U$, where $\|Q\|_1$ is the norm of the function Q in $L^1(D')$.

References

[1] J. Väisälä. *Lectures on n-Dimensional Quasiconformal Mappings.* Lecture Notes in Math. 229. Berlin etc., Springer–Verlag, 1971.

[2] O. Martio, S. Rickman and J. Väisälä. Distortion and singularities of quasiregular mappings. *Ann. Acad. Sci. Fenn. Ser. A1,* 465, 1–13, 1970.

[3] S. Rickman. *Quasiregular mappings.* Springer-Verlag, Berlin, 1993.

[4] O. Martio, V. Ryazanov, U. Srebro and E. Yakubov. *Moduli in Modern Mapping Theory.* & Springer Science + Business Media, LLC : New York, 2009.

[5] V. Gol'dshtein, L. Gurov and A. Romanov. Homeomorphisms that induce monomorphisms of Sobolev spaces. *Israel J. Math.,* 91, 31–60, 1995.

[6] A. Menovschikov and A. Ukhlov. Composition operators on Sobolev spaces and Q-homeomorphisms. *Comput. Methods Funct. Theory,* 24, 149–162, 2024.