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Contact information

Website: https://sconferences.com

E-mail: info@sconferences.com

ABSOLUTE CONTINUITY AND SINGULARITY OF A QUATERNIONIC MEASURE

Kolomiiets Tamila
Doctor of Philosophy, Senior Lecturer at the Department of of Mathematical Analysis, Business Analysis and Statistics
Zhytomyr Ivan Franko State University

Abstract
In this paper the concepts of absolutely continuous ω_a and singular ω_s quaternion-valued measures relative to the classical real-valued measure μ are introduced and their properties are presented. Analogues of Lebesgue decomposition theorem, Radon-Nikodym theorem and one of its consequences for the quaternion-valued measure ω are proved.

Keywords: *quaternion algebra, quaternion-valued measure, absolutely continuous quaternionic measures, singular quaternionic measures, Lebesgue theorem, Radon-Nikodym theorem.*

Introduction

Recently, the real-valued measure theory [1] has many generalizations, in particular to complex and hypercomplex measure theories. For example, the seminal paper [2] generalized the notion of a classical real measure μ to a complex measure w and studied its properties. The generalization of some of the ideas of [2] to a quaternion-valued measure, i.e., a measure with values in the algebra of quaternions [3], is the subject of publications [4-5]. In this article, we highlight some properties of a quaternion-valued measure.

Main part

Let X be a non-empty set and let $\mathfrak{M}$ be a σ-algebra of subsets of X.

Definition 1. *Let $\mathfrak{M}$ be a σ-algebra of subsets of a set X. A quaternionic measure ω on a measurable space $(X,\mathfrak{M})$ is a quaternion-valued function on $\mathfrak{M}$ such that for any collection of sets $\{A_n, n\in\mathbb{N}\}\subset\mathfrak{M}$ that $A_n\cap A_m=\emptyset$ whenever $n\neq m$ we have*

$$\omega\left(\bigcup_{n=1}^{\infty} A_n\right)=\sum_{n=1}^{\infty}\omega(A_n). \tag{1}$$

Since the union of sets A_n is not changed if the subscripts are permuted, every rearrangement of series (1) must converge to

$$\omega\left(\bigcup_{n=1}^{\infty} A_n\right).$$

For this reason, we assume that the series converges absolutely.

Let μ be a positive measure on a measurable space $(X,\mathfrak{M})$ and w be a quaternionic measure on $(X,\mathfrak{M})$.

Definition 2. *We say that ω is absolutely continuous with respect to μ if $\mu(A)=0$ implies $\omega(A)=0$ for $A\in\mathfrak{M}$. We write $\omega\ll\mu$.*

Definition 3. *Given a quaternionic measure ω on a measurable space $(X,\mathfrak{M})$, assume that there is a set $F\in\mathfrak{M}$ such that $\omega(A)=\omega(A\cap F)$ for every $A\in\mathfrak{M}$, we say that ω is concentrated on F. This is equivalent to say that $\omega(A)=0$ whenever $A\cap F=0$.*

Let ω_1, ω_2 be quaternionic measures on $\mathfrak{M}$ and suppose there exist a pair of disjoint sets F, G such that ω_1 is concentrated on F and ω_2 is concentrated on G. Then we say that ω_1 and ω_2 are mutually singular, and write $\omega_1\perp\omega_2$.

Theorem 1. Properties of mutually singular quaternionic measures. *Suppose ω,ω_1,ω_2 are quaternionic measures and μ is a positive measure, then:*

1. *If ω is concentrated on F, so is $\mathrm{var}[\omega]$.*
2. *If $\omega_1\perp\omega_2$ then $\mathrm{var}[\omega_1]\perp\mathrm{var}[\omega_2]$.*
3. *If $\omega_1\perp\mu$ and $\omega_2\perp\mu$, then $(\omega_1+\omega_2)\perp\mu$.*
4. *If $\omega_1\ll\mu$ and $\omega_2\ll\mu$, then $(\omega_1+\omega_2)\ll\mu$.*
5. *If $\omega\ll\mu$, then $\mathrm{var}[\omega]\ll\mu$.*

6. If $\omega_1 \ll \mu$ and $\omega_2 \perp \mu$, then $\omega_1 \perp \omega_2$.

7. If $\omega \ll \mu$ and $\omega \perp \mu$ then $\omega = 0$ identically.

Proof.

1. If $A \cap F = \emptyset$ then for any partition $\{A_n, n \in \mathbb{N}\}$ of A we have $\omega(A_n) = 0$ for every $n \in \mathbb{N}$ and hence $\mathrm{var}[\omega](A) = 0$ for any A.

2. This follows from 1.

3. There is a set $B \in \mathfrak{M}$ on which μ is concentrated. There are F, $G \in \mathfrak{M}$ such that ω_1 is concentrated on F and ω_2 is concentrated on G. If $A \subset (F \cup G)^c = F^c \cap G^c$ then

$$(\omega_1 + \omega_2)(A) = \omega_1(A) + \omega_2(A) = 0.$$

This means that $\omega_1 + \omega_2$ is concentrated on $F \cup G$, but it is clear that $B \subset (F \cup G)^c$, hence

$$(\omega_1 + \omega_2) \perp \mu.$$

4. Follows directly from the definitions.

5. Suppose $\mu(A) = 0$ and $\{A_n, n \in \mathbb{N}\}$ is a partition of A. Then $\mu(A_n) = 0$ and since $\omega \ll \mu$ then $\omega(A_n) = 0$ for every $n \in \mathbb{N}$; hence

$$\sum_{n=1}^{\infty} |\omega(A_n)| = 0.$$

This implies that $\mathrm{var}[\omega](A) = 0$.

6. Since $\omega_2 \perp \mu$ there is a set $E \in \mathfrak{M}$ such that $\mu(E) = 0$ and ω_2 is concentrated on E. Since $\omega_1 \ll \mu$, then $\omega_1(A) = 0$ for every $A \in \mathfrak{M}$ such that $A \subset E$ and hence ω_1 is concentrated on E^c.

7. It follows from 6 that $\omega \perp \omega$. Hence $\omega = 0$.■

Theorem 2 (Lebesgue). Decompostition of a quaternionic measure. *Let λ be a signed real σ-finite measure on a measurable space $(X, \mathfrak{M})$ and let w be a quaternionic measure on $(X, \mathfrak{M})$. Then there exists a unique pair of quaternionic measures w_a and w_s such that*

$$w = w_a + w_s, w_a \ll \lambda, w_s \perp \lambda. \tag{2}$$

The pair w_a, w_s is called the Lebesgue decomposition of w w.r.t. λ, where w_a is the absolutely continuous part and w_s is the singular part of the decomposition.

Proof. Since w is a quaternionic finite measure on $(X, \mathfrak{M})$, we have $w = \lambda_0 + I\lambda_1 + J\lambda_2 + K\lambda_3$, with λ_k, $k = 0,1,2,3$ real finite signed measures. By applying Lebesgue's decomposition theorem to each λ_k, we obtain $\lambda_k = \lambda_a^{(k)} + \lambda_s^{(k)}$, where $\lambda_a^{(k)} \ll \lambda$ and $\lambda_s^{(k)} \perp \lambda$. By putting

$$w_a = \lambda_a^{(0)} + I\lambda_a^{(1)} + J\lambda_a^{(2)} + K\lambda_a^{(3)}$$

and

$$w_s = \lambda_s^{(0)} + I\lambda_s^{(1)} + J\lambda_s^{(2)} + K\lambda_s^{(3)}$$

we conclude the proof of the existence of the pair w_a, w_s. Suppose that there is another pair w'_a, w'_s, which satisfies (2), then

$$w'_a - w_a = w_s - w'_s.$$

It is easily seen that $w'_a - w_a \ll \lambda$ and $w_s - w'_s \perp \lambda$.

Hence, considering item 7 of Theorem 1 we have $w'_a - w_a = w_s - w'_s = 0$.■

Theorem 3 (Radon-Nikodym). *Let μ be a positive σ-finite measure on a measurable space $(X, \mathfrak{M})$, let w be a quaternionic measure on $(X, \mathfrak{M})$ and let w_a be absolutely continuous part of the Lebesgue decomposition of w w.r.t. μ. Then there is a measurable quaternionic function h on X such that for every set $A \in \mathfrak{M}$*

$$w_a(A) = \int_A h d\mu,$$

where h is uniquely defined up to a μ-null set.

Remark 1. *Recall that a quaternionic function is measurable if the preimage of any borelian set belongs to $\mathfrak{M}$.*

Proof. Since $w_a \ll \mu$, taking into account that $w_a(\cdot) := \lambda_a^{(0)}(\cdot) + I\lambda_a^{(1)}(\cdot) + J\lambda_a^{(2)}(\cdot) + K\lambda_a^{(3)}(\cdot)$, where $\lambda_a^{(k)}$ are signed measures, we have that $\lambda_a^{(k)} \ll \mu$ for each $k = 0,1,2,3$. Taking into account Radon-Nikodym Theorem for signed measures there exist measurable functions h_k such that

$$\lambda_a^{(k)}(A) = \int_A h_k d\mu, \quad \forall A \in \mathfrak{M}, \qquad k = 0,1,2,3.$$

Hence

$$w_a(A) = \int_A \big(h_0(x) + Ih_1(x) + Jh_2(x) + Kh_3(x)\big) d\mu(x). ■$$

Remark 2. *The function* $h(x) := h_0(x) + Ih_1(x) + Jh_2(x) + Kh_3(x)$ *will be called the Radon-Nikodym derivative of* w_a *w.r.t* μ *and it is denoted by* $dw_a/d\mu$.

In the quaternionic case the Radon-Nikodym theorem has many corollaries and we give one of them.

Theorem 4. *Let* w *be a quaternionic measure on a measurable space* $(X, \mathfrak{M})$. *Then there exists a measurable function* h *such that* $|h(x)| = 1$ *for all* $x \in X$ *and*

$$\frac{dw}{d\mathrm{var}[w]} = h.$$

Proof. Since $w \ll \mathrm{var}[w]$ *it follows from Theorem 3 that there is a measurable function* h *such that* $dw/d\mathrm{var}[w] = h$.

For a positive real p *let us consider* $S_p := \{x \in X : |h(x)| < p\}$. *Then for any partition* $\{A_n\}$ *of* S_p *we have:*

$$\sum_{n=1}^{\infty} |w(A_n)| = \sum_{n=1}^{\infty} \left| \int_{A_n} h(x) d\mathrm{var}[w](x) \right| \le p \sum_{n=1}^{\infty} \mathrm{var}[w](A_n) = p\mathrm{var}[w](S_p).$$

Hence $\mathrm{var}[w](S_p) \le p\mathrm{var}[w](S_p)$. *If* $p < 1$ *then* $\mathrm{var}[w](S_p) = 0$. *Therefore,* $|h(x)| \ge 1$ *a.e. On the other hand for* $A \in \mathfrak{M}$ *such that* $\mathrm{var}[w](A) > 0$ *we have:*

$$\frac{1}{\mathrm{var}[w](A)} \left| \int_A h(x) d\mathrm{var}[w](x) \right| = \frac{|w(A)|}{\mathrm{var}[w](A)} \le 1.$$

Thus, the integral

$$I_A(h) = \frac{1}{\mathrm{var}[w](A)} \int_A h(x) d\mathrm{var}[w](x)$$

lies in a 4-D ball $B_1(0)$ *of radius* 1 *for each* $A \in \mathfrak{M}$ *such that* $\mathrm{var}[w](A) > 0$. *Suppose* $B_r(a)$ *is a ball of radius* r *and with center at the point* a *such that* $B_r(a) \cap B_1(0) = \emptyset$. *Let us show that* $\mathrm{var}[w](C) = 0$, *where* $C = h^{-1}(B_r(a))$.

Indeed, if $\mathrm{var}[w](C) > 0$ *then*

$$|I_C(h) - a| = \frac{1}{\mathrm{var}[w](C)} \left| \int_C (h(x) - a) d\mathrm{var}[w](x) \right| \le \frac{1}{\mathrm{var}[w](C)} \int_C |h(x) - a| d\mathrm{var}[w](x) \le r,$$

which is impossible since $I_C(f) \in B_1(0)$ *and we conclude that* $|h(x)| \le 1$ *(a. e.) Therefore,*

$$|h(x)| = 1 \text{ (a. e.)}.$$

Let $N := \{x \in X : |h(x)| \ne 1\}$. *Since as it is shown* $\mathrm{var}[w](N) = 0$ *we redefine* h *on* N *so that* $h(x)=1$ *for all* $x \in N$ *and obtain a function with the desired properties.* ■

Conclusion

The obtained results can be used in the course of research for problems of measure theory, random process theory, and statistical physics.

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