On equicontinuity of families of mappings with one normalization condition by the prime ends

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Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm}\Gamma$, if $\int\limits_{\gamma} \rho(x)|dx| \geqslant 1$ for each (locally rectifiable) $\gamma \in \Gamma$. We define the quantity

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^n(x)\, dm(x) \qquad (1)$$

and call $M(\Gamma)$ *a modulus* of Γ; here m stands for the n-dimensional Lebesque measure, see [1, 6.1].

Given sets E and F and a domain D in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, we denote $\Gamma(E, F, D)$ the family of all paths $\gamma : [0, 1] \to \overline{\mathbb{R}^n}$ joining E and F in D, that is, $\gamma(0) \in E$, $\gamma(1) \in F$ and $\gamma(t) \in D$ for all $t \in [0, 1]$.

An *end* of a domain D is an equivalence class of chains of cross-cuts of D. We say that an end K is a *prime end* if K contains a chain of cross-cuts $\{\sigma_m\}$, such that

$$\lim_{m\to\infty} M(\Gamma(C,\sigma_m,D))=0$$

for some continuum C in D. Set $\mathbb{B}^n:=\{x\in\mathbb{R}^n : |x|<1\}$. We say that the boundary of a domain D in $\mathbb{R}^n$ is *locally quasiconformal* if every point $x_0\in\partial D$ has a neighborhood U that admit a conformal mapping φ onto the unit ball $\mathbb{B}^n\subset\mathbb{R}^n$ such that $\varphi(\partial D\cap U)$ is the intersection of $\mathbb{B}^n$ and a coordinate hyperplane, see e.g. [2], cf. [3]. We say that a bounded domain D in $\mathbb{R}^n$ is *regular* if D may be mapped quasiconformally onto a bounded domain with a locally quasiconformal boundary. If $\overline{D}_P$ is the completion of a regular domain D by its prime ends and g_0 is a quasiconformal mapping of a domain D_0 with locally quasiconformal boundary onto D, then this mapping naturally determines the metric $\rho_0(p_1,p_2)=\left|\widetilde{g_0}^{-1}(p_1)-\widetilde{g_0}^{-1}(p_2)\right|$, where $\widetilde{g_0}$ is the extension of g_0 onto $\overline{D_0}$. Let $x_0\in\overline{D}$, $x_0\neq\infty$, $S(x_0,r)=\{x\in\mathbb{R}^n : |x-x_0|=r\}$, $A=A(x_0,r_1,r_2)=\{x\in\mathbb{R}^n : r_1<|x-x_0|<r_2\}$.

Let $f:D\to\mathbb{R}^n$, $n\geqslant 2$, and let $Q:\mathbb{R}^n\to[0,\infty]$ be a Lebesgue measurable function such that $Q(y)\equiv 0$ for $y\in\mathbb{R}^n\setminus f(D)$. Let $A=A(y_0,r_1,r_2)$ and let $\Gamma_f(y_0,r_1,r_2)$ denotes the family of all paths $\gamma:[a,b]\to D$ such that

$$f(\gamma)\in\Gamma(S(y_0,r_1),S(y_0,r_2),A(y_0,r_1,r_2))\,,$$

i.e., $f(\gamma(a))\in S(y_0,r_1)$, $f(\gamma(b))\in S(y_0,r_2)$, and $f(\gamma(t))\in A(y_0,r_1,r_2)$ for any $a<t<b$.

We say that, f *satisfies the inverse Poletsky inequality* at a point $y_0\in f(D)$, if the relation

$$M(\Gamma_f(y_0,r_1,r_2))\leqslant\int\limits_{f(D)\cap A(y_0,r_1,r_2)} Q(y)\cdot\eta^n(|y-y_0|)\,dm(y) \qquad (2)$$

holds for any Lebesgue measurable function $\eta:(r_1,r_2)\to[0,\infty]$ satisfying the relation

$$\int\limits_{r_1}^{r_2}\eta(r)\,dr\geqslant 1\,. \qquad (3)$$

We say that the boundary of D is *weakly flat* at a point $x_0\in\partial D$ if, for every number $P>0$ and every neighborhood U of the point x_0, there is a neighborhood $V\subset U$ such that $M(\Gamma(E,F,D))\geqslant P$ for all continua E and F in D intersecting ∂U and ∂V. We say that the boundary ∂D is weakly flat if the corresponding property holds at every point of the boundary.

Given domains $D,D'\subset\mathbb{R}^n$, $n\geqslant 2$, points $a\in D$, $b\in D'$ and a Lebesgue measurable function $Q:D'\to[0,\infty]$ denote $\mathfrak{S}_{a,b,Q}(D,D')$ a family of all open discrete and closed mappings f of D onto D', satisfying the relation (2) for any $y_0\in D'$, while $f(a)=b$. The following statement holds.

Theorem 1. *Assume that, D has a weakly flat boundary, any component of which does not degenerate into a point. If $Q\in L^1(D')$ and D' is regular, then any $f\in\mathfrak{S}_{a,b,Q}(D,D')$ has a continuous extension $\overline{f}:\overline{D}\to\overline{D'}_P$, $\overline{f}(\overline{D})=\overline{D'}_P$, and, in addition, a family $\mathfrak{S}_{a,b,Q}(\overline{D},\overline{D'})$ which consists of all extended mappings $\overline{f}:\overline{D}\to\overline{D'}_P$, is equicontinuous in $\overline{D}$.*

The result mentioned above is published in [4].

References

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