# On the inverse Poletsky inequality with a cotangent dilatation

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The following definitions are from [1]. A path $\gamma$ in $\mathbb{R}^n$ is a continuous mapping $\gamma : \Delta \to \mathbb{R}^n$ where $\Delta$ is an interval in $\mathbb{R}$. Its locus $\gamma(\Delta)$ is denoted by $|\gamma|$. Given a family $\Gamma$ of paths $\gamma$ in $\mathbb{R}^n$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for $\Gamma$, abbr. $\rho \in \operatorname{adm}\Gamma$, if

$$\int\limits_{\gamma} \rho(x)|dx| \geqslant 1$$

for each (locally rectifiable) $\gamma \in \Gamma$. Given $p \geqslant 1$, the *p-modulus* of $\Gamma$ is defined by the relation

$$M_p(\Gamma) := \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x) dm(x) \tag{1}$$

interpreted as $+\infty$ if $\operatorname{adm}\Gamma=\varnothing$.

We will need the following definitions related to paths, their lengths and mappings defined on them, see [2, section 8]. If $\gamma:\Delta\to\mathbb{R}^n$ is a locally rectifiable path, then there is the unique nondecreasing length function $l_\gamma$ of $\Delta$ onto a length interval $\Delta_\gamma\subset\mathbb{R}$ with a prescribed normalization $l_\gamma(t_0)=0\in\Delta_\gamma$, $t_0\in\Delta$, such that $l_\gamma(t)$ is equal to the length of the subpath $\gamma|_{[t_0,t]}$ of $\gamma$ if $t>t_0$, $t\in\Delta$, and $l_\gamma(t)$ is equal to minus length of $\gamma|_{[t,t_0]}$ if $t<t_0$, $t\in\Delta$. Let $g:|\gamma|\to\mathbb{R}^n$ be a continuous mapping, and suppose that the path $\widetilde{\gamma}=g\circ\gamma$ is also locally rectifiable. Then there is a unique non-decreasing function $L_{\gamma,g}:\Delta_\gamma\to\Delta_{\widetilde{\gamma}}$ such that $L_{\gamma,g}\left(l_\gamma(t)\right)=l_{\widetilde{\gamma}}(t)$ for all $t\in\Delta$. A path $\gamma$ in $D$ is called here a (whole) *lifting* of a path $\widetilde{\gamma}$ in $\mathbb{R}^n$ under $f:D\to\mathbb{R}^n$ if $\widetilde{\gamma}=f\circ\gamma$.

Further, we use the notation $I$ for the segment $[a,b]$. Given a closed rectifiable path $\gamma:I\to\mathbb{R}^n$, we define a length function $l_\gamma(t)$ by the rule $l_\gamma(t)=S\left(\gamma,[a,t]\right)$, where $S(\gamma,[a,t])$ is the length of the path $\gamma|_{[a,t]}$. Let $\alpha:[a,b]\to\mathbb{R}^n$ be a rectifiable path in $\mathbb{R}^n$, $n\geqslant 2$, and $l(\alpha)$ be its length. A *normal representation* $\alpha^0$ of $\alpha$ is defined as a path $\alpha^0:[0,l(\alpha)]\to\mathbb{R}^n$ which can be got from $\alpha$ by change of parameter such that $\alpha(t)=\alpha^0\left(S\left(\alpha,[a,t]\right)\right)$ for every $t\in[0,l(\alpha)]$. Such a normal representation always exists and is unique (see [1, Theorem 2.4]).

The following definition may be found in [1, 2.5, item 2, section I]. Let $\alpha:[a,b]\to\mathbb{R}^n$ be a closed rectifiable path in $\mathbb{R}^n$, $n\geqslant 2$. A mapping $f:|\alpha|\to\mathbb{R}^n$ is said to be *absolutely continuous on* $\alpha$, if the function $f\circ\alpha^0$ is absolutely continuous on $[0,l(\alpha)]$, where $l(\alpha)$ denotes the length of $\alpha$, and $\alpha^0$ is its normal representation.

In the following, we say that some property $P$ holds for *$p$-almost all paths in the domain $D$* if this property may be violated only for some family $\Gamma_0$ of paths in $D$ such that $M_p(\Gamma_0)=0$, where $M_p(\Gamma_0)$ denotes the $p$-module of the family of paths $\Gamma_0$ defined in (1). We will say that the mapping $f:D\to\mathbb{R}^n$ has the *$ACP$-property with respect to $p$-modulus,* write $f\in ACP_p$, if the length function $L_{\gamma,f}$ is absolutely continuous on all closed intervals $\Delta_\gamma$ for $p$-almost all paths $\gamma$ in $D$.

Let $X$ and $Y$ be two spaces with measures $\mu$ and $\mu'$, respectively. We say that a mapping $f:X\to Y$ has *$N$-property of Luzin*, if from the condition $\mu(E)=0$ it follows that $\mu'(f(E))=0$. Similarly, we say that a mapping $f:X\to Y$ has *$N^{-1}$-Luzin property,* if from the condition $\mu'(E)=0$ it follows that $\mu(f^{-1}(E))=0$.

Let $x\in D$ be a differentiability point of $f$. We set

$$l(f'(x))\;=\;\min_{h\in\mathbb{R}^n\setminus\{0\}}\frac{|f'(x)h|}{|h|}\,,\quad \|f'(x)\|\;=\;\max_{h\in\mathbb{R}^n\setminus\{0\}}\frac{|f'(x)h|}{|h|}\,,\quad J(x,f)=\det f'(x)\,.$$

Given sets $E$ and $F$ and a given domain $D$ in $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$, we denote by $\Gamma(E,F,D)$ the family of all paths $\gamma:[0,1]\to\overline{\mathbb{R}^n}$ joining $E$ and $F$ in $D$, that is, $\gamma(0)\in E$, $\gamma(1)\in F$ and $\gamma(t)\in D$ for all $t\in(0,1)$. Everywhere below, unless otherwise stated, the boundary and the closure of a set are understood in the sense of the extended Euclidean space $\overline{\mathbb{R}^n}$. Let $x_0\in\overline{D}$, $x_0\neq\infty$,

$$B(x_0,r)=\{x\in\mathbb{R}^n:|x-x_0|<r\}\,,\quad \mathbb{B}^n=B(0,1)\,,$$

$$S(x_0,r)=\{x\,\in\,\mathbb{R}^n:|x-x_0|=r\}\,,S_i=S(x_0,r_i)\,,\quad i=1,2\,,$$

$$A=A(x_0,r_1,r_2)=\{x\,\in\,\mathbb{R}^n:r_1<|x-x_0|<r_2\}\,.$$

Let $f:D\to\mathbb{R}^n$, $n\geqslant 2$, and let $Q:\mathbb{R}^n\to[0,\infty]$ be a Lebesgue measurable function such that $Q(y)\equiv 0$ for $y\in\mathbb{R}^n\setminus f(D)$. Let $A=A(y_0,r_1,r_2)$ and $\Gamma_f(y_0,r_1,r_2)$ denotes the family of all paths $\gamma:[a,b]\to D$ such that $f(\gamma)\in\Gamma(S(y_0,r_1),S(y_0,r_2),A(y_0,r_1,r_2))$, i.e., $f(\gamma(a))\in S(y_0,r_1)$,

$f(\gamma(b)) \in S(y_0, r_2)$, and $f(\gamma(t)) \in A(y_0, r_1, r_2)$ for any $a < t < b$. We say that $f$ *satisfies the inverse Poletsky inequality at $y_0 \in f(D)$ with respect to $p$-modulus,* if the relation

$$M_p(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_A Q(y) \cdot \eta^p(|y - y_0|)\, dm(y) \tag{2}$$

holds for any $0 < r_1 < r_2 < r_0 := \sup\limits_{y \in f(D)} |y - y_0|$ and any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that $\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1$. A mapping $f : D \to \mathbb{R}^n$ is called *weakly light,* if, for any $y \in \mathbb{R}^n$, each connected component $\{f^{-1}(y)\}$ does not contain a non-degenerate path (see, e.g., Remark 8.3 in [2]).

**Theorem 1.** *Let $p > 1$, and let $f : D \to \mathbb{R}^n$ be a weakly light mapping which is differentiable a.e. and has Luzin $N$- and $N^{-1}$-properties with respect to the Lebesgue measure in $\mathbb{R}^n$, besides that, $f \in ACP_p(D)$. Let $y_0 \in \overline{f(D)} \setminus \{\infty\}$. Set*

$$K_{CT,p,y_0}(y, f) = \sum\limits_{x \in f^{-1}(y)} \frac{\left(\sup\limits_{|h|=1} \left|\left(f'(x)h, \frac{f(x)-y_0}{|f(x)-y_0|}\right)\right|\right)^p}{|J(x, f)|}\,. \tag{3}$$

*Then $f$ satisfies the inverse Poletsky inequality 2 at $y_0$ for $Q_*(y) := K_{CT,p,y_0}(y, f)$.*

The result mentioned above is published in [3].

## References


[1] Väisälä J. *Lectures on n-Dimensional Quasiconformal Mappings.* Lecture Notes in Math. 229. Berlin etc., Springer–Verlag, 1971.

[2] Martio O., Ryazanov V., Srebro U. and Yakubov E. *Moduli in Modern Mapping Theory.* Springer Science + Business Media, LLC : New York, 2009.

[3] Sevost'yanov E., Targonskii V. On the Inverse Poletsky Inequality with a Cotangent Dilatation. *Comput. Methods Funct. Theory.* 2023. https://doi.org/10.1007/s40315-023-00495-3