On Beltrami equations with inverse conditions and hydrodynamic normalization

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Abstract. A *Beltrami equation with two characteristics* is a differential equation of the form

$$f_{\overline{z}} = \mu(z) \cdot f_z + \nu(z) \cdot \overline{f_z}\,, \tag{1}$$

where $\mu = \mu(z)$ and $\nu = \nu(z)$ are given measurable functions with $|\mu(z)| < 1$ and $|\nu(z)| < 1$ a.a. Let $\mu : D \to \mathbb{D}$ and $\nu : D \to \mathbb{D}$ be functions such that the relation $|\mu(z)| + |\nu(z)| < 1$ holds for almost any $z \in D$. We will consider that $\mu(z) = \nu(z) \equiv 0$ for any $z \in \mathbb{C} \setminus D$. Fix $n \geqslant 1$ and set

$$\mu_n(z) = \begin{cases} \mu(z), & z \in \mathbb{C}, K_{\mu,\nu}(z) \leqslant n, \\ 0\,, & \text{otherwise in } \mathbb{C}, \end{cases} \qquad \nu_n(z) = \begin{cases} \nu(z), & z \in \mathbb{C}, K_{\mu,\nu}(z) \leqslant n, \\ 0\,, & \text{otherwise in } \mathbb{C}\,. \end{cases} \tag{2}$$

Let $f_n : \mathbb{C} \to \mathbb{C}$ be a homeomorphic solution of the equation $(f_n)_{\overline{z}} = \mu_n(z) \cdot (f_n)_z + \nu_n(z) \cdot \overline{(f_n)_z}$. Set $g_n(z) := f_n^{-1}(z)$. Observe that f_n is conformal at the neighborhood of the infinity, so there is a continuous extension $f_n : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Thus $f_n(\mathbb{C}) = \mathbb{C}$ and $f_n(\infty) = \infty$. Note that, $g_n : \mathbb{C} \to \mathbb{C}$ is quasiconformal, in particular, g_n is almost everywhere differentiable in $\mathbb{C}$. It may be showed that, $f_n(z) = a_n z + b_n + o(1)$ as $z \to \infty$, where $a_n, b_n \in \mathbb{C}$ and $a_n \neq 0$. We may consider that $a_n = 1$ and $b_n = 0$ for any $n \in \mathbb{N}$. Note that such a function f_n is unique.

Let

$$K_{\mu_{g_n}}(w) = \frac{|(g_n)_w|^2 - |(g_n)_{\overline{w}}|^2}{(|(g_n)_w| - |(g_n)_{\overline{w}}|)^2}\,, \qquad K_{I,p}(w, g_n) = \frac{|(g_n)_w|^2 - |(g_n)_{\overline{w}}|^2}{(|(g_n)_w| - |(g_n)_{\overline{w}}|)^p}\,. \tag{3}$$

Theorem. Let D be a domain in $\mathbb{C}$ such that $\overline{D}$ is a compact set in $\mathbb{C}$, let $\mu : \mathbb{C} \to \mathbb{D}$ and $\nu : \mathbb{C} \to \mathbb{D}$ be Lebesgue measurable functions vanishing outside D such that the relation $|\mu(z)| + |\nu(z)| < 1$ holds for almost any $z \in D$. In addition, let μ_n, ν_n, f_n and g_n as above, $n = 1, 2, \ldots,$. Let $Q : \mathbb{C} \to [1, \infty]$ be a Lebesgue measurable function. Assume that the following conditions hold: 1) for each $0 < r_1 < r_2 < 1$ and $y_0 \in \mathbb{C}$ there is a set $E \subset [r_1, r_2]$ of positive linear Lebesgue measure such that the function Q is integrable over the circles $S(y_0, r)$ for any $r \in E$; 2) there exist a number $1 < p \leqslant 2$ such that, for any bonded domain $G \subset \mathbb{C}$ there exists a constant $M = M_G > 0$ such that $\int\limits_G K_{I,p}(w, g_n)\, dm(w) \leqslant M$ for all $n = 1, 2, \ldots$, where $K_{I,p}(w, g_n)$ is defined in (3); 3) the inequality $K_{\mu_{g_n}}(w) \leqslant Q(w)$ holds for a.e. $w \in \mathbb{C}$, where $K_{\mu_{g_n}}$ is defined in (3). Then the equation (1) has a continuous $W^{1,p}_{\rm loc}(\mathbb{C})$-solution f in $\mathbb{C}$ such that $f(z) = z + \varepsilon(z)$, where $\varepsilon(z) \to 0$ as $z \to \infty$.

Corollary. In particular, the conclusion of Theorem holds if, in this theorem, we abandon condition 1), accept condition 3), and replace condition 2) with the requirement $Q \in L^1_{\rm loc}(\mathbb{C})$. If G is some bounded domain in $\mathbb{C}$ and K is a compactum in G, then there is some domain $G' \subset \mathbb{C}$ and a function Q' equal Q in G' and vanishing outside G' such that Q' is integrable in $\mathbb{C}$ and the relation $|f(x) - f(y)| \leqslant \frac{C}{\log^{1/2}\left(1+\frac{r_0}{2|x-y|}\right)}$ holds for any $x, y \in K$, where $C = C(K, \|Q'\|_1, G) > 0$ is some constant depending only on K, G and $\|Q'\|_1$, $\|Q'\|_1$ denotes L^1-norm of Q' in $\mathbb{R}^n$, and $r_0 = d(K, \partial G)$.

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