On quasilinear Beltrami equations and tangential dilatation

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Abstract. Below we consider that a mapping f is sense-preserving. Given a mapping $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$, we set $f_{\overline{z}} = (f_x + if_y)/2$ ł $f_z = (f_x - if_y)/2$. We say that, a function $\nu = \nu(z, w) : D \times \mathbb{C} \to \mathbb{D}$ satisfies *Caratheodory conditions,* if ν is Lebesgue measurable over $z \in D$ for every $w \in \mathbb{C}$, and ν is continuous by w for almost all $z \in D$. Let $\mu = \mu(z, w)$ and $\nu = \nu(z, w)$ satisfy Caratheodory conditions and, in addition, $|\mu(z, w)| + |\nu(z, w)| < 1$ for all $w \in \mathbb{C}$ and almost all $z \in D$. The *maximal dilatation corresponding to μ and ν* is defied by the equality $K_{\mu,\nu}(z, w) = \frac{1+|\mu(z,w)|+|\nu(z,w)|}{1-|\mu(z,w)|-|\nu(z,w)|}$. The *tangential dilatation* corresponding to the functions $\mu(z, w)$ and $\nu(z, w)$ with respect to the point $z_0 \in \mathbb{C}$ and $\theta \in [0, 2\pi)$ is called the quantity

$$K^T_{\mu,\nu}(z, z_0, w, \theta) = \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\left(\mu(z, w) + \nu(z, w)e^{i\theta}\right)\right|^2}{1 - |\mu(z, w) + \nu(z, w)e^{i\theta}|^2}$$

whenever $z \in D$ is a differentiability point of f. The *quasilinear Beltrami equation with two characteristics* is defined by the formula

$$f_{\overline{z}} = \mu(z, f(z)) \cdot f_z + \nu(z, f(z)) \cdot \overline{f_z}\,. \qquad (1)$$

The mapping $f : D \to \mathbb{C}$ is called *a regular solution of the equation (1),* if $f \in W^{1,1}_{\rm loc}$ and $J(z, f) \neq 0$ almost everywhere in D. We say that, a locally integrable function $\varphi : D \to \mathbb{R}$ has a *finite mean oscillation* at the point x_0 (we write: $\varphi \in FMO(x_0)$), if $\limsup\limits_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} \int\limits_{B(x_0,\,\varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon|\, dm(x) < \infty$, where $\overline{\varphi}_\varepsilon = \frac{1}{\pi\varepsilon^2} \int\limits_{B(x_0,\,\varepsilon)} \varphi(x)\, dm(x)$. Let D be a domain in $\mathbb{C}$, $z_0 \in D$, and let $Q^{(1)}_{z_0} : D \to [0, \infty]$ be a Lebesgue measurable function that is zero outside the domain D. Set $q^{(1)}_{z_0}(r) = \frac{1}{2\pi} \int\limits_0^{2\pi} Q^{(1)}_{z_0}(z_0 + re^{\,i\varphi})\, d\varphi$.

Theorem. *Let D be a domain in $\mathbb{C}$, and let the functions $\mu = \mu(z, w)$ and $\nu = \nu(z, w)$ satisfy the Caratheodory conditions and, in addition, $|\mu(z, w)| + |\nu(z, w)| < 1$ for all $w \in \mathbb{C}$ and almost all $z \in D$. Suppose that, there exists a function $Q : D \to [1, \infty]$ such that $K_{\mu,\nu}(z, w) \leqslant Q(z) \in L^1_{\rm loc}(D)$ for almost all $z \in D$ and all $w \in \mathbb{C}$. Assume that, for any $z_0 \in D$ there is a function $Q^{(1)}_{z_0} : D \to [0, \infty]$ such that the inequality $K^T_{\mu,\nu}(z, z_0, w, e^{i\theta}) \leqslant Q^{(1)}_{z_0}(z)$ holds for almost all $z \in D$, all $w \in \mathbb{C}$ and all $\theta \in [0, 2\pi)$. Suppose that, one of two conditions holds: either $Q^{(1)}_{z_0} \in FMO(D)$, or*

$$\int\limits_0^{\delta(z_0)} \frac{dr}{rq^{(1)}_{z_0}(r)} = \infty$$

for some $\delta(z_0) < \operatorname{dist}(z_0, \partial D)$ and each $z_0 \in D$, where $q^{(1)}_{z_0}(r)$ as above. Then the equation (1) has a regular homeomorphic solution f of the class $W^{1,1}_{\rm loc}$ in D such that $f^{-1} \in W^{1,2}_{\rm loc}(f(D))$. Moreover, f has a homeomorphic extension in $\mathbb{C}$, which is conformal outside the domain D, and it may be chosen such that $f(0) = 0,\ f(1) = 1$.

Keywords: Beltrami equations; Quasiconformal mappings; Mappings with a finite distortion.