On global behavior of mappings in metric spaces in terms of prime ends

Evgeny Sevost'yanov

Zhytomyr Ivan Franko State University, Zhytomyr, Ukraine; Institute of Applied Mathematics and Mechanics of NAS of Ukraine, Kyiv, Ukraine

Given a metric space (X, d, μ) with a measure μ, a *domain* in X is an open path-connected set in X. We call a bounded connected set $E \subsetneq \Omega$ an *acceptable* set if $\overline{E} \cap \partial\Omega \neq \varnothing$. We call a sequence $\{E_k\}_{k=1}^{\infty}$ of acceptable sets a chain if it satisfies the following conditions: 1. $E_{k+1} \subset E_k$ for all $k = 1, 2, \ldots,$ 2. $\mathrm{dist}\,(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k) > 0$ for all $k = 1, 2, \ldots,$ 3. The impression $\bigcap\limits_{k=1}^{\infty} \overline{E_k} \subset \partial\Omega$. We say that a chain $\{E_k\}_{k=1}^{\infty}$ divides the chain $\{F_k\}_{k=1}^{\infty}$ if for each k there exists l_k such that $E_{l_k} \subset F_k$. Two chains are equivalent if they divide each other. A collection of all mutually equivalent chains is called an *end* and denoted $[E_k]$, where $\{E_k\}_{k=1}^{\infty}$ is any of the chains in the equivalence class. The impression of $[E_k]$, denoted $I[E_k]$, is defined as the impression of any representative chain. We say that an end $[E_k]$ is a *prime end* if it is not divisible by any other end. The collection of all prime ends is called the *prime end boundary* and is denoted E_Ω. In what follows, we set $\overline{\Omega}_P := \Omega \cup E_\Omega$. Given a family of paths Γ in X, a Borel function $\varrho : X \to [0, \infty]$ is called *admissible* for Γ, abbr. $\varrho \in \operatorname{adm}\Gamma$, if $\int\limits_\gamma \varrho\, ds \geqslant 1$ for all (locally rectifiable) $\gamma \in \Gamma$. We denote by $\Gamma(E, F, G)$ the family of all continuous curves $\gamma : [0, 1] \to X$ such that $\gamma(0) \in E$, $\gamma(1) \in F$, and $\gamma(t) \in G$ for all $t \in (0, 1)$. Everywhere further (X, d, μ) and (X', d', μ') are metric spaces with metrics d and d' and locally finite Borel measures μ and μ', correspondingly. We will assume that μ is a Borel measure such that $0 < \mu(B) < \infty$ for all balls B in X. Given $p \geqslant 1$, the p-modulus of the family Γ is the number $M_p(\Gamma) = \inf\limits_{\varrho \in \operatorname{adm}\Gamma} \int\limits_X \varrho^p(x)\, d\mu(x)$. Let G and G' be domains with finite Hausdorff dimensions α and $\alpha' \geqslant 1$ in X and X', and let $Q : G \to [0, \infty]$ be a measurable function. Given $x_0 \in \partial G$, denote $S_i := S(x_0, r_i)$, $i = 1, 2$, where $0 < r_1 < r_2 < \infty$. We say that a mapping $f : G \to G'$ is a *ring Q-mapping at a point $x_0 \in \partial G$*, if the inequality $M_{\alpha'}(f(\Gamma(S_1, S_2, A))) \leqslant \int\limits_{A \cap G} Q(x)\eta^\alpha(d(x, x_0))\, d\mu(x)$ holds for any ring $A = A(x_0, r_1, r_2) = \{x \in X : r_1 < d(x, x_0) < r_2\}$, $0 < r_1 < r_2 < \infty$, and any measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that $\int\limits_{r_1}^{r_2} \eta(r) dr \geqslant 1$. Given $\delta > 0$, $D \subset X$, a continuum $A \subset D$ and a measurable function $Q : D \to [0, \infty]$, denote $\mathfrak{F}_{Q,\delta,A}(D)$ the family of all ring Q-homeomorphisms $f : D \to X' \setminus K_f$ in D, such that $f(D)$ is some open set in X' and $d'(K_f) = \sup\limits_{x,y \in K_f} d'(x, y) \geqslant \delta$ and $d'(f(A)) \geqslant \delta$, where $K_f \subset X'$ is a continuum.

Theorem. *Let D and $D'_f := f(D)$, $f \in \mathfrak{F}_{Q,\delta,A}(D)$, be domains with finite Hausdorff dimensions α and $\alpha' \geqslant 2$ in spaces (X, d, μ) and (X', d', μ'), respectively, and let X' be a domain with finite Hausdorff dimension $\alpha' \geqslant 2$. Assume that X is complete and supports an α-Poincaré inequality, and that the measure is doubling. Let D be a bounded domain which is finitely connected at the boundary, and let $Q : X \to (0, \infty)$ be a locally integrable function. Assume that, $Q \in FMO(\overline{D})$. If $D'_f := f(D)$ and X' are* *equi-uniform domains over* $f \in \mathfrak{F}_{Q,\delta,A}(D)$ *and* $\overline{D'_f}$ *are compacts in* X', *then* $\mathfrak{F}_{Q,\delta,A}(D)$ *is equicontinuous in* $\overline{D}_P$.

e-mail: esevostyanov2009@gmail.com