On boundary behavior of ring Q-mappings in terms of prime ends

Evgeny Sevost'yanov

Zhytomyr Ivan Franko State University, Zhytomyr, Ukraine
esevostyanov2009@gmail.com

By definition, a chain of cross-cuts $\{\sigma_m\}$ determines a *chain of domains* $d_m \subset D$ such that $\partial\, d_m \cap D \subset \sigma_m$ and $d_1 \supset d_2 \supset \cdots \supset d_m \supset \cdots$. Two chains of cross-cuts $\{\sigma_m\}$ and $\{\sigma_k'\}$ are said to be equivalent if for every $m \in \mathbb{N}$ the domain d_m contains all domains d_k' except finitely many of them, and for every $k \in \mathbb{N}$ the domain d_k' also contains all domains d_m except finitely many. An *end* of a domain D is an equivalence class of chains of cross-cuts of D. We say that an end K is a *prime end* if K contains a chain of cross-cuts $\{\sigma_m\}$, such that $\lim\limits_{m\to\infty} M(\Gamma(C,\sigma_m,D)) = 0$ for some continuum C in D, where M is the modulus of the family $\Gamma(C,\sigma_m,D)$.

We say that the boundary of a domain D in $\mathbb{R}^n$ is *locally quasiconformal* if every point $x_0 \in \partial D$ has a neighborhood U that admits a conformal mapping φ onto the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ such that $\varphi(\partial D \cap U)$ is the intersection of $\mathbb{B}^n$ and a coordinate hyperplane. We say that a bounded domain D in $\mathbb{R}^n$ is *regular* if D can be mapped quasiconformally onto a domain with a locally quasiconformal boundary. If $\overline{D}_P$ is the completion of a regular domain D by its prime ends and g_0 is a quasiconformal mapping of a domain D_0 with locally quasiconformal boundary onto D, then this mapping naturally determines the metric $\rho_0(p_1,p_2) = |\tilde{g_0}^{-1}(p_1) - \tilde{g_0}^{-1}(p_2)|$, where $\tilde{g_0}$ is the extension of g_0 onto $\overline{D_0}$. Let $x = (x_1,\ldots,x_n)$ and $f(x) = (f_1(x),\ldots,f_n(x))$. Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and $f : D \to \mathbb{R}^n$ be a continuous mapping. A mapping $f : D \to \mathbb{R}^n$ is said to be *discrete* if the preimage $f^{-1}(y)$ of every point $y \in \mathbb{R}^n$ consists of isolated points, and *open* if the image of every open set $U \subset D$ is open in $\mathbb{R}^n$. A mapping f is closed if the image of every closed set $U \subset D$ is closed in

$f(D)$. Given a domain D and two sets E and F in $\overline{\mathbb{R}^n}$, $n \geq 2$, $\Gamma(E, F, D)$ denotes the family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ which join E and F in D, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $a<t<b$. Denote

$$S(x_0, r_i) = \{x \in \mathbb{R}^n : |x - x_0| = r_i\}, \quad i = 1, 2,$$

$$A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1<|x - x_0|<r_2\}.$$

Given a (Lebesgue) measurable function $Q : D \to [0, \infty]$ with $Q(x) = 0$ for $x \notin D$, a mapping $f : D \to \mathbb{R}^n$ is called *ring Q-mapping at a point $x_0 \in \overline{D}$* if

$$M(f(\Gamma(S_1, S_2, D))) \leqslant \int\limits_{A(x_0,r_1,r_2)\cap D} Q(x) \cdot \eta^n(|x - x_0|)\, dm(x)$$

for any $A(x_0, r_1, r_2)$, $0<r_1<r_2<r_0$, some $r_0>0$ and for every Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ with $\int\limits_{r_1}^{r_2} \eta(r)dr \geqslant 1$. The following results hold.

Theorem 1. *Let $n \geqslant 2$, let D be a regular domain in $\mathbb{R}^n$, and let D' be a bounded domain with locally quasiconformal boundary. Assume that $f : D \to D'$ is open, discrete and closed ring Q-mapping at ∂D, $D' = f(D)$. Then f has a continuous extension $f : \overline{D}_P \to \overline{D'}_P$ such that $f(\overline{D}_P) = \overline{D'}_P$, whenever*

$$\int\limits_{\varepsilon}^{\varepsilon_0} \frac{dt}{t q_{x_0}^{\frac{1}{n-1}}(t)} < \infty\,, \qquad \int\limits_{0}^{\varepsilon_0} \frac{dt}{t q_{x_0}^{\frac{1}{n-1}}(t)} = \infty \tag{1}$$

for every $x_0 \in \partial D$, where

$$q_{x_0}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{|x-x_0|=r} Q(x)\, d\mathcal{H}^{n-1}.$$

Theorem 2. *The statement of Theorem 1 is true, if instead of the conditions (1) we require that $Q \in FMO(x_0)$ for every $x_0 \in \partial D$.*

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