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Educational Issues
Въпроси на преподаването

TEACHING FUTURE MATHEMATICS TEACHERS TO SOLVE PROBLEMS WITH PARAMETERS: UKRAINIAN EXPERIENCE

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Abstract. The professional development of Mathematics teachers is one of the important areas of the research. We consider the ability to solve problems with parameters as a significant indicator of competence of a teacher of Mathematics. We propose a model of purposeful formation of future teachers of Mathematics of the concept of parameter in the process of studying the discipline "Problems with Parameters", which was created by the authors specifically for this purpose. The research began in 2002 and it involved 290 students from the Faculty of Physics and Mathematics of one of the state Universities in Ukraine. The paper represents a part of the research on learning to solve linear, quadratic and fractional equations with parameters. The authors highlighted typical errors of students during the study of these topics and created a system of means to prevent them.

Keywords: parameters; linear equation with parameters; quadratic equation with parameters; fractional equation with parameters; mathematics student teachers

1. Introduction

Solving problems with parameters has always attracted us with its challenges and, at the same time, with its potential opportunities for the development of mathematical thinking and research skills of both students and teachers (Šedivý 1976, p. 121). On the one hand, the concept of a parameter is an important concept, and the ability to operate with it indicates high-level mathematical competence (Gonda & Tirpáková 2021, p. 179). On the other hand, a parameter is a complex concept. According to the annual official reports of the National Center for Educational Quality Assessment on state certification, on average, about 1% of all high school graduates start solving problems with parameters, and only 0.29% solve them correctly.

In the school course of Mathematics, students encounter parameters (explicitly or implicitly) quite often: in families of equations, in families of inequalities and their systems, in families of functions, in formulas for the nth term of numerical sequences, in some geometric formulas, in geometric problems, in text problems (Ursini & Trigueros 2004, p. 361; Bloedy-Vinner 2001,

p. 178), in problems of Probability Law (Sondore & Daugulis 2022). It should be noted that in our country, the ability to solve problems with parameters is not required by the curriculum in the secondary school (grades 5-9, age 10-14); it is not a program requirement in high school (grades 10-11, age 15-16) for those who study Mathematics at the standard level, but it is a program requirement for those who study at the specialized or advanced level.

The process of forming the concept of a "parameter" of students is long and difficult. This is emphasized in the vast majority of studies related to this concept. One of the reasons for this is the relation of the parameter to unknowns and variables (Furinghetti & Paola 1994; Bloedy-Vinner 1994, p. 91; Bloedy-Vinner 2001, p. 188; Ursini & Trigueros 2004, p. 366; Bardini et al. 2005, p. 135; Postelnicu & Postelnicu 2015, p. 458), i.e., to concepts whose content is also quite complex (Arcavi 1994). It is well known that in Algebra we operate with numbers and letters. Depending on the context, letters can denote (as symbols) and stand for (in terms of meaning) different concepts: generalized numbers, variables, unknown quantities, parameters. In turn, a parameter can assume the functions of a number, a variable, and an unknown quantity. Moreover, the role of a parameter changes in the process of solving a problem that contains a parameter (Bloedy-Vinner 1994; Bloedy-Vinner 2001). This introduces uncertainty into the content and scope of the concept of "parameter" and is the source of numerous mistakes made by students when solving the corresponding exercises. The second reason for the difficulty of forming this concept is that the parameter belongs to the concepts of the highest cognitive level, the ability to work with it requires a high level of algebraic thinking (Godino et al. 2014). In particular, this is due to the fact that the concept of a parameter is a second-order structure. That is, when we operate with the concept of a parameter, we inevitably come to second-order functions (where values are also functions) or second-order quantifiers (Bloedy-Vinner 2001, p. 182). "Identifying parameters is only the tip of the iceberg, the real difficulty lies not with identifying parameters, but with parameters in action, that is to say when solving problems with parameters" (Postelnicu & Postelnicu 2016, p. 453). For example, solving equations with parameters causes significant difficulties for high school students and even for Mathematics teachers, and these difficulties increase if the equation is not presented in a standard form and if the parameters and variables are not used in their usual context (Ilany & Hassidov 2014, p. 967).

Considering the value of forming the concept of a parameter, the studies that have been processed offer methods and means for its effective formation. Let us highlight the following:

1. Explain to pupils (students) the purpose of letters and draw their attention to sources of information to clarify this role. It is also important to show how the roles of parameters change during the solution and emphasize the temporary nature of such roles (Bloedy-Vinner 2001, p. 184). Note that we systematically applied this tool throughout our study and were convinced of its effectiveness.

2. Use practical (real-life) tasks to form the concept of parameter, since tasks with parameters are an integral part of tasks related, for example, to physics, chemistry, economics, finance, writing computer programs (Karlashchuk 2000, p. 67; Gonda & Tirpáková 2021, p. 192). We consider it a powerful tool, however, we have had little experience in its application. The reason is that it is quite difficult to find mathematical modeling tasks (learning tasks) that would reveal the essence of the parameter and would not require deep subject knowledge, for example, in physics or engineering.

3. Parameterize physical problems (Emanovský & Gonda 2018). This is an interesting technique that has a great didactic potential, and not only for physical tasks, in our opinion. Note that we applied it to the parameterisations of problems with economic content. According to our observations, students were always engaged in such activities willingly and with interest.

4. Involve a variety of technological means as effective tools for forming the concept of parameter. These tools are especially actively offered in the last two decades: spreadsheets (Green 2008; Daher 2011), symbolic calculator (Drijvers 2001), graphing calculator (Abramovich & Norton 2006), dynamic geometric environment, based on GeoGebra (Fahlgren 2017). Note that we used GeoGebra in our study. This tool proved to be useful in solving problems with parameters in a graphical way.

Our own experience of teaching Mathematics at school, teaching students at University, conversations with school mathematics teachers, led to the need for purposeful training of students, future teachers of Mathematics, to solve problems with parameters. Teachers' professional development has always been and is now also identified as an important domain of Mathematics education research (Bakker et al. 2021, p. 8).

The purpose of the study was to create a course to teach students to form the concept of a parameter and solve problems with parameters. We set the tasks: 1) to form the course in content, i.e. to select algebraic topics, tasks from which it is expedient to consider; 2) to create a system of tasks with parameters for students to solve in class and for independent work; 3) to select a system of teaching tools for this course and recommendations for their use; 4) to propose a time frame for the course; 5) to compose texts of written works on individual topics, blocks of topics and the course as a whole to determine the level of competence of students to solve tasks with parameters.

2. Materials and Methods

Future teachers of Mathematics from the Faculty of Physics and Mathematics our state university (it is the only higher education institution in our region that trains teachers of mathematics) were involved in this study. A total of 290 students participated in the research, and one group of students participated in the study which, depending on the year of enrollment, ranges from 12 to 30 students. Specifically, in the 2022 – 2023 academic year, there were 20 students, and in the 2023 – 2024 academic year, there were 15 students. Their initial level of mathematical competence was generally low. The exception was, on average, one to two students per group.

At the beginning of the study, in 2002, we created and implemented the author's course, which was aimed at teaching students to solve problems with parameters from different sections of Elementary Mathematics. 32 academic hours were allocated for its study. During the years, while this experimental course was transformed into a compulsory educational component, we formed, first of all, an understanding of the goals of studying this course by students and rational means of achieving these goals. Therefore, significant changes were made in the course content, in the sequence of study of individual topics and in the system of means for teaching students to solve problems with parameters. For example, in the beginning this course included problems with parameters, which covered most of the topics of Algebra (Trigonometry, logarithms, derivative, etc.), but during the next ten years the volume of topics was significantly reduced.

Since 2016, the educational program of the first educational degree (Bachelor's degree) introduced the discipline "Problems with parameters" as a mandatory subject for students of the Faculty of Physics and Mathematics, who are future teachers of Mathematics. During the next seven years, under the influence of the annual analysis of the research results, we changed the total number of hours for studying this discipline and the ratio of the number of hours allocated to classroom and independent work of students. Also, new and improved approaches to the formation of the concept of parameter were introduced.

3. General (annual) research pattern

Stage 1. At the beginning of the study of the discipline "Problems with Parameters", every year, the students took a test that assessed their level of mathematical competence in Elementary mathematics (algebra). The test included questions about their school experience of solving problems with parameters. It is clear that all students, at one level or another, had already formed knowledge (e.g. knowledge of elementary functions, the area of definition of equations, the module, etc.) and skills in Elementary Mathematics (e.g. the ability to open parentheses, to find similar summands, etc.). The students had little or no experience in solving problems with parameters.

Stage 2. These are, in fact, lecture and practical classes. During the study of the discipline "Problems with Parameters" students performed written works: 8 tests (each lasting from 40 to 60 minutes), 2 modular tests (each lasting 1 hour 20 minutes). Each test had from 4 to 8 variants. The number and texts of these papers were the same since 2019. Sometimes we additionally conducted small independent works. Thus, we pursued two main goals. The first one was to check the students' competence in understanding the concept of parameter and the ability to work with it. The second one was to correct and improve elements of the methodology of teaching students to solve problems with parameters. As a result, each time we made a list of typical mistakes of students and analyzed them. After that we came up with a "counteraction" to each item on the list in order to prevent or at least reduce the number of such errors in the future.

Stage 3. At the end of the study of this discipline by the respective group of students, after the examination, we administered Google Forms to the students.

On the basis of a comprehensive analysis of students' questionnaires on the educational component "Problems with Parameters", conversations with students, observation of how students perceive the new material about parameters, how they justify their actions when performing problems with parameters, how they react to the proposed texts of tasks of control works (replies regarding the complexity, time allocated for their performance, etc.), how they get acquainted with the results of written works, analyze the mistakes made, etc., we made our conclusions about the effectiveness of the educational component "Problems with Parameters" (table 1).

Table 1. Data from educational programs on the number of hours in the discipline "Problems with parameters"

Year of SC	Number of hours/ ECTS credits for studying the discipline	Number of hours for lecture classes	Number of hours for practical training	Number of hours for independent work
2016 – 2018	90/3	20	40	60
2019	120/4	20	40	60
2020	120/4	10	30	80
2021 – 2024	120/4	20	44	56

Here are examples of such changes. In the educational program for 2019, the number of credits for studying this discipline was increased from three to four, because the vast majority of students recognized the time factor as an important condition for the formation of the concept of the parameter. In the educational program for 2020, the number of academic hours was reduced.

Explanation. Students in the process of studying the discipline and in written surveys constantly drew our attention to the lack of literature related to the theoretical and practical aspects of solving problems with parameters. Therefore, we wrote and published a manual "Problems with parameters in the school course of mathematics". In our opinion, this manual should have reduced the classroom load and shifted the emphasis on independent work of students. The decision turned out to be wrong, as it turned out after analyzing the results of written work and questionnaires. Therefore, in the educational program for 2021 we practically returned to the previous distribution of hours, even slightly increasing the number of academic hours. We consider this distribution to be optimal for the time being. We would also like to note that in a number of cases, particularly under the influence of external circumstances (e.g. years of COVID-19 epidemic), the means of studying "Problems with parameters" were adjusted in the process of its study.

Now the content of the discipline "Problems with Parameters", according to the curriculum, is divided into two modules (table 2), it is studied by students in the third year, during one semester (in the sixth semester), its final control is a written exam.

This paper presents a part of the study related to the study of the first three topics of the first module of the curriculum of the discipline "Problems with Parameters": "Linear Equations with Parameters", 'Quadratic Equations with Parameters', 'Fractional Equations with Parameters'.

The research questions for this part of the study were as follows:

1) Is it possible to algorithmize students' activities in solving problems on these topics?

2) Is the analytical method of solving problems from this unit the main method?

3) Is it advisable to use the graphical method to solve problems on certain topics?

4) Is there a correlation between the level of skills and knowledge in elementary mathematics and the ability of students to solve problems with parameters from this unit?

5) Is there a correlation between students' ability to solve problems with parameters from this unit and their performance in the discipline "Problems with parameters" in general?

Table 2. Distribution of academic hours in the discipline "Problems with parameters" in the academic year 2023/2024

Content		Class hours
Mod. 1.	*Rational equations and inequalities with parameters*	34
Topic 1.	Linear equations with parameters	4
Topic 2.	Quadratic equations with parameters	4
Topic 3	Fractional equations with parameters	4
Topic 4.	Linear inequalities with parameters	4
Topic 5.	Quadratic inequalities with parameters	6
Topic 6.	Fractional inequalities with parameters	6
Topic 7.	Exercises with parameters that are related to the location of the roots of a quadratic trinomial	6
Mod. 2.	*Systems of equations and inequalities with parameters. Equations and inequalities with modules*	30
Topic 8.	Systems of linear and nonlinear equations with parameters	6
Topic 9.	Systems of linear and nonlinear inequalities with parameters	6
Topic 10.	Equations with modules and parameters	6
Topic 11.	Inequalities with modules and a parameter	6
Topic 12.	Equations and inequalities with parameters from different sections of elementary mathematics	6
	Total	64

6) Can it be argued that students' competence in this block of topics is decisive for students' ability to solve equations with parameters in various topics of elementary mathematics?

4. Results

By the beginning of studying the first block of topics of this discipline (these are "Linear equations with parameters", "Quadratic equations with parameters", "Fractional equations with parameters") all students without exception knew how to solve linear, quadratic and fractional equations without a parameter. Despite the fact that the groups of students who studied the discipline "Problems with parameters" annually differed in the level of mathematical competence, in the number of people in the group, however, the same set of errors regarding the ability to solve problems related to the topics defined above was always clearly observed. The frequency of these errors decreased every year, but it was not possible to achieve that students did not make them at all.

We identified the following types of difficulties and mistakes that students made when solving linear, quadratic, and fractional equations with parameters.

Type 1: They solve with respect to a parameter, not with respect to a variable.

Type 2: Do not see the structure of a linear or quadratic equation.

Type 3: They do not take into account that when dividing by a parameter or by an expression with a parameter in equations, it is necessary to analyze cases when the expression is equal to zero and not equal to zero.

Type 4. Substitute individual values of the parameter, then solve without the parameter. And, most often, zero is substituted.

Type 5. They do not consider the case of linearity in the equation, which can be both linear and quadratic depending on the parameter.

Type 6. They do not find out at what values of the parameter the discriminant is positive, negative or equal to zero, but simply declare these cases.

Type 7. Represent the root of the equation through a parameter or through an expression with a parameter, not a specific number. For example, if a quadratic equation considers the case when the discriminant is zero.

Type 8. It is not determined for which values of the parameter the roots of a fractional equation belong to the corresponding domain of the equation.

Type 9. They cannot write down the correct answer, although they have solved the problem correctly, because they cannot "assemble" the answers found at different stages of the solution.

Type 10: They do not where the difference between the situations is when the equation with a parameter has no solutions and where it does not make sense (when the problem is not defined at all).

Type 11: They cannot figure out how to solve the equation if it is necessary to find the values of the parameter at which a certain condition or conditions are met, and not just a requirement to solve the equation for all values of the parameter.

Type 12: They make computational errors.

In the course of the study, we have selected and created a system of tools that we recommend to use to overcome difficulties and avoid mistakes in the process of forming the concept of a parameter.

Method 1: Use capital letters A, B, C to represent the coefficients in a linear and quadratic equation.

This tool is effective. This is evidenced by the results of students' written work and questionnaires. With the help of this tool, most students quickly learned to see the structure of a linear or quadratic equation with a parameter (similar to what they did when they learned to solve the corresponding equations without a parameter).

Method 2. Use an axis (we call it the "parameter axis" or "parameter line") to indicate the solutions that have been found at different stages of the solution.

The parameter axis is a line (similar to a coordinate line) on which the solutions found are labeled depending on the values of the parameter. It should also be added that in some types of exercises that students will be introduced to during the course of the course "Problems with Parameters", the parameter line acts in a different capacity - as an intermediate but important link in the solution – e.g. when solving equations with modules, systems of inequalities, sets of inequalities, etc.

Method 3. Pay attention to the domain of the parameter when solving equations.

A parameter in an equation may be subject to certain restrictions due to the valid values it can take. That is, the parameter cannot take all real values, but only valid values if the parameter in an equation that is linear or reducible to a linear equation is contained in the denominator of a fraction, for example, $\frac{x}{a}+\frac{a}{3}+\frac{x+a}{a+3}=1$, under the sign of the square root, $(2-\sqrt{a})\,x=1$ for example, under the sign of the logarithm, for example, $(2-\log_a 25)\,x=-3$, etc. In the above examples, these are the following constraints, respectively: $a\neq 0$ and $a\neq -3$; $a\in[0;+\infty)$; $a\ \in(0;1)\cup(1;+\infty)$.

For those values of the parameter when the problem is undefined, in the answer is advisable to write: "the problem is undefined" or "the problem has no meaning".

Method 4. Pay attention to the form of writing the answer depending on the requirements of the problem.

If the problem requirement sounds like: "Solve the equation for each value of the parameter", then the answer is given as a compound sentence: "If ____, then ___". If the task requires finding the values of a parameter at which a certain condition or conditions are fulfilled, the answer should contain only the corresponding values of the parameter.

Method 5. Demonstrate the "controlling role" of the parameter in different types of equations. Show the parameter as a second-order function.

For example, the equation $(2a-1)x^2+x+a+5=0$ when $a\neq 0.5$ is a quadratic equation with the variable x and the parameter a, where its coefficients are $A=2a-1, B=1, C=a+5$; when $a=0.5$ it is a linear equation.

For example, $ax = 5$ is a linear equation with the variable x and the parameter a. Depending on the values of the parameter, we get new equations. We represent this in different forms: using symbols (for example, if $a = -9$, then $-9x = 5$; if $a = \pi$, then $\pi x = 5$ or $f(a; ax = 5)$; $f(1; x = 5)$; $f(-9; -9x = 5)$; $f(1.5; 1.5x = 5)$) or a table (table 3).

Table 3

a	1	-9	1.5	0	1/3	π
$ax = 5$	$x = 5$	$-9x = 5$	$1.5x = 5$	$0x = 5$	$\frac{1}{3}x = 5$	$\pi x = 5$

Method 6. Use didactic materials in the form of theoretical instructions to guide students in the process of solving rational equations. These tools are also convenient to use during online learning. For this purpose, we have created and prescribed:

1) an indicative plan for solving a linear equation;
2) an approximate plan for solving a quadratic equation;
3) an approximate plan for solving a fractional equation;
4) an algorithm that is convenient to use at the stage of systematization and generalization of knowledge on this topic.

Method 7. Use didactic materials in the form of practical instructions and hints for students:

1) samples of ready-made complete solutions to basic tasks with detailed explanations of all steps;
2) forms of solved basic tasks with blanks that should be filled in.

Method 8. Use the graphical method of solving equations in the xOy coordinate system, and especially in the xOa coordinate system.

Students can draw graphs both by hand and using the GeoGebra software. It is clear that the graphical method is not very effective for solving linear equations and quadratic equations. Its use often requires a lot of time. However, we have seen that this activity is very useful for forming the concept of a parameter. It also helps to use the graphical method more freely and easily when solving more complex problems with a parameter in other areas of mathematics.

We will show the results of the study related to the topics "Linear Equations with Parameters", "Quadratic equations with parameters", "Fractional equations with parameters" through the prism of students' study of the discipline "Problems with parameters" in the academic year 2023 – 2024. The group consisted of 15 students (14 girls and 1 boy).

Stage 1. Data on the level of knowledge and skills in elementary mathematics of students are presented on fig. 1.

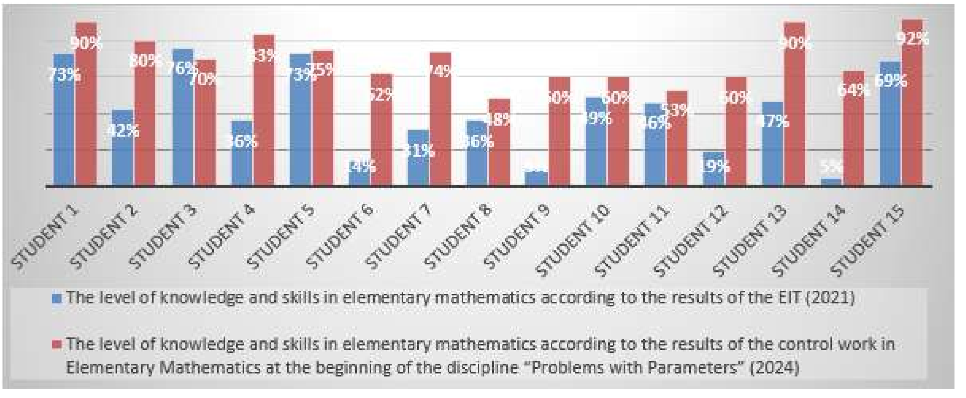

Figure 1

School experience in solving problems with parameters was little (fig. 2).

Figure 2

Stage 2. At the end of the study of the topics "Linear equations with parameters", "Quadratic equations with parameters", and "Fractional equations with parameters," students wrote a final test lasting 40 minutes with the following

Task: (Variant 1) Solve for all values of the parameter a the equations: 1) $a^2x + 1 = x + a$; 2) $(a - 5)x^2 + 3ax - (a - 5) = 0$; 3) $x + 1/a = a + 1/x$; for all values of the patameter a .

The results of the students are presented in table 4.

Table 4. Results of the control work on the topic "Rational Equations with Parameters"

Number of the student who com- pleted the work	Number of points for the entire test Max 50 (100%)	Number of points for solving a linear equation with parameters Max 10 (20%)	Number of points for solving a quadratic equation with parameters Max 15 (30%)	Number of points for solving a fractional equation with parameters Max 25 (50%)
1	45 (90%)	10 (100%)	15 (100%)	20 (80%)
2	40 (80%)	10 (100%)	13 (86.67%)	17 (68%)
3	34 (68%)	10 (100%)	15 (100%)	9 (36%)
4	35 (70%)	2 (20%)	15 (100%)	18 (72%)
5	39 (78%)	10 (100%)	15 (100%)	14 (56%)
6	25 (50%)	10 (100%)	5 (33.33%)	10 (40%)
7	42 (84%)	10 (100%)	15 (100%)	17 (68%)
8	17.5 (35%)	7 (70%)	8 (53.33%)	2.5 (10%)
9	17.5 (35%)	8 (80%)	6.5 (43.33%)	2.5 (10%)
10	17.5 (35%)	5.5 (55%)	8 (53.33%)	4 (16%)
11	35 (70%)	10 (100%)	15 (100%)	10 (40%)
12	17.5 (35%)	10 (100%)	5 (33.33%)	2.5 (10%)
13	40 (80%)	10 (100%)	15 (100%)	15 (60%)
14	30 (60%)	5 (50%)	15 (100%)	10 (40%)
15	48 (96%)	10 (100%)	1 (100%)	23 (92%)

Let's show how some typical mistakes are manifested in specific student works during this test.

Student A understands, according to her, how to solve equations on the topic in general, but makes mistakes when solving them. Despite the fact that the task clearly stated that a was a parameter, the student mistakenly identified the equation as linear and solved it accordingly (fig. 3).

Moreover, she also did not analyze whether the expression into which she divided both parts of the equation could be equal to zero. She made a similar mistake when solving another equation in the same writing (fig. 4).

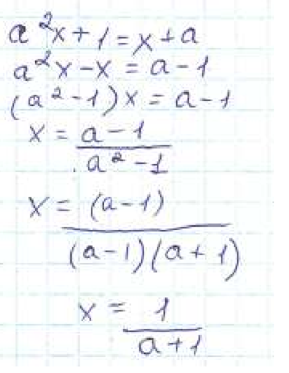

Figure 3 **Figure 4**

Analyzing the level of students' competence in the topic "Rational equations with parameters", we noticed the connection between these results and their level of competence in elementary mathematics at the beginning of the discipline "Problems with Parameters", as well as the connection with the results of studying this discipline in general. Let's look at this in more detail.

The graph on fig. 5 shows the correlation between the levels of competence in elementary mathematics based on the results of university studies (2024) and the levels of students' competence in the topic "Rational Equations with Parameters" (2024).

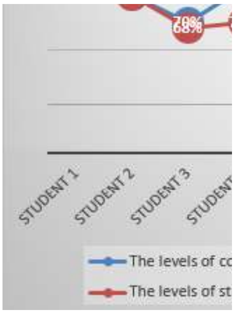

Figure 5

Table 5. Spearman's rank coefficient comparing the ordered levels of competence in Elementary mathematics based on the results of University studies (2024) – Series A, and competence in the topic "Rational equations with parameters" (2024) – Series B

N	Rank Series A	Rank	Rank Series B	Rank	d	d^2
1	90%	13.5	90%	14	0.5	0.25
2	80%	11	80%	11.5	-0.5	0.25
3	70%	8	68%	7	1	1
4	83%	12	70%	8.5	3.5	12.25
5	75%	10	74%	10	0	0
6	62%	6	50%	5	1	1
7	74%	9	84%	13	−4	16
8	48%	1	35%	2.5	−1.5	2.25
9	60%	4	35%	2.5	1.5	2.25
10	60%	4	35%	2.5	1.5	2.25
11	53%	2	70%	8.5	−6.5	42.25
12	60%	4	35%	2.5	1.5	2.25
13	90%	13.5	80%	11.5	2	4
14	64%	7	60%	6	1	1
15	92%	15	98%	15	0	0
						Σ 87

We will use Spearman's test to statistically verify this correlation (Kartashov 2008, p. 406; Bosniuk 2020, p. 59). Let's formulate the following hypotheses:

H_0 – the correlation between the ordered values of students' levels of competence in Elementary mathematics based on the results of university studies (2024) and the levels of students' competence in the topic "Rational equations with parameters" (2024) does not differ from zero;

H_1 – the correlation between the ordered values of students' levels of competence in Elementary mathematics based on the results of university studies (2024) and the levels of students' competence in the topic "Rational equations with parameters" (2024) is statistically significantly different from zero.

All calculations related to the calculation and squaring of the differences between the ranks of different levels of students' competence are presented in table 5.

Since there are identical groups of ranks in the rank series being compared, corrections for identical ranks must be made before calculating the rank correlation coefficients $T_a = \frac{\sum(a^3-a)}{12}$ and $T_b = \frac{\sum(b^3-b)}{12}$, where a is the volume of each group of equal ranks in the rank series A and b is the volume of each group of equal ranks in the rank series B. Here:

$$T_a = \frac{(3^3-3) + (2^3-2)}{12} = 2.5; T_b = \frac{(4^3-4) + (2^3-2) + (2^3-2)}{12} = 6.$$

We determine the empirical value of r_s by the formula:

$$r_{s\ \exp} = 1 - \frac{6\sum(d^2) + T_a + T_b}{N(N^2-1)}.$$

Here:

$$r_{sexp} = 1 - \frac{6 \cdot 87 + 2.5 + 6}{15(15^2-1)} = 1 - \frac{522 + 8.5}{15 \cdot 224} = 1 - \frac{530.5}{3360} = 0.8421$$

According to the tables in (Bosniuk 2020, p. 130), we determine the critical values of r_s for $N = 15$:

$$r_s = \begin{cases} 0.514 \ (\rho \leq 0.05), \\ 0.641 \ (\rho \leq 0.01) \end{cases}$$

$$r_{s\ \exp} > r_s(\rho \leq 0.01)$$

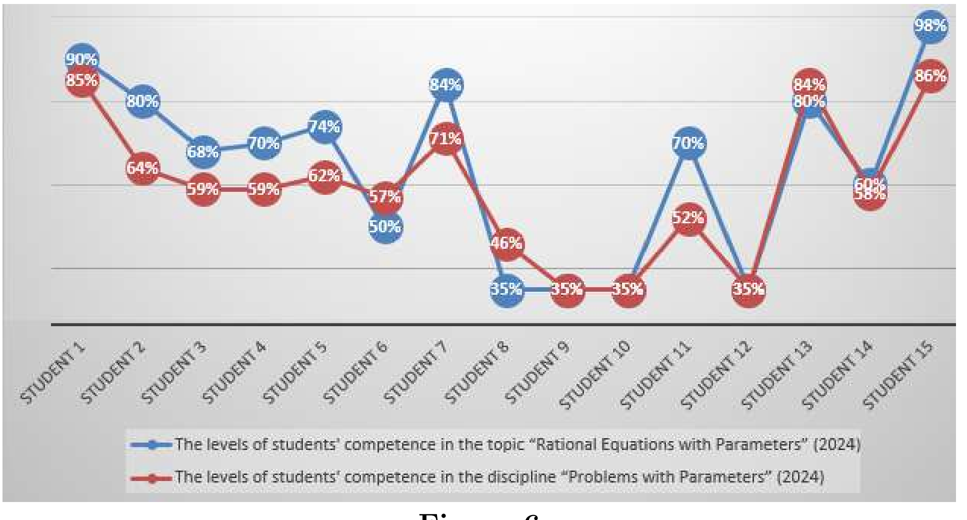

Figure 6

Therefore, the hypothesis H_0 is rejected. The hypothesis H_1 is accepted. The correlation between the ordered values of students' levels of competence in Elementary mathematics based on the results of university studies (2024) and the levels of students' competence in the topic "Rational equations with parameters" (2024) is statistically significant ($\rho \leq 0,01$) and is positive.

The graph on fig. 6 shows the correlation between students ability to solve problems with parameters from this block and their performance in the discipline "Problems with parameters" in general.

Table 6. Spearman's rank coefficient comparing levels of competence in the topic "Rational equations with parameters" – Series A, and competence in the discipline "Problems with parameters" (2024) – Series B

N	Rank Series A	Rank	Rank Series B	Rank	d	d^2
1	90%	14	85%	14	0	0
2	80%	11.5	64%	11	0.5	0.25
3	68%	7	59%	3.5	3.5	12.25
4	70%	8.5	59%	3.5	5	25
5	74%	10	62%	10	0	0
6	50%	5	57%	6	−1	1
7	84%	13	71%	12	1	1
8	35%	2.5	46%	4	−1.5	2.25
9	35%	2.5	35%	2	0.5	0.25
10	35%	2.5	35%	2	0.5	0.25
11	70%	8.5	52%	5	3.5	12.25
12	35%	2.5	35%	2	0.5	0.25
13	80%	11.5	84%	13	−1.5	2.25
14	60%	6	58%	7	−1	1
15	98%	15	86%	15	0	0
						Σ 58

Let's use Spearman's test again to statistically verify this correlation. Let's formulate the following hypotheses:

H_0 – the correlation between the ordered values of the levels of students' competence in the topic "Rational equations with parameters" (2024) and the levels of students' competence in the discipline "Problems with parameters" (2024) does not differ from zero;

H_1 – the correlation between the ordered values of the levels of students' competence in the topic "Rational equations with parameters" (2024) and the levels of students' competence in the discipline "Problems with parameters" (2024) is statistically significantly different from zero.

All calculations are presented in table 6.

Since there are identical groups of ranks in the rank series being compared, corrections for identical ranks must be made before calculating the rank correlation coefficients $T_a = \frac{\sum (a^3 - a)}{12}$ and $T_b = \frac{\sum (b^3 - b)}{12}$, where a is the volume of each group of equal ranks in the rank series A and b is the volume of each group of equal ranks in the rank series B.

Here:

$$T_a = \frac{\left(4^3 - 4\right) + \ \left(2^3 - 2\right) + \left(2^3 - 2\right)}{12} = 6, \quad T_b = \frac{3^3 - 3}{12} = 2.$$

We determine the empirical value of r_s by the formula:

$$r_{s\ \text{exp}} = 1 - \frac{6 \sum \left(d^2\right) + T_a + T_b}{N(N^2 - 1)}.$$

Here:

$$r_{sexp} = 1 - \frac{6 \cdot 58 + 6 + 2}{15(15^2 - 1)} = 1 - \frac{348 + 8}{15 \cdot 224} = 1 - \frac{356}{3360} = 0.94.$$

According to the tables in (Bosniuk 2020, p. 130), we determine the critical values of r_s for $N = 15$:

$$r_s = \begin{cases} 0.514 \ (\rho \leq 0.05), \\ 0.641 \ (\rho \leq 0.01) \end{cases}$$

$$r_{s\ \text{exp}} > r_s(\rho \leq 0.01)$$

Therefore, hypothesis H_0 is rejected and Hypothesis H_1 is accepted. The correlation between the ordered values of students' competence levels in the topic "Rational equations with parameters" (2024) and the levels of students' competence in the discipline "Problems with parameters" (2024) is statistically significant ($\rho \leq 0.01$) and is positive.

Stage 3. Let us present some results of the students' questionnaire at the end of the course "Problems with parameters", which complement the idea of the study and its individual results.

Students indicated which tools and methods of learning to solve problems with parameters they preferred (fig. 7).

In particular, traditionally, means 1 and 2 from the list of means we presented in our article are highly appreciated (we received such feedback every year). As usual, all students highly appreciated the opportunities of the discipline "Problems with parameters" for their professional development (fig. 8, fig. 10) and emphasized the influence of their own skills in Elementary mathematics on the ability to solve problems with parameters (fig. 9).

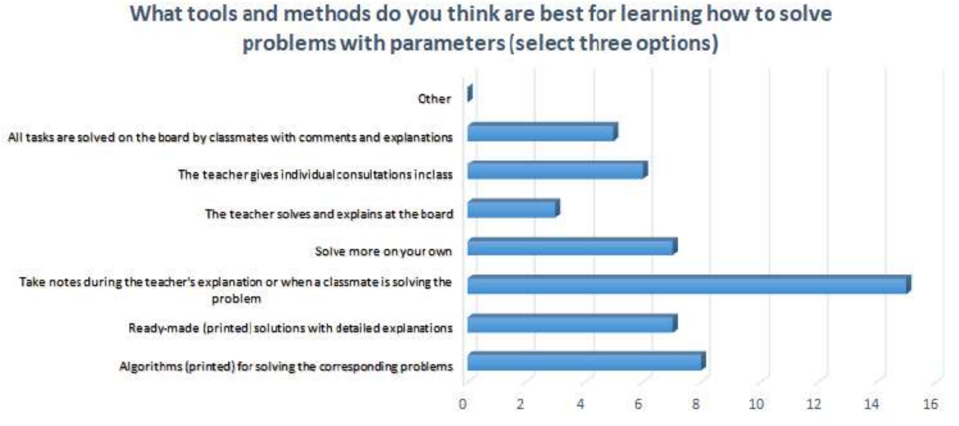

Figure 7

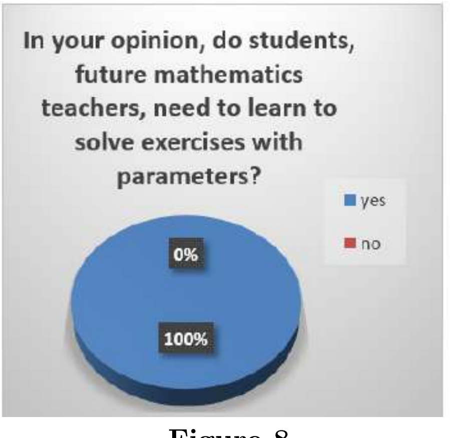

Figure 8

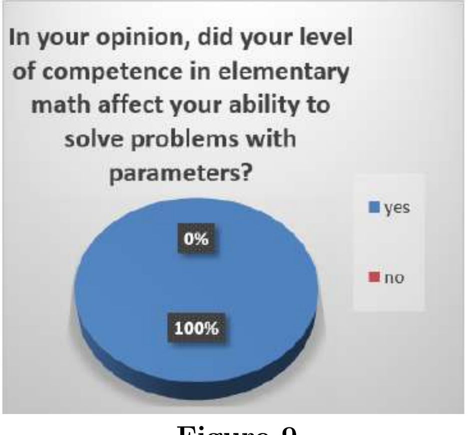

Figure 9

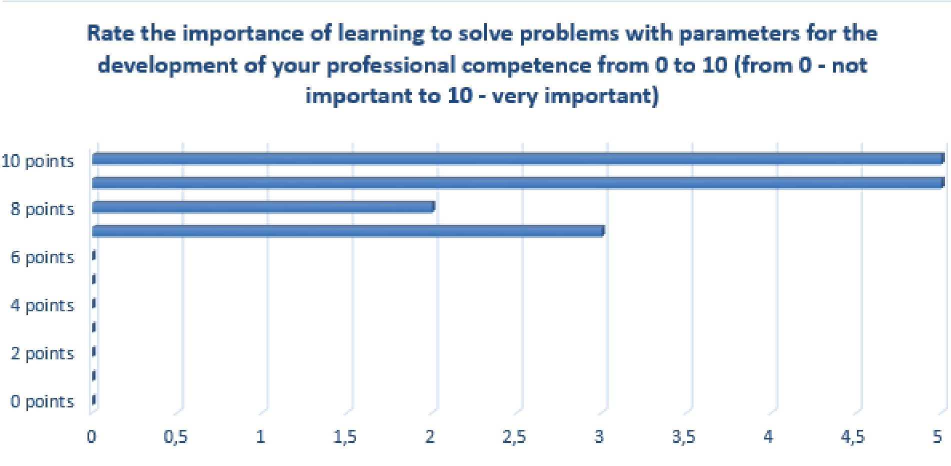

Figure 10

4. Discussions & conclusions

Let's proceed to the answers to the research questions on the study of the first block of topics of the curriculum of the discipline "Problems with parameters": "Linear equations with parameters", "Quadratic equations with parameters", "Fractional equations with parameters".

Regarding 1. We believe that the results of the study allow us to state that it is possible to algorithmize the students' activity of solving problems that are related only to the topics: "Linear equations with parameters", "Quadratic equations with parameters".

Regarding 2. Analytical method of problem solving is the main method for problems from this block. It allows reducing the difficulties in the perception and learning to solve linear, quadratic and fractional equations with parameters and in the process of forming the concept of parameter, because it quite naturally "repeats" the basic steps of solving these equations without a parameter.

Regarding 3. As a result of the study, we were convinced that the graphical method for solving problems of certain topics is advisable to use at the stage when students have generally formed tentative plans for solving linear, quadratic and fractional equations with parameters. The graphical method helps to form the idea that the parameter not only can take on the role of a number or an unknown quantity, but also the role of a variable quantity. Particularly valuable in this respect was the application of the graphical method in the variable-parameter coordinate system (xOa).

Regarding 4 and 5. The correlation between the level of skills and knowledge in Elementary mathematics and the students' ability to solve problems with parameters from this block and the correlation between the students' ability to solve problems with parameters from this block and their academic performance in the discipline "Problems with parameters" in general is statistically proved.

Regarding 6. The question of whether the students' competence in this block of topics is determinant for students' ability to solve equations with parameters from different topics of Elementary Mathematics requires further research. At present, we are inclined to the negative answer. Most likely, this is only one of the conditions. However, students' competence of the topics defined above is the foundation for studying the following topics and forming the concept of a parameter.

We should also note that the study did not have the opportunity to separate experimental and control groups, since for a long time our faculty has been recruiting only one group of students each year who decided to become math teachers. This profession is not in demand among young people, although the shortage of teachers, in particular, of math, is currently very high. Of course, it would be useful and interesting to compare the experience and results of other researchers in this area. At the moment, we have not been able to find information about such studies.

In our opinion, the educational component "Problems with parameters" created by us can be effectively used to teach students to solve problems with parameters. Moreover, it is quite possible to use its individual parts as separate modules during the study of educational components already existing in the curriculum. For example, to the educational component "Elementary mathematics", add the module "Rational equations with parameters". We are ready to provide the learning tools we have created for this purpose (a three-level system of tasks with answers to them; texts of tests, instructions, etc.).

Our research is complete, but we are going to test certain topics and blocks of the created course in schools. Also, starting in 2021, we have started working with Math teachers over professional development courses to familiarize them with our work.

In conclusion, during the years of the pandemic, the issue of solving problems with parameters has receded into the background, one might even say, disappeared from the attention of scientists, researchers, and practicing math teachers. This is understandable, as education in general and mathematics in particular have suffered considerable educational losses during the pandemic. Over time, education has faced new challenges. Nevertheless, we believe that research related to the formation of such a deep multi-stage concept as a parameter should not be abandoned, but rather highlighted and cultivated. The process of solving problems with parameters, in our opinion, is potentially creative and developing for any personality

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