On boundary behavior of unclosed mappings with moduli inequality

Evgeny Sevost'yanov
(Zhytomyr Ivan Franko State University; Institute of Applied Mathematics and Mechanics, Slov'yans'k)
E-mail: esevostyanov2009@gmail.com

Victoria Desyatka
(Zhytomyr Ivan Franko State University)
E-mail: victoriazehrer@gmail.com

Let D be a domain in $\overline{\mathbb{R}^n}$ and let $b \in \partial D$. Then D has property P_1 at b if the following condition is satisfied: If E and F are connected subsets of D such that $b \in \overline{E} \cup \overline{F}$, then $M(\Gamma(E, F, D)) = \infty$, where M denotes the (conformal) modulus of families of paths in $\mathbb{R}^n$ (see the definition below), and $\Gamma(E, F, D)$ is a family of paths joining E and F in D (see e.g. [1, Definition 17.5]). The following results hold.

Theorem A. *Suppose that $f : D \to D'$ is a quasiconformal mapping and that D has property P_1 at $\in \partial D$. Then $C(f, b)$ contains at most one point at which D' is finitely connected (see* [1, Theorem 17.13]*).*

Theorem B. *Let $f : D \to \mathbb{R}^n$ be quasiregular mapping with $C(f, \partial D) \subset \partial f(D)$. If D is locally connected at a point $b \in \partial D$ and $D' = f(D)$ is qc accessible at some point $y \in C(f, b)$, then $C(f, b) = \{y\}$ (see e.g.* [2, Theorem 4.2]*, cf.* [3, Theorem 4.2]*).*

We give some generalization of Theorems **A** and **B**. Recall some definitions. A Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for the family Γ of paths γ in $\mathbb{R}^n$, if the relation $\int\limits_{\gamma} \rho(x)\, |dx| \geqslant 1$ holds for all (locally rectifiable) paths $\gamma \in \Gamma$. In this case, we write: $\rho \in \mathrm{adm}\, \Gamma$. Let $p \geqslant 1$, then *p-modulus* of Γ is defined by the equality $M_p(\Gamma) = \inf\limits_{\rho \in \mathrm{adm}\, \Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x)\, dm(x)$. Let $x_0 \in \mathbb{R}^n$, $0 < r_1 < r_2 < \infty$,

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}\,, \quad B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\} \qquad (1)$$

and $A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$. Let $S_i = S(x_0, r_i)$, $i = 1, 2$, where spheres $S(x_0, r_i)$ centered at x_0 of the radius r_i are defined in (1). Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function satisfying the condition $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus D$. Let $p \geqslant 1$. Due to [4], a mapping $f : D \to \overline{\mathbb{R}^n}$ is called a *ring Q-mapping at the point $x_0 \in \overline{D} \setminus \{\infty\}$ with respect to p-modulus*, if the condition

$$M_p(f(\Gamma(S_1, S_2, D))) \leqslant \int\limits_{A \cap D} Q(x) \cdot \eta^p(|x - x_0|)\, dm(x) \qquad (2)$$

holds for some $r_0(x_0)>0$, all $0<r_1<r_2<r_0$ and all Lebesgue measurable functions $\eta:(r_1,r_2)\rightarrow [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2}\eta(r)\,dr\geqslant 1\,. \tag{3}$$

Recall that a mapping $f:D\rightarrow {\Bbb R}^n$ is called *discrete* if the pre-image $\{f^{-1}\left(y\right)\}$ of each point $y\,\in\,{\Bbb R}^n$ consists of isolated points, and *is open* if the image of any open set $U\subset D$ is an open set in ${\Bbb R}^n$. Later, in the extended space $\overline{{\Bbb R}^n}={\Bbb R}^n\cup\{\infty\}$ we use the *spherical (chordal) metric* $h(x,y)=|\pi(x)-\pi(y)|$, where π is a stereographic projection $\overline{{\Bbb R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1},\frac{1}{2})$ in ${\Bbb R}^{n+1}$, namely,

$$h(x,\infty)=\frac{1}{\sqrt{1+{|x|}^2}}\,,\quad h(x,y)=\frac{|x-y|}{\sqrt{1+{|x|}^2} \sqrt{1+{|y|}^2}}\,, \quad x\ne \infty\ne y$$

(see [1, Definition 12.1]). Further, the closure $\overline{A}$ and the boundary ∂A of the set $A\subset\overline{{\Bbb R}^n}$ we understand relative to the chordal metric h in $\overline{{\Bbb R}^n}$. Given a mapping $f:D\,\rightarrow\,{\Bbb R}^n$, we denote $C(f, x):=\{y\in \overline{{\Bbb R}^n}:\exists\,x_k\in D: x_k\rightarrow x, f(x_k)\rightarrow y, k\rightarrow\infty\}$ and $C(f, \partial D)=\bigcup\limits_{x\in \partial D}C(f, x)\,.$ In what follows, ${\rm Int\,}A$ denotes the set of inner points of the set $A\subset \overline{{\Bbb R}^n}$. Recall that the set $U\subset\overline{{\Bbb R}^n}$ is neighborhood of the point z_0, if $z_0\in {\rm Int\,}A$. Due to [4], we say that a function ${\varphi}:D\rightarrow{\Bbb R}$ has a *finite mean oscillation* at a point $x_0\in D$, write $\varphi\in FMO(x_0)$, if $\limsup\limits_{\varepsilon\rightarrow 0}\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0,\,\varepsilon)} |{\varphi}(x)-\overline{{\varphi}}_{\varepsilon}|\ dm(x)<\infty$, where $\overline{{\varphi}}_{\varepsilon}=\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0,\,\varepsilon)} {\varphi}(x) dm(x)\,.$ Let $Q:{\Bbb R}^n\rightarrow [0,\infty]$ be a Lebesgue measurable function. We set $Q^{\,\prime}(x)=\left\{\begin{array}{rr} Q(x), & Q(x)\geqslant 1\,,\\ 1, & Q(x)<1\,. \end{array} \right.$ Denote by $q^{\,\prime}_{x_0}$ the mean value of $Q^{\,\prime}(x)$ over the sphere $|x-x_0|=r$, that means,

$$q^{\,\prime}_{x_0}(r):=\frac{1}{\omega_{n-1}r^{n-1}}\int\limits_{|x-x_0|=r}Q^{\,\prime}(x)\,d\mathcal{H}^{n-1}\,. \tag{4}$$

Note that, using the inversion $\psi(x)=\frac{x}{|x|^2}$, we may give the definition of FMO as well as the quantity in (4) for $x_0=\infty$. We say that the boundary ∂D of a domain D in ${\Bbb R}^n$, $n\geqslant 2$, is *strongly accessible at a point $x_0\in \partial D$ with respect to the p-modulus* if for each neighborhood U of x_0 there exist a compact set $E\subset D$, a neighborhood $V\subset U$ of x_0 and $\delta>0$ such that $M_p(\Gamma(E,F, D))\geqslant \delta$ for each continuum F in D that intersects ∂U and ∂V.

Theorem 1. *([5]). Let $p\geqslant 1$, let D and $D^{\,\prime}$ be domains in ${\Bbb R}^n$, $n\geqslant 2$, $f:D\rightarrow D^{\,\prime}$ be an open discrete mapping satisfying relations (2)–(3) at the point $b\in \partial D$ with respect to p-modulus, $f(D)=D^{\,\prime}$. In addition, assume that 1) the set $E:=f^{\,-1}(C(f, \partial D))$ is nowhere dense in D and D is finitely connected on E, i.e., for any $z_0\in E$ and any neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V}\subset \widetilde{U}$ of z_0 such that $(D\cap \widetilde{V})\setminus E$ consists of finite number of components; 2) for any neighborhood U of b there is a neighborhood $V\subset U$ of b such that: 2a) $V\cap D$ is connected, 2b) $(V\cap D)\setminus E$ consists at most of m components, $1\leqslant m<\infty$, 3) $D^{\,\prime}\setminus C(f, \partial D)$ consists of finite components, each of them has a strongly accessible boundary with respect to p-modulus. Suppose that at least one of the following conditions is satisfied: 4_1) a function Q has a finite mean oscillation at the* *point* b; $4_2)$ $q_b(r)=O\left(\left[\log\frac{1}{r}\right]^{n-1}\right)$ *as* $r\rightarrow 0$; $4_3)$ *the condition* $\int\limits_{0}^{\delta(b)}\frac{dt}{t^{\frac{n-1}{p-1}}q_b^{\prime\,\frac{1}{p-1}}(t)}=\infty$ *holds for some* $\delta(b)>0$. *Then* f *has a continuous extension to* b.

If the above is true for any point $b\in\partial D$, *the mapping* f *has a continuous extension* $\overline{f}:\overline{D}\rightarrow\overline{D^{\prime}}$, *moreover,* $\overline{f}(\overline{D})=\overline{D^{\prime}}$.

References

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