# On the prime ends extension of unclosed inverse mappings

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The following statements contain itself some results on prime end boundary extension of quasiconformal mappings.

**Theorem A.** *Under a quasiconformal mapping $f$ of a collared domain $D_0$ onto a domain $D$, there exists a one-to-one correspondence between the boundary points of $D_0$ and the prime ends of $D$. Moreover, the cluster set $C(f,b)$, $b\in\partial D_0$, coincides with the impression $I(P)$ of the corresponding prime end $P$ of $D$ (see* [1, Theorem 4.1]*).*

Given $f:D\rightarrow D'$, we set $C(f,x):=\{y\in\overline{\mathbb{R}^n}: \exists\, x_k\in D: x_k\rightarrow x, f(x_k)\rightarrow y, k\rightarrow\infty\}$ and $C(f,\partial D)=\bigcup\limits_{x\in\partial D}C(f,x)\,.$

**Theorem B.** *Let $f:D\rightarrow\mathbb{R}^n$ be quasiregular mapping with $C(f,\partial D)\subset\partial f(D)$. If $D$ is locally connected at a point $b\in\partial D$ and $D'=f(D)$ is qc accessible at some point $y\in C(f,b)$, then $C(f,b)=\{y\}$ (see* [2, Theorem 4.2]*).*

The goal of this abstract is to consider mappings which are not closed. Let $y_0\in\mathbb{R}^n$, $0<r_1<r_2<\infty$ and $A=A(y_0,r_1,r_2)=\left\{y\in\mathbb{R}^n: r_1<|y-y_0|<r_2\right\}$. Given sets $E$, $F\subset\overline{\mathbb{R}^n}$ and a domain $D\subset\mathbb{R}^n$ we denote by $\Gamma(E,F,D)$ a family of all paths $\gamma:[a,b]\rightarrow\overline{\mathbb{R}^n}$ such that $\gamma(a)\in E,\gamma(b)\in F$ and $\gamma(t)\in D$ for $t\in(a,b)$. If $f:D\rightarrow\mathbb{R}^n$, $y_0\in f(D)$ and $0<r_1<r_2<d_0=\sup\limits_{y\in f(D)}|y-y_0|$, then by $\Gamma_f(y_0,r_1,r_2)$ we denote the family of all paths $\gamma$ in $D$ such that $f(\gamma)\in\Gamma(S(y_0,r_1),S(y_0,r_2),A(y_0,r_1,r_2))$. Let $Q:\mathbb{R}^n\rightarrow[0,\infty]$ be a Lebesgue measurable function. We say that $f$ *satisfies Poletsky inverse inequality* at the point $y_0\in f(D)$, if the relation

$$M(\Gamma_f(y_0,r_1,r_2))\leqslant\int\limits_{A(y_0,r_1,r_2)\cap f(D)}Q(y)\cdot\eta^n(|y-y_0|)\,dm(y) \tag{1}$$

holds for any Lebesgue measurable function $\eta:(r_1,r_2)\rightarrow[0,\infty]$ such that

$$\int\limits_{r_1}^{r_2}\eta(r)\,dr\geqslant 1\,. \tag{2}$$

Recall that a mapping $f:D\rightarrow\mathbb{R}^n$ is called *discrete* if the pre-image $\{f^{-1}(y)\}$ of each point $y\in\mathbb{R}^n$ consists of isolated points, and *is open* if the image of any open set $U\subset D$ is an open set in $\mathbb{R}^n$. Later, in the extended space $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$ we use the spherical (chordal) metric $h$ (see [3, Definition 12.1]). Further, the closure $\overline{A}$ and the boundary $\partial A$ of the set $A\subset\overline{\mathbb{R}^n}$ we understand relative to the chordal metric $h$ in $\overline{\mathbb{R}^n}$.

The boundary of $D$ is called *weakly flat* at the point $x_0\in \partial D$, if for every $P>0$ and for any neighborhood $U$ of the point $x_0$ there is a neighborhood $V\subset U$ of the same point such that $M(\Gamma(E, F, D))>P$ for any continua $E, F\subset D$ such that $E\cap \partial U\ne\varnothing\ne E\cap \partial V$ and $F\cap \partial U\ne\varnothing\ne F\cap \partial V$. The boundary of $D$ is called weakly flat if the corresponding property holds at any point of the boundary $D$. Consider the following definition, see e.g. [1]. The boundary of a domain $D$ in $\mathbb{R}^n$ is said to be *locally quasiconformal* if every $x_0\in\partial D$ has a neighborhood $U$ that admits a quasiconformal mapping $\varphi$ onto the unit ball $\mathbb{B}^n\subset\mathbb{R}^n$ such that $\varphi(\partial D\cap U)$ is the intersection of $\mathbb{B}^n$ and a coordinate hyperplane. The sequence of cuts $\sigma_m$, $m=1,2,\ldots,$ is called *regular,* if $\overline{\sigma_m}\cap\overline{\sigma_{m+1}}=\varnothing$ for $m\in\mathbb{N}$ and, in addition, $d(\sigma_m)\to 0$ as $m\to\infty$. If the end $K$ contains at least one regular chain, then $K$ will be called *regular*. We say that a bounded domain $D$ in $\mathbb{R}^n$ is *regular*, if $D$ can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in $\mathbb{R}^n$, and, besides that, every prime end in $D$ is regular. Note that space $\overline{D}_P=D\cup E_D$ is metric, which can be demonstrated as follows. If $g:D_0\rightarrow D$ is a quasiconformal mapping of a domain $D_0$ with a locally quasiconformal boundary onto some domain $D$, then for $x, y\in \overline{D}_P$ we put:

$$\rho(x, y):=|g^{-1}(x)-g^{-1}(y)|\,, \tag{3}$$

where the element $g^{-1}(x)$, $x\in E_D$, is to be understood as some (single) boundary point of the domain $D_0$. The specified boundary point is unique, see e.g. [1, Theorem 4.1]. It is easy to verify that $\rho$ in (3) is a metric on $\overline{D}_P$.

Let $E\subset\overline{D}$. We say that $D$ is *finitely connected at the point* $z_0\in E$, if for each neighborhood $\widetilde{U}$ of $z_0$ there is a neighborhood $\widetilde{V}\subset\widetilde{U}$ of $z_0$ such that $(D\cap\widetilde{V})\setminus E$ consists of finite number of components. We say that $D$ is *finitely connected on* $E$, if $D$ is finitely connected at every point $z_0\in E$. The following theorem is true.

**Theorem 1.** *Let $D$ and $D^{\,\prime}$ be domains in $\mathbb{R}^n$, $n\geqslant 2$, and let $D$ be a domain with a weakly flat boundary. Suppose that $f$ is open discrete mapping of $D$ onto $D^{\,\prime}$ satisfying the relation (1) at each point $y_0\in\overline{D^{\,\prime}}$. In addition, assume that the following conditions are fulfilled:*

*1) for each point $y_0\in\partial D^{\,\prime}$ there is $0<r_0:=\sup\limits_{y\in D^{\,\prime}}|y-y_0|$ such that for any $0<r_1<r_2<r_0:=\sup\limits_{y\in D^{\,\prime}}|y-y_0|$ there exists a set $E\subset[r_1, r_2]$ of positive linear Lebesgue measure such that $Q$ is integrable on $S(y_0, r)$ for $r\in E$;*

*2) $D^{\,\prime}$ is a regular domain and, in addition, $D^{\,\prime}$ is finitely connected on $C(f, \partial D)\cap D^{\,\prime}$, i.e., for each point $z_0\in C(f, \partial D)\cap D^{\,\prime}$ and for any neighborhood $U$ of this point there exists a neighborhood $V\subset U$ of this point such that the set $V\setminus C(f, \partial D)$ consists of a finite number of components;*

*3) the set $f^{\,-1}(C(f, \partial D)\cap D^{\,\prime})$ is nowhere dense in $D$;*

*4) the set $D^{\,\prime}$ is finitely connected in $E_{D^{\,\prime}}:=\overline{D^{\,\prime}}_P\setminus D^{\,\prime}$, i.e., for any $P_0\in E_{D^{\,\prime}}$ and any neighborhood $U$ of $P_0$ in $\overline{D^{\,\prime}}_P$ there is a neighborhood $V\subset U$ such that $V\setminus C(f, \partial D)$ consists of finite number of components.*

*Then the mapping $f$ has a continuous extension $\overline{f}:\overline{D}\rightarrow\overline{D^{\,\prime}}_P$ by the metric $\rho$ defined in (3). Moreover, $\overline{f}(\overline{D})=\overline{D^{\,\prime}}_P$.*

## References


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