On lower distance estimates for one class of homeomorphisms

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Everywhere further, (X, d, μ) and (X', d', μ') are metric spaces with metrics d and d' and locally finite Borel measures μ and μ', correspondingly. Let G and G' be domains with finite Hausdorff dimensions α and $\alpha' \geqslant 2$ in (X, d, μ) and (X', d', μ'), respectively. For $x_0 \in X$ and $r > 0$, $B(x_0, r)$ and $S(x_0, r)$ denote the ball $\{x \in X : d(x, x_0) < r\}$ and the sphere $\{x \in X : d(x, x_0) = r\}$, correspondingly. Put

$$d(E) := \sup_{x,y\in E} d(x, y)\,.$$

Given $0 < r_1 < r_2 < \infty$, denote $A = A(x_0, r_1, r_2) = \{x \in X : r_1 < d(x, x_0) < r_2\}$. Let $p \geqslant 1$ and $q \geqslant 1$, and let $Q : G \to [0, \infty]$ be a measurable function. Similarly to [1, Ch. 7], a homeomorphism $f : G \to G'$ is called a *ring Q-homeomorphism at a point $x_0 \in \overline{G}$ with respect to (p, q)-moduli*, if the inequality

$$M_p(f(\Gamma(S(x_0, r_1), S(x_0, r_2), A(x_0, r_1, r_2)))) \leqslant \int\limits_{A(x_0,r_1,r_2)\cap G} Q(x) \cdot \eta^q(d(x, x_0))\, d\mu(x) \qquad (1)$$

holds for all $0 < r_1 < r_2 < r_0 := d(G)$ and each measurable function $\eta : (r_1, r_2) \to [0, \infty]$ with

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\ . \qquad (2)$$

We say that $f : G \to G'$ is a *ring Q-homeomorphism at a point $x_0 \in \overline{G}$*, if the latter is true for $p = \alpha'$ and $q = \alpha$. For $X = X' = \mathbb{R}^n$, $n \geqslant 2$, we set $d(x, y) = d'(x, y) = |x - y|$, and $\mu = \mu' = m$, where m is the Lebesgue measure. Due to [2], a domain D in $\mathbb{R}^n$ is called a *quasiextremal distance domain* (a *QED-domain for short)* if

$$M(\Gamma(E, F, \mathbb{R}^n)) \leqslant A \cdot M(\Gamma(E, F, D)) \qquad (3)$$

for some finite number $A \geqslant 1$ and all continua E and F in D. In the same way, one can define quasiextremal distance domains in an arbitrary metric measure space.

Given a compact set K in a domain D, we set $d(K, \partial D) = \inf\limits_{x\in K, y\in\partial D} d(x, y)$. If $\partial D = \varnothing$, we set $d(K, \partial D) = \infty$.

Given a domain D in $\mathbb{R}^n$, $n \geqslant 2$, a Lebesgue measurable function $Q : D \to [0, \infty]$, a compact set $K \subset D$ and numbers $A \geqslant 1, \delta > 0$ denote by $\mathfrak{F}^{A,\delta}_{K,Q}(D)$ a family of all mappings $f : D \to \mathbb{R}^n$ satisfying the relations (1)–(2) at any point $x_0 \in D$ with $d(x, y) = d'(x, y) = |x-y|$ and $\mu = \mu' = m$, where m is the Lebesgue measure, such that $D_f := f(D)$ is a QED-domain with A in (3) and, in addition, $d(f(K), \partial D_f) \geqslant \delta$. The following result holds.

Theorem 1. *If $Q \in L^1(D)$, then there exist constants $C, C_1 > 0$ such that*

$$|f(x) - f(y)| \geqslant C_1 \cdot \exp\left\{-\frac{\|Q\|_1 A}{C|x-y|^n}\right\} \tag{4}$$

for all $x, y \in K$ and every $f \in \mathfrak{F}_{K,Q}^{A,\delta}(D)$.

Theorem 1 admits an analogue in metric spaces, which we will now formulate.

Let X be a metric space. We say that the condition of the *complete divergence of paths* is satisfied in $D \subset X$ if for any different points y_1 and $y_2 \in D$ there are some w_1, $w_2 \in \partial D$ and paths $\alpha_2 : (-2, -1] \to D$, $\alpha_1 : [1, 2) \to D$ such that 1) α_1 and α_2 are subpaths of some geodesic path $\alpha : [-2, 2] \to X$, that is, $\alpha_2 := \alpha|_{(-2,-1]}$ and $\alpha_1 := \alpha|_{[1,2)}$; 2) 2) the geodesic path α joins the points w_2, y_2, y_1 and w_1 such that $\alpha(-2) = w_2$, $\alpha(-1) = y_2$, $\alpha(1) = y_1$, $\alpha(2) = w_1$.

Note that the condition of the complete divergence of the paths is satisfied for an arbitrary bounded domain D' of the Euclidean space $\mathbb{R}^n$. Let (X, μ) be a metric space with measure μ and of Hausdorff dimension n. For each real number $n \geqslant 1$, we define *the Loewner function* $\Phi_n : (0, \infty) \to [0, \infty)$ on X as

$$\Phi_n(t) = \inf\{M_n(\Gamma(E, F, X)) : \Delta(E, F) \leqslant t\}\,, \tag{5}$$

where the infimum is taken over all disjoint nondegenerate continua E and F in X and

$$\Delta(E, F) := \frac{\mathrm{dist}\,(E, F)}{\min\{d(E), d(F)\}}\,.$$

A pathwise connected metric measure space (X, μ) is said to be a *Loewner space* of exponent n, or an n-Loewner space, if the Loewner function $\Phi_n(t)$ is positive for all $t > 0$ (see [1, Section 2.5] or [3, Ch. 8]). Observe that, $\mathbb{R}^n$ and $\mathbb{B}^n \subset \mathbb{R}^n$ are Loewner spaces (see [3, Theorem 8.2 and Example 8.24(a)]).

Given a domain D in X, $n \geqslant 2$, a measurable function $Q : D \to [0, \infty]$, a compact set $K \subset D$ and numbers $A, \delta > 0$ denote by $\mathfrak{F}_{K,Q}^{A,\delta}(D)$ a family of all mappings $f : D \to X'$ satisfying the relations (1)–(2) at any point $x_0 \in D$, such that $D_f := f(D)$ is a compact QED-subdomain of X' with A in (3) and, in addition, $d'(f(K), \partial D_f) \geqslant \delta$. The following result holds.

Theorem 2. *Let (X, d, μ) and (X', d', μ') be metric spaces with metrics d and d' and locally finite Borel measures μ and μ', correspondingly. Assume that, the condition of the complete divergence of paths is satisfied in a domain $D \subset X$. Let X' be a n-Loewner space in which the relation $\mu(B_R) \leqslant C^* R^n$ holds for some constant $C^* \geqslant 1$, for some exponent $n > 0$ and for all closed balls B_R of radius $R > 0$. If $Q \in L^1(D)$, then there exist constants $C, C_1 > 0$ such that*

$$d'(f(x), f(y)) \geqslant C_1 \cdot \exp\left\{-\frac{\|Q\|_1 A}{C d^n(x, y)}\right\} \tag{6}$$

for all $x, y \in K$ and every $f \in \mathfrak{F}_{K,Q}^{A,\delta}(D)$.

References

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