On Koebe-Bloch theorem for mappings with inverse Poletsky inequality

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Let us recall the formulation of the classical Koebe theorem.

Theorem A. *Let $f : \mathbb{D} \to \mathbb{C}$ be an univalent analytic function such that $f(0) = 0$ and $f'(0) = 1$. Then the image of f covers the open disk centered at 0 of radius one-quarter, that is, $f(\mathbb{D}) \supset B(0, 1/4)$.*

The main fact contained in the paper is the statement that something similar has been done for a much more general class of spatial mappings. Below $dm(x)$ denotes the element of the Lebesgue measure in $\mathbb{R}^n$. Everywhere further the boundary ∂A of the set A and the closure $\overline{A}$ should be understood in the sense of the extended Euclidean space $\overline{\mathbb{R}^n}$. Recall that, a Borel function $\rho : \mathbb{R}^n \to [0,\infty]$ is called *admissible* for the family Γ of paths γ in $\mathbb{R}^n$, if the relation

$$\int\limits_{\gamma} \rho(x)\,|dx| \geqslant 1$$

holds for all (locally rectifiable) paths $\gamma \in \Gamma$. In this case, we write: $\rho \in \operatorname{adm}\Gamma$. The *modulus* of Γ is defined by the equality

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^n(x)\,dm(x)\,.$$

Let $y_0 \in \mathbb{R}^n$, $0 < r_1 < r_2 < \infty$ and

$$A = A(y_0, r_1, r_2) = \{y \in \mathbb{R}^n : r_1 < |y - y_0| < r_2\}\,.$$

Given $x_0 \in \mathbb{R}^n$, we put $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$, $\mathbb{B}^n = B(0,1)$, $S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$. A mapping $f : D \to \mathbb{R}^n$ is called *discrete* if the pre-image $\{f^{-1}(y)\}$ of any point $y \in \mathbb{R}^n$ consists of isolated points, and *open* if the image of any open set $U \subset D$ is an open set in $\mathbb{R}^n$. Given sets E, $F \subset \overline{\mathbb{R}^n}$ and a domain $D \subset \mathbb{R}^n$ we denote by $\Gamma(E,F,D)$ the family of all paths $\gamma : [a,b] \to \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in (a,b)$. Given a mapping $f : D \to \mathbb{R}^n$, a point $y_0 \in \overline{f(D)} \setminus \{\infty\}$, and $0 < r_1 < r_2 < r_0 = \sup\limits_{y \in f(D)} |y - y_0|$, we denote by $\Gamma_f(y_0, r_1, r_2)$ a family of all paths γ in D such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$. Let $Q : \mathbb{R}^n \to [0,\infty]$ be a Lebesgue measurable function. We say that f *satisfies the inverse Poletsky inequality at a point* $y_0 \in \overline{f(D)} \setminus \{\infty\}$ if the relation

$$M(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} Q(y)\cdot \eta^n(|y - y_0|)\,dm(y) \tag{1}$$

holds for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\,dr \geqslant 1\,. \tag{2}$$

The relations (1) are proved for different classes of mappings, see e.g. [2].

Set $q_{y_0}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{S(y_0,r)} Q(y)\,d\mathcal{H}^{n-1}(y)$, where ω_{n-1} denotes the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. We say that a function $\varphi : D \to \mathbb{R}$ has a *finite mean oscillation* at a point $x_0 \in D$, write $\varphi \in FMO(x_0)$, if $\limsup\limits_{\varepsilon \to 0} \frac{1}{\Omega_n \varepsilon^n} \int\limits_{B(x_0,\,\varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon|\;dm(x) < \infty$, where $\overline{\varphi}_\varepsilon = \frac{1}{\Omega_n \varepsilon^n} \int\limits_{B(x_0,\,\varepsilon)} \varphi(x)\,dm(x)$ and Ω_n is the volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. We also say that a function $\varphi : D \to \mathbb{R}$ has a finite mean oscillation at $A \subset \overline{D}$, write $\varphi \in FMO(A)$, if φ has a finite mean oscillation at any point $x_0 \in A$. Let h be a chordal metric in $\overline{\mathbb{R}^n}$,

$$h(x,\infty) = \frac{1}{\sqrt{1+{|x|}^2}}\,, \quad h(x,y) = \frac{|x-y|}{\sqrt{1+{|x|}^2}\sqrt{1+{|y|}^2}}\,, \qquad x \ne \infty \ne y\,,$$

and let $h(E) := \sup\limits_{x,y \in E} h(x,y)$ be a chordal diameter of a set $E \subset \overline{\mathbb{R}^n}$ (see, e.g., [1, Definition 12.1]).

Given a continuum $E \subset D$, $\delta > 0$ and a Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty]$ we denote by $\mathfrak{F}_{E,\delta}(D)$ the family of all open discrete mappings $f : D \to \mathbb{R}^n$, $n \geqslant 2$, satisfying relations (1)–(2) at any point $y_0 \in \overline{\mathbb{R}^n}$ such that $h(f(E)) \geqslant \delta$. The following statement holds, cf. [3].

Theorem 1. *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and let $B(x_0, \varepsilon_1) \subset D$ for some $\varepsilon_1 > 0$.*

Assume that, $Q \in L^1(\mathbb{R}^n)$ and, in addition, one of the following conditions hold:

1) $Q \in FMO(\overline{\mathbb{R}^n})$;

2) for any $y_0 \in \overline{\mathbb{R}^n}$ there is $\delta(y_0) > 0$ such that

$$\int\limits_0^{\delta(y_0)} \frac{dt}{t q_{y_0}^{\frac{1}{n-1}}(t)} = \infty\,. \tag{3}$$

Then there is $r_0 > 0$, which does not depend on f, such that

$$f(B(x_0, \varepsilon_1)) \supset B_h(f(x_0), r_0) \qquad \forall\ f \in \mathfrak{F}_{E,\delta}(D)\,,$$

where $B_h(f(x_0), r_0) = \{w \in \overline{\mathbb{R}^n} : h(w, f(x_0)) < r_0\}$.

Remark 2. The condition $Q \in FMO(\infty)$ of the condition (3) for $y_0 = \infty$ must be understood as follows: these conditions hold for $y_0 = \infty$ if and only if the function $\widetilde{Q} := Q\left(\frac{y}{|y|^2}\right)$ satisfies similar conditions at the origin.

The result mentioned above is obtained in [4].

References

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