On hyperholomorphic functions of spatial variable

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Let $\mathbb{H}(\mathbb{C})$ be the algebra of complex quaternions $\sum_{k=0}^{3} a_k \boldsymbol{i}_k$, where $\{a_k\}_{k=0}^{3}$ are complex numbers, $\boldsymbol{i}_0 = 1$ be the unit, $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$ be the quaternion units, i.e. they satisfy the multiplicative rule $\boldsymbol{i}_1^2 = \boldsymbol{i}_2^2 = \boldsymbol{i}_3^2 = \boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3 = -1$. Let $z := \sum_{k=1}^{3} z_k \boldsymbol{i}_k$ be a point of Euclidean space $\mathbb{R}^3$ with the basic set $\{\boldsymbol{i}_k\}_{k=1}^{3}$ and let Ω be a domain in $\mathbb{R}^3$.

Definition. Function $f := \sum_{k=0}^{3} f_k \boldsymbol{i}_k$, where $f_k : \Omega \to \mathbb{C}$, is called left-hyperholomorphic or right-hyperholomorphic if its components $\{f_k\}_{k=0}^{3}$ are $\mathbb{R}^3$-differentiable functions and the condition $\sum_{k=1}^{3} \boldsymbol{i}_k \frac{\partial f}{\partial z_k} = 0$ or $\sum_{k=1}^{3} \frac{\partial f}{\partial z_k} \boldsymbol{i}_k = 0$ respectively holds true in the domain Ω.

In known formerly publications (see e. g. [1]) similar definitions included the more strong condition on components of function f to have continuous partial derivatives.

By $\Gamma_{z,\delta}$ denote the set of points ζ contained in Γ such that $|\zeta - z| \leq \delta$. The next theorem (see [2]) is a quaternion analog of the Cauchy theorem from complex analysis.

Theorem. Let Ω be a bounded domain with the piece-wise smooth boundary Γ such that for all points z from $\mathbb{R}^3$ and for all $\delta > 0$ the diameter of the set $\Gamma_{z,\delta}$ divided by its square measure is bounded by a positive constant. Let function $f : \overline{\Omega} \to \mathbb{H}(\mathbb{C})$ be continuous in $\overline{\Omega}$ and right-hyperholomorphic in Ω and let function $g : \overline{\Omega} \to \mathbb{H}(\mathbb{C})$ be continuous in $\overline{\Omega}$ and left-hyperholomorphic in Ω. Then

$$\iint_{\Gamma} f(z)\, \nu(z)\, g(z)\, ds = 0,$$

where $\nu(z) := \sum_{k=1}^{3} \nu_k(z) \boldsymbol{i}_k$ is the unit normal vector to the surface Γ.

References

1. Blaya R. A., Reyes J. B., Shapiro M. On the Laplasian vector fields theory in domains with rectifiable boundary. *Mathematical Methods in the Applied Sciences.* 2006; **29**: 1861 – 1881.

2. Gerus O. F. On hyperholomorphic functions of spatial variable. *Ukrain. Mat. Zh.* 2011; **63**(4): 459 – 465 (Russian).