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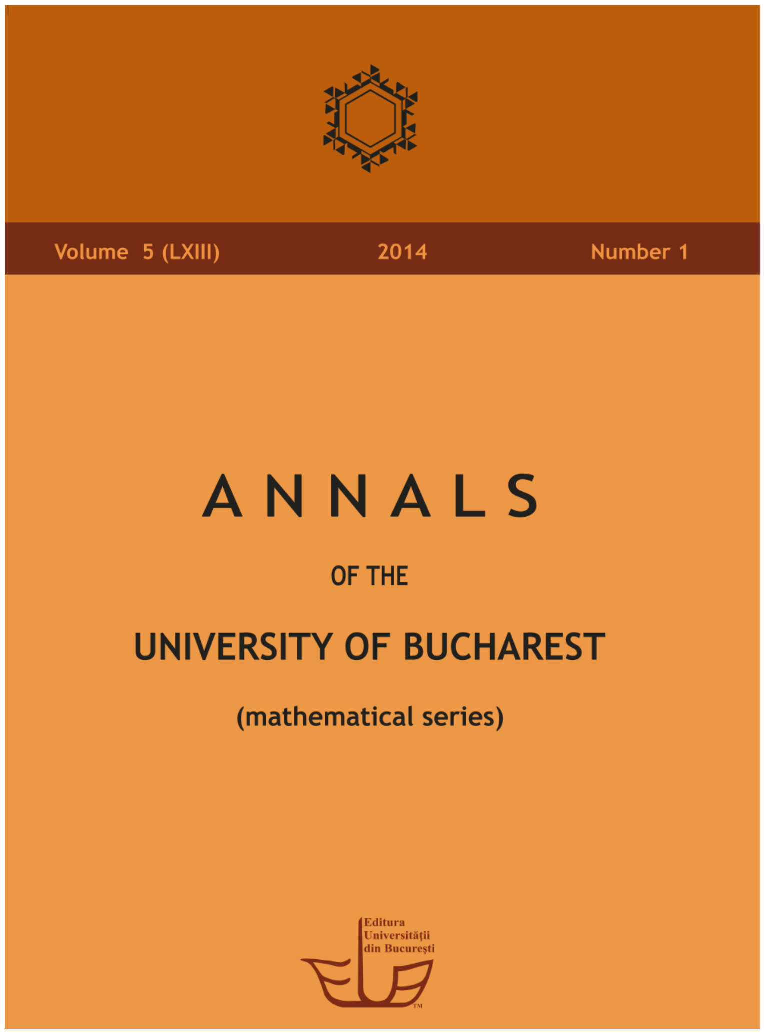

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Distortion estimates under mappings with controlled p-module

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Distortion estimates under mappings with controlled p-module

Anatoly Golberg, Ruslan Salimov and Evgeny Sevost'yanov

Dedicated to Professor Cabiria Andreian Cazacu on the occasion of her Birthday

Abstract - We study open discrete mappings preserving integrally quasiinvariant the weighted p-module and provide conditions ensuring the local Hölder continuity of such mappings with respect to euclidian distances and to their logarithms. The inequalities defining the continuity are sharp with respect to the order.

Key words and phrases : p-module, mappings with bounded and finite distortion, Q-mappings, lipschitz mappings, Hölder continuity, equicontinuity, Orlicz-Sobolev classes.

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1. Introductory remarks

In this paper, we continue investigation of properties of mappings with controlled p-module. The main characterization of these mappings requires extension of underlying quasiinvariance property of p-moduli (of appropriate order) under quasiconformal and quasiisomertic mappings. This approach involves the integral restrictions for the growth of moduli of families of curves of type

$$\mathcal{M}_p(f(\Gamma(S_1, S_2, A))) \le \int\limits_A Q(x)\,\eta^p(|x-x_0|)\,dm(x). \tag{1.1}$$

Here the test subdomains $A \subset G$ are spherical rings $A = A(r_1, r_2, x_0) = \{\, x \in G : r_1 < |x - x_0| < r_2\}$, $0 < r_1 < r_2 < r_0 := \text{dist}\,(x_0, \partial G)$, and η is arbitrary measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)dr \ge 1, \tag{1.2}$$

and $Q : G \to [0, \infty]$ is a given measurable function. The point x_0 is fixed in G.

The mappings satisfying (1.1) are called *ring* (p,Q)*-mappings at the point* x_0. We also say that a mapping is ring (p,Q)-mapping in the domain G if it is ring (p,Q)-mapping at any $x_0 \in G$. The study of such homeomorphisms was started in [9]; on their differential and geometric properties see [10], [12]. Such mappings in $\mathbb{R}^n$ are close to bilipschitz mappings (see e.g. [8], [28], [30]). Note also that right-hand side in (1.1) can be treated as a weighted p-module; cf. [1], [32], [5]. For the study of mappings admitting close modular descriptions, we also refer to [2], [3], [6], [16], [18], [23], [25], [33], [34].

The *p-module* $(p \geq 1)$ of a family Γ of curves in $\mathbb{R}^n$, $n \geq 2$, is defined by

$$\mathcal{M}_p(\Gamma) = \inf_{\varrho \in \operatorname{adm}\Gamma} \int_{\mathbb{R}^n} \varrho^p(x)\, dm(x)\,, \tag{1.3}$$

where the infimum is taken over all Borel functions $\varrho : \mathbb{R}^n \to [0,\infty]$ with

$$\int_\gamma \varrho(x)\,|dx| \geq 1, \qquad \forall\, \gamma \in \Gamma;$$

such ϱ are called *admissible* for Γ (abbr. $\varrho \in \operatorname{adm}\Gamma$). Here m stands for n-dimensional Lebesgue measure in $\mathbb{R}^n$.

It is known that homeomorphisms $f : G \to \mathbb{R}^n$ of a domain $G \in \mathbb{R}^n$ $(n \geq 2)$ admitting *quasiinvariance* of p-module for $n-1 < p < n$, i.e. such that

$$K^{-1}\mathcal{M}_p(\Gamma) \leq \mathcal{M}_p(f(\Gamma)) \leq K\,\mathcal{M}_p(\Gamma), \tag{1.4}$$

are *lipschitzian* which means that

$$\limsup_{x \to x_0} \frac{|f(x)-f(x_0)|}{|x-x_0|} \leq C \quad \text{for all} \quad x_0 \in G.$$

It was established in [8] that, in fact, it suffices to apply only the right-hand side in (1.4) and take the constant $C = K^{1/(n-p)}$.

We consider much more general class of open discrete mappings satisfying (1.4) in some integral sense and establish for those the logarithmic version of the classical Hölder's continuity,

$$\limsup_{x \to x_0} |f(x)-f(x_0)| \left(\log \frac{1}{|x-x_0|}\right)^{\alpha} \leq C, \quad \alpha > 0. \tag{1.5}$$

Clearly, any Hölder continuous mapping with a positive exponent satisfies (1.5). Thus, the inequality (1.5) provides a natural extension of the Hölder continuity and can be regarded as a logarithmic Hölder continuity.

For many questions concerning quasiconformal mappings and their generalizations it would be interesting to have criteria for the Lipschitz or Hölder continuity or giving more general regularity conditions in a prescribed point or on a given set; see, e.g. [14], [15], [17], [20], [30]. Our main results state that for every $n-1<p<n$ and any locally integrable Q with the exponent exceeding $n/(n-p)$ the corresponding mapping is Hölder continuous while for the degree $n/(n-p)$ it is logarithmically Hölder continuous. An illustrative example shows that the degree $n/(n-p)$ cannot be reduced without additional restrictions of the majorant. Other results concern the equicontinuity and normality of the mapping families. In particular, it is established that a family of mappings of finite distortion from the Orlicz-Sobolev class is eqicontinuous and normal.

2. p-capacity of condensers and related estimates

We need some basic definitions and auxiliary results.

2.1. p-capacity

Recall that a pair $\mathcal{E}=(A,\,C)$, where A is an open set in $\mathbb{R}^n$, and C is a compact subset of A, is called *condenser* in $\mathbb{R}^n$. A quantity

$$\operatorname{cap}_p \mathcal{E} \quad = \quad \operatorname{cap}_p\,(A,\,C) \quad = \quad \inf_{u\,\in\,W_0(\mathcal{E})} \int\limits_A |\nabla u|^p\;dm(x)\,,$$

where $W_0(\mathcal{E})=W_0\,(A,\,C)$ is a family of all nonnegative absolutely continuous on lines (ACL) functions $u: A\to\mathbb{R}$ with compact support in A and such that $u(x)\geq 1$ on C, is called *p-capacity* of the condenser $\mathcal{E}$.

For the general properties of p-capacities and their relation to the mapping theory, we refer, e.g. to [13] and [24]. In particular, when $1<p<n$,

$$\operatorname{cap}_p \mathcal{E} \geq n\Omega_n^{\frac{p}{n}}\left(\frac{n-p}{p-1}\right)^{p-1}[m(C)]^{\frac{n-p}{n}}\;, \tag{2.1}$$

where Ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, and $m(C)$ is the n-dimensional Lebesgue measure of C.

Another lower estimate of p-capacity of a condenser $\mathcal{E}=(A,C)$ in $\mathbb{R}^n$ is given by

$$\operatorname{cap}_p \mathcal{E} = \operatorname{cap}_p\,(A,C) \geq \left(c_1\frac{(d(C))^p}{(m(A))^{1-n+p}}\right)^{\frac{1}{n-1}}\;, \qquad p>n-1, \tag{2.2}$$

where c_1 depends only on n and p and $d(C)$ denotes the diameter of C (see [21, Proposition 6]).

A curve γ in $\mathbb{R}^n$ ($\overline{\mathbb{R}^n}$) is a continuous mapping $\gamma:\Delta\to\mathbb{R}^n$ ($\overline{\mathbb{R}^n}$), where Δ is an interval in $\mathbb{R}$. Its locus $\gamma(\Delta)$ is denoted by $|\gamma|$. Let Γ be a family of curves γ in $\mathbb{R}^n$. The p-module of the family Γ is defined by (1.3). For the basic features of $\mathcal{M}_p(\Gamma)$, we refer to [33, 6.1]; see also [34], [25]. Here we recall only some of them.

If $\operatorname{adm}\Gamma = \varnothing$ we set $\mathcal{M}_p(\Gamma) = \infty$. Note that $\mathcal{M}_p(\varnothing) = 0$; $\mathcal{M}_p(\Gamma_1) \leq \mathcal{M}_p(\Gamma_2)$ whenever $\Gamma_1 \subset \Gamma_2$, and moreover $\mathcal{M}_p\left(\bigcup\limits_{i=1}^{\infty} \Gamma_i\right) \leq \sum\limits_{i=1}^{\infty} \mathcal{M}_p(\Gamma_i)$ (see [33, Theorem 6.2]).

We say that Γ_1 is *minorized* by Γ_2 and write $\Gamma_2 < \Gamma_1$ if every $\gamma \in \Gamma_1$ has a subcurve which belongs to Γ_2. The relation $\Gamma_2 < \Gamma_1$ implies that $\operatorname{adm}\Gamma_2 \subset \operatorname{adm}\Gamma_1$ and therefore $\mathcal{M}_p(\Gamma_1) \leq \mathcal{M}_p(\Gamma_2)$.

Let E_0, E_1 are two sets in $D \subset \overline{\mathbb{R}^n}$. Denote by $\Gamma(E_0, E_1, D)$ a family of al curves joining E_0 and E_1 in D. For such families, we need the following statement given in [25, Proposition 10.2, Ch. II].

Proposition 2.1. *Let $\mathcal{E} = (A, C)$ be a condenser in $\mathbb{R}^n$ and let Γ_E be the family of all curves of the form $\gamma : [a, b) \to A$ with $\gamma(a) \in C$ and $|\gamma| \cap (A \setminus F) \neq \varnothing$ for every compact $F \subset A$. Then* $\operatorname{cap}_q \mathcal{E} = \mathcal{M}_q\left(\Gamma_{\mathcal{E}}\right)$.

Note that Proposition 2.1 allows us to give a natural extension of p-capacity of a condenser $E \subset \overline{\mathbb{R}^n}$ by $\operatorname{cap}_q E = \mathcal{M}_q\left(\Gamma_E\right)$.

2.2. Upper bound for p-capacity

Denote $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ and let $\mathbb{B}^n = B(0,1)$. For any Lebesgue measurable function $Q : G \to [0, \infty]$ one can define (for almost all $r > 0$) the mean

$$q_{x_0}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{S(x_0,r)} Q(x)\, d\mathcal{H}^{n-1}\,,$$

where $S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$ and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

For a spherical ring $A(x_0, r_1, r_2)$ centered at x_0 with radii r_1 and r_2, denote

$$I \;=\; I(x_0, r_1, r_2) \;=\; \int\limits_{r_1}^{r_2} \frac{dr}{r^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)}\,. \tag{2.3}$$

A necessary and sufficient condition for a discrete open mapping f to be ring Q-mapping with respect to p-module is given by

Proposition 2.2. (see [31]) *Let G be a domain in $\mathbb{R}^n$, let $Q : G \to [0, \infty]$ be a locally integrable function in G and let $\mathcal{E} = \left(B(x_0, r_2), \overline{B(x_0, r_1)}\right)$, $0 < r_1 < r_2 < \operatorname{dist}(x_0, \partial D)$ be a condenser. An open discrete mapping*

$f : G \to \mathbb{R}^n$ is ring (p, Q)-mapping at a point $x_0 \in G$ if and only if for any $0 < r_1 < r_2 < d_0 = \operatorname{dist}(x_0, \partial G)$, the following inequality

$$\operatorname{cap}_p f(\mathcal{E}) \ \leq \ \frac{\omega_{n-1}}{I^{p-1}}$$

holds, where I is defined by (2.3).

Note that the minimum of the integrals in (1.1) among η satisfying (1.2) is attained on the metric

$$\eta_0(r) = \frac{1}{I r^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)}\,.$$

2.3. Equicontinuity and normality

Let $f : G \to \mathbb{R}^n$ be a discrete open mapping, $\beta : [a, b) \to \mathbb{R}^n$ be a curve, and $x \in f^{-1}(\beta(a))$. A curve $\alpha : [a, c) \to G$ is called a *maximal f-lifting* of β starting at x, if (1) $\alpha(a) = x$; (2) $f \circ \alpha = \beta|_{[a,\,c)}$; (3) for $c < c' \leq b$, there is no curves $\alpha' : [a, c') \to G$ such that $\alpha = \alpha'|_{[a,\,c)}$ and $f \circ \alpha' = \beta|_{[a,\,c')}$. The assumption on f yields that every curve β with $x \in f^{-1}(\beta(a))$ has a maximal f-lifting starting at x (see [25, Corollary II.3.3], [22, Lemma 3.12]).

Let (X, d_X) and (Y, d_Y) be two metric spaces with distances d_X and d_Y, respectively. A family $\mathcal{F}$ of continuous mappings $f : X \to Y$ is called *equicontinuous at a point* $x_0 \in X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ for all $f \in \mathcal{F}$ and any $x \in X$ provided that $d_X(x, x_0) < \delta$. The family $\mathcal{F}$ is called *equicontinuous in* X if $\mathcal{F}$ is equicontinuous for every $x_0 \in X$.

We define the chordal metric h in $\overline{\mathbb{R}^n}$ by

$$h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}, \quad h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}\,, \quad x \neq \infty \neq y\,,$$

and the *chordal diameter* of a set $E \subset \overline{\mathbb{R}^n}$ by

$$h(E) = \sup_{x\,, y \in E} h(x, y)\,, \tag{2.4}$$

(see, e.g. [34]), and set $(X, d_X) = (G, |\cdot|)$ and $(Y, d_Y) = \left(\overline{\mathbb{R}^n}, h\right)$. Here G is a domain in $\mathbb{R}^n$, $|\cdot|$ and h denote the Euclidean and chordal metrics, respectively.

The following notion is closely related to equicontinuity. A family $\mathcal{F}$ is called *normal* if for any sequence $\{f_m\}$ of continuous mappings $f_m : X \to Y$ there exists a subsequence $\{f_{m_k}\}$ that converges uniformly on each compact set $E \subset X$.

The following well-known Ascoli's theorem provides a sufficient condition for an equicontinuous family $\mathcal{F}$ to be normal; see e.g. [33], [26].

Proposition 2.3. *If T is a separable topological space and Y is a compact metric space, then every equicontinuous family $\mathcal{F}$ of mappings $f : T \to Y$ is a normal family.*

3. Distortion estimates for integrable majorant

One of the interesting problems in geometric function theory is to find the conditions insuring the Hölder continuity of mappings. For the Q-homeomorphisms with respect to conformal module, Q of BMO-class (bounded mean oscillation), FMO-class (finite mean oscillation) and L^α see, e.g. [23].

In this section we impose only integrability conditions on the majorant Q in a domain G and establish that $Q \in L^\alpha_{\mathrm{loc}}(G)$ with an exponent $\alpha < n/(n-p)$ implies the Hölder continuity of open discrete ring (p, Q)-mappings with degree $1 - n/(n-p)\alpha$. When Q is locally integrable with the exponent $n/(n-p)$, one can derive a stronger inequality. As was mentined above, this kind of continuity can be regarded as a logarithmic Hölder continuity.

Denote as usual for $\alpha > 0$,

$$\|Q\|_\alpha = \left(\int\limits_{B(x,\delta)} Q^\alpha(x)\, dm(x) \right)^{\frac{1}{\alpha}} . \tag{3.1}$$

Theorem 3.1. *Let G a domain in $\mathbb{R}^n$, $n \geq 2$, and $x \in G$. Let $f : G \to \mathbb{R}^n$ be an open discrete ring (p, Q)-mapping at x_0, $n-1 < p < n$, with $Q(x) \in L^\alpha_{\mathrm{loc}}$, $\alpha > \frac{n}{n-p}$. Then for any pair of points $x, x_0 \in G$ such that $|x - x_0| < \delta$, $\delta = \frac{1}{4}\mathrm{dist}\,(x_0, \partial G)$, the following inequality holds*

$$|f(x) - f(x_0)| \leq \lambda_p \|Q\|_\alpha^{\frac{1}{n-p}} |x - x_0|^{1 - \frac{n}{\alpha(n-p)}}, \tag{3.2}$$

with a constant λ_p depending only on n, p and α.

Proof. Consider a spherical ring $A = A(x_0, \varepsilon_1, \varepsilon_2)$ centered at $x_0 \in G$ and radii $\varepsilon_1, \varepsilon_2$, $0 < \varepsilon_1 < \varepsilon_2 < \delta$, such that $A(x_0, \varepsilon_1, \varepsilon_2) \subset G$. Then $\mathcal{E} = \left(B\,(x_0, \varepsilon_2)\,, \overline{B\,(x_0, \varepsilon_1)}\right)$ and $f(\mathcal{E}) = \left(f(B\,(x_0, \varepsilon_2)), f(\overline{B\,(x_0, \varepsilon_1)})\right)$ are both condensers located in G and G^*, respectively. By Proposition 2.1,

$$\mathrm{cap}_p\, f(\mathcal{E}) = \mathcal{M}_p(\Gamma_{f(\mathcal{E})})\,,$$

where $\Gamma_{f(\mathcal{E})}$ stands for a family of curves described in this statement. Let Γ^* be a family of all maximal liftings of $\Gamma_{f(\mathcal{E})}$ starting in $B\,(x_0, \varepsilon_1)$. Note that $\Gamma^* \subset \Gamma_{\mathcal{E}}$ and $\Gamma_{\mathcal{E}} > \Gamma(S(x_0, \varepsilon_1), S(x_0, \varepsilon_2 - \varepsilon), A(x_0, \varepsilon_1, \varepsilon_2))$ for sufficiently small $\varepsilon > 0$. Now for every measurable function $\eta : [\varepsilon_1, \varepsilon_2 - \varepsilon] \to [0, \infty]$ satisfying

$$\int_{\varepsilon_1}^{\varepsilon_2-\varepsilon} \eta(r)dr \geq 1 \,, \tag{3.3}$$

the inequality (1.1) and Proposition 2.1 yield

$$\mathrm{cap}_p\, f(\mathcal{E}) \leq \int_{A(x_0,\varepsilon_1,\varepsilon_2)} Q(x) \cdot \eta^p(|x - x_0|)dm(x)\,. \tag{3.4}$$

The function

$$\eta(t) \;=\; \begin{cases} \frac{1}{\varepsilon_2-\varepsilon-\varepsilon_1}, & t \in [\varepsilon_1, \varepsilon_2 - \varepsilon], \\ 0, & t \notin [\varepsilon_1, \varepsilon_2 - \varepsilon], \end{cases}$$

satisfies (3.3) and by (3.4),

$$\mathrm{cap}_p\, f(\mathcal{E}) \leq \frac{1}{(\varepsilon_2 - \varepsilon_1 - \varepsilon)^p} \int_{A(x_0,\varepsilon_1,\varepsilon_2)} Q(x)\; dm(x)\,.$$

Letting $\varepsilon \to 0$, one obtains

$$\mathrm{cap}_p\, f(\mathcal{E}) \leq \frac{1}{(\varepsilon_2 - \varepsilon_1)^p} \int_{A(x_0,\varepsilon_1,\varepsilon_2)} Q(x)\; dm(x)\,,$$

and by the Hölder inequality,

$$\mathrm{cap}_p\, f(\mathcal{E}) \leq \frac{(\Omega_n \varepsilon_2^n)^{\frac{\alpha-1}{\alpha}}}{(\varepsilon_2 - \varepsilon_1)^p} \|Q\|_\alpha. \tag{3.5}$$

Choosing $\varepsilon_1 = 2\varepsilon$ and $\varepsilon_2 = 4\varepsilon$, one gets the upper bound for p-capacity

$$\mathrm{cap}_p\, (f(B(x_0, 4\varepsilon)), f(\overline{B(x_0, 2\varepsilon)}) \leq\; \gamma_1 \|Q\|_\alpha \varepsilon^{\frac{\alpha n - \alpha p - n}{\alpha}}\,. \tag{3.6}$$

On the other hand, one can derive from the inequality (2.1) the following lower bound

$$\mathrm{cap}_p\, (f(B(x_0, 4\varepsilon)), f(\overline{B(x_0, 2\varepsilon)}) \geq \gamma_2 \left[m(f(B(x_0, 2\varepsilon)))\right]^{\frac{n-p}{n}}\,, \tag{3.7}$$

where γ_2 is a positive constant depending only on p and n. Combining the estimates (3.6) and (3.7), one gets the upper bound for the image of the ball $B(x_0, 2\varepsilon)$,

$$m(f(B(x_0, 2\varepsilon))) \leq \gamma_3 \|Q\|_\alpha^{\frac{n}{n-p}} \varepsilon^{\frac{(\alpha n - \alpha p - n)n}{\alpha(n-p)}}\,, \tag{3.8}$$

where γ_3 is also a constant depending only on p and n.

Now, letting in (3.5) $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = 2\varepsilon$, one obtains

$$\mathrm{cap}_p\left(f(B(x_0, 2\varepsilon)), f(\overline{B(x_0,\varepsilon)})\right) \le \gamma_4 \|Q\|_\alpha \varepsilon^{\frac{\alpha n - \alpha p - n}{\alpha}}, \tag{3.9}$$

and after applying the lower bound (2.2),

$$\left(\mathrm{cap}_p\left(f(B(x_0, 2\varepsilon)), f(\overline{B(x_0,\varepsilon)})\right)\right)^{n-1} \ge \gamma_5 \frac{d^p(f(\overline{B(x_0,\varepsilon)})}{m^{p-1}(f(B(x_0,2\varepsilon)))}. \tag{3.10}$$

The inequalities (3.8), (3.9) and (3.10) result in

$$d(f(\overline{B(x_0,\varepsilon)}) \le \gamma \|Q\|_\alpha^{\frac{1}{n-p}} \varepsilon^{1-\frac{n}{\alpha(n-p)}}$$

with a constant γ depending only on p, α and n.

Now the desired estimate (3.2) follows from the obvious inequality

$$d(f(\overline{B(x_0,\varepsilon)})) \ge |f(x) - f(x_0)| \quad \text{for} \;\; x \in S(x_0, \varepsilon).$$

□

Theorem 3.2. *Let G be a domain in $\mathbb{R}^n$, $n \ge 2$, and $x_0 \in G$. Suppose $Q \in L_{\mathrm{loc}}^{\frac{n}{n-p}}$. Then for every open discrete ring (p,Q)-mapping $f : G \to \mathbb{R}^n$ at x_0, $n-1 < p < n$, the following estimate*

$$|f(x) - f(x_0)| \left(\log \frac{1}{|x - x_0|}\right)^{\frac{p(n-1)}{n(n-p)}} \le C_{n,p} \, \|Q\|_{\frac{n}{n-p}}^{\frac{1}{n-p}}, \qquad |x - x_0| < r_0,$$

holds for any $x \in G$ provided that $|x - x_0| < \delta = \min\{1, \mathrm{dist}^4(x_0, \partial G)\}$; here $C_{n,p}$ is a positive constant depending only on n and p.

Proof. Consider a spherical ring $A(x_0, \varepsilon_1, \varepsilon_2) = \{x : \;\varepsilon_1 < |x - x_0| < \varepsilon_2\}$ with radii $0 < \varepsilon_1 < \varepsilon_2$ such that $A(x_0, \varepsilon_1, \varepsilon_2) \subset G$. Since $\mathcal{E} = \left(B(x_0, \varepsilon_2), \overline{B(x_0, \varepsilon_1)}\right)$ is a condenser, (3.4) yields

$$\mathrm{cap}_p f(\mathcal{E}) \le \int\limits_{A(x_0,\varepsilon_1,\varepsilon_2)} Q(x) \cdot \eta^p(|x - x_0|) dm(x)$$

for any $\eta : [\varepsilon_1, \varepsilon_2 - \varepsilon] \to [0, \infty]$ obeying (3.3). Obviously, for sufficiently small $\varepsilon > 0$, the function

$$\eta(t) = \begin{cases} \frac{1}{t \log \frac{\varepsilon_2 - \varepsilon}{\varepsilon_1}}, & \text{if } t \in (\varepsilon_1, \varepsilon_2 - \varepsilon), \\ 0, & \text{otherwise} \end{cases}$$

satisfies (3.3); hence,

$$\mathrm{cap}_p f(\mathcal{E}) \le \log^{-p} \frac{\varepsilon_2 - \varepsilon}{\varepsilon_1} \int\limits_{A(x_0,\varepsilon_1,\varepsilon_2)} \frac{Q(x)}{|x - x_0|^p} \, dm(x). \tag{3.11}$$

Letting in (3.11) $\varepsilon \to 0$, one gets

$$\mathrm{cap}_p\, f(\mathcal{E}) \le \log^{-p} \frac{\varepsilon_2}{\varepsilon_1} \int\limits_{A(x_0,\varepsilon_1,\varepsilon_2)} \frac{Q(x)}{|x-x_0|^p}\, dm(x)\,.$$

After applying to the integral in the right-hand side the Hölder inequality, one gets

$$\mathrm{cap}_p\, f(\mathcal{E}) \le \log^{-p} \frac{\varepsilon_2}{\varepsilon_1} \left(\int\limits_{B(x_0,r_0)} Q^{\frac{n}{n-p}}(x)\, dm(x) \right)^{\frac{n-p}{n}} \left(\int\limits_{A(x_0,\varepsilon_1,\varepsilon_2)} \frac{dm(x)}{|x-x_0|^n} \right)^{\frac{p}{n}},$$

and since $\int\limits_{A(x_0,\varepsilon_1,\varepsilon_2)} \frac{dm(x)}{|x-x_0|^n} = \omega_{n-1} \log(\varepsilon_2/\varepsilon_1)$,

$$\mathrm{cap}_p\, f(\mathcal{E}) \le \omega_{n-1}^{\frac{p}{n}} \left(\int\limits_{B(x_0,r_0)} Q^{\frac{n}{n-p}}(x)\, dm(x) \right)^{\frac{n-p}{n}} \log^{\frac{p(1-n)}{n}} \frac{\varepsilon_2}{\varepsilon_1} \tag{3.12}$$

Choosing $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = \sqrt{\varepsilon}$, we have

$$\mathrm{cap}_p\, f(\mathcal{E}) \le C_1 \left(\int\limits_{B(x_0,r_0)} Q^{\frac{n}{n-p}}(x)\, dm(x) \right)^{\frac{n-p}{n}} \log^{\frac{p(1-n)}{n}} \frac{1}{\varepsilon} \tag{3.13}$$

with C_1 depending only on p.

On the other hand, from (2.2),

$$\left(\mathrm{cap}_p\, \left(f(B(x_0,\sqrt{\varepsilon})), f(\overline{B(x_0,\varepsilon)})\right)\right)^{n-1} \ge C_2 \frac{d^p(f(\overline{B(x_0,\varepsilon)}))}{m^{p-n+1}(f(B(x_0,\sqrt{\varepsilon})))}, \tag{3.14}$$

where C_2 is a positive constant depending only on p and n.

Now, combining (3.13) and (3.14), we have

$$\frac{d^p(f(B(x_0,\varepsilon)))}{m^{p-n+1}(f(B(x_0,\sqrt{\varepsilon})))} \le C_3 \left(\int\limits_{B(x_0,r_0)} Q^{\frac{n}{n-p}}(x)\, dm(x) \right)^{\frac{n-p}{n}} \log^{-\frac{p(1-n)^2}{n}} \frac{1}{\varepsilon}\,. \tag{3.15}$$

To find an upper bound for $m(f(B(x_0,\sqrt{\varepsilon})))$ in (3.15), pick in (3.12) $\varepsilon_1 = \sqrt{\varepsilon}$ and $\varepsilon_2 = \sqrt[4]{\varepsilon}$. Then one derives

$$\operatorname{cap}_p(f(B(x_0,\sqrt[4]{\varepsilon})), f(\overline{B(x_0,\sqrt{\varepsilon})})) \le C_4 \left(\int\limits_{B(x_0,r_0)} Q^{\frac{n}{n-p}}(x)\, dm(x) \right)^{\frac{n-p}{n}} \log^{\frac{p(1-n)}{n}} \frac{1}{\varepsilon}. \tag{3.16}$$

The capacity in the left-hand side of (3.16) is estimated by (2.1)

$$\operatorname{cap}_p(f(B(x_0,\sqrt[4]{\varepsilon})), f(\overline{B(x_0,\sqrt{\varepsilon})})) \ge C_5 \left[m(f(B(x_0,\sqrt{\varepsilon}))) \right]^{\frac{n-p}{n}}; \tag{3.17}$$

here C_5 depends only on p and n. Combining (3.16) and (3.17), we derive the desired estimate

$$m(f(B(x_0,\sqrt{\varepsilon}))) \le C_6 \left(\log \frac{1}{\varepsilon} \right)^{\frac{p(1-n)}{n-p}} \int\limits_{B(x_0,r_0)} Q^{\frac{n}{n-p}}(x)\, dm(x)$$

with a constant C_6 depending only on p and n.

Substituting this into (3.15), one obtains the estimate

$$d(f(B(x_0,\varepsilon))) \left(\log \frac{1}{\varepsilon} \right)^{\frac{p(n-1)}{n(n-p)}} \le C_{p,n} \left(\int\limits_{B(x_0,r_0)} Q^{\frac{n}{n-p}}(x)\, dm(x) \right)^{\frac{1}{n}}$$

which implies

$$|f(x)-f(x_0)| \left(\log \frac{1}{|x-x_0|} \right)^{\frac{p(n-1)}{n(n-p)}} \le d(f(B(x_0,\varepsilon))) \left(\log \frac{1}{\varepsilon} \right)^{\frac{p(n-1)}{n(n-p)}} \le C_{p,n} \, \|Q\|_{\frac{n}{n-p}}^{\frac{1}{\frac{n-p}{n}}},$$

where $\|Q\|_{\frac{n}{n-p}}$ is the norm (3.1) defined over $B(x_0,r_0)$ and $C_{p,n}$ is a positive constant depending only on p and n. □

The lower bound for the exponent, namely $\alpha = n/(n-p)$, is sharp in the following sense: this degree $n/(n-p)$ can be reduced only by adding more restrictions on the majorant Q. This will be illustrated in the following example and in the next section.

Fix any p, $n-1<p<n$, consider a mapping $f:\mathbb{B}^n \to \mathbb{B}^n$, defined by

$$f(x) = \frac{x}{|x|} \left(1 + \frac{n-p}{p-1} \log \frac{1}{|x|} \right)^{-\frac{p-1}{n-p}}, \quad x \neq 0, \quad \text{and} \quad f(0)=0. \tag{3.18}$$

We show that f is a ring (p,Q)-homeomorphism at the origin with $Q(x) = |x|^{p-n}$. Using the spherical coordinates (ρ,ψ_i) and (r,φ_i), $i=1,\dots,n-1$, for the image and the inverse image, respectively, one can rewrite (3.18) by

$$f(x)=\left\{\rho=\left(1+\frac{n-p}{p-1}\log\frac{1}{r}\right)^{-\frac{p-1}{n-p}}, \psi_i=\varphi_i, 0<r<1, 0\le \varphi_i<2\pi\right\}, f(0)=0.$$

The p-inner dilatation of this mapping can be calculated similar to [11], and one obtains that this dilatation coefficient is equal to r^{p-n}. Thus by [11, Theorem 4.1], $Q(x) = |x|^{p-n}$. On the other hand, a direct calculation yields

$$\lim_{x\to 0}|f(x)|\left(\log\frac{1}{|x|}\right)^{\frac{p-1}{n-p}} = \left(\frac{p-1}{n-p}\right)^{\frac{p-1}{n-p}}.$$

This shows that the mapping f is neither Hölder continuous with any exponent α nor logarithmically Hölder continuous with the exponent $p(n-1)/n(n-p)$, because

$$\frac{p-1}{n-p} < \frac{p(n-1)}{n(n-p)},$$

and, moreover, the estimate given by Theorem 3.2 is sharp in the order.

The questions concerning equicontinuity and normality of various classes of mappings are of a special interest. For the classical quasiconformal mappings we refer to [33], for more general mappings quasiconformal in the mean see [26] (cf. [23]).

Theorem 3.3. *Let G and G^* be two domains in $\mathbb{R}^n$, $n\ge 2$, and let $\mathcal{F}_Q$ be a family of open discrete ring (p,Q)-mappings $f: G\to G^*$ at x_0, $n-1<p<n$, with $Q(x)\in L^\alpha(G)$, $\alpha\ge \frac{n}{n-p}$. Then the family $\mathcal{F}_Q$ is equicontinuous at x_0.*

This theorem follows from Theorems 3.1 and 3.2.

Corollary 3.1. *Let $\mathcal{F}_Q$ be a family of all open discrete ring (p,Q)-mappings $f: G\to G^*$ in G, $n-1<p<n$, with $Q(x)\in L^\alpha(G)$, $\alpha\ge\frac{n}{n-p}$. Then the family $\mathcal{F}_Q$ is normal.*

Here f is assumed to be a mapping between metric spaces $(X,d_X) := (G,|\cdot|)$ and $(Y,d_Y)=\left(\overline{\mathbb{R}^n},h\right)$, where G is a domain in $\mathbb{R}^n$, and $|\cdot|$ and h stand for Euclidean metric in $\mathbb{R}^n$ and chordal metric in $\overline{\mathbb{R}^n}$, respectively.

Observe, that the assumption $\alpha\ge n/(n-p)$ is essential. Indeed, the following theorem shows that reducing this lower bound cannot insure equicontinuity.

Theorem 3.4. *Given $\varepsilon \in (0, n/(n-p))$, there exist a function $Q = Q_\varepsilon : \mathbb{B}^n \to [0, \infty]$ and a sequence of bounded ring (p, Q)-homeomorphisms $f_m : \mathbb{B}^n \to \mathbb{R}^n$ at $x_0 = 0$ such that $Q \in L^{\frac{n}{n-p}-\varepsilon}(\mathbb{B}^n)$ and f_m is not equicontinuous at $x_0 = 0$.*

Proof. For a given $0 < \alpha < 1$, we set

$$Q(x) = \frac{1}{|x|^{(\alpha-1)(p-1)+n-1}}\,; \tag{3.19}$$

then

$$\int\limits_{\mathbb{B}^n} Q^s(x)dm(x) < \infty,$$

whenever $s < \frac{n}{(\alpha-1)(p-1)+n-1}$. Letting $\alpha \to 0$, one gets $Q \in L^s$ for any s sufficiently close to $n/(n-p)$. This provides the integrability of Q with any degree ε, $0 < \varepsilon < n/(n-p)$.

Now consider the sequence of homeomorphisms $f_m : \mathbb{B}^n \to \mathbb{B}^n$ defined by

$$f_m(x) = \frac{x}{|x|}\varphi_m(|x|)\,, \quad f_m(0) = 0\,, \tag{3.20}$$

where

$$\varphi_m(s) = \left(1 + \frac{n-p}{p-1}\int\limits_s^1 \frac{dt}{t^{\frac{n-1}{p-1}} q_{0,m}^{\frac{1}{p-1}}(t)}\right)^{\frac{p-1}{p-n}}, \quad p \in (1, n)\,, \tag{3.21}$$

$$q_{0,m}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{|x|=r} Q_m(x)\, dS\,,$$

$$Q_m(x) \quad = \quad \begin{cases} Q(x), & |x| > 1/m\,, \\ 1\,, & |x| \le 1/m\,, \end{cases}$$

To show that f_m is a ring (p, Q)-homeomorphism at $x_0 = 0$, pick arbitrary $r_1, r_2 \in \mathbb{R}$ such that $0 < r_1 < r_2 < 1$ and consider a condenser $E = \left(B(0, r_2), \overline{B(0, r_1)}\right)$. Clearly,

$$f_m(E) = \left(B(0, \varphi_m(r_2)), \overline{B(0, \varphi_m(r_1))}\right)$$

also is a condenser for any m, and its p-capacity can be calculated by

$$\mathrm{cap}_p f_m(E) = \omega_{n-1}\left(\frac{p-1}{n-p}\right)^{1-p}\left(\varphi_m^{\frac{p-n}{p-1}}(r_1) - \varphi_m^{\frac{p-n}{p-1}}(r_2)\right)^{1-p}, \ 1 < p < n, \tag{3.22}$$

(cf. [8, p. 177]). Substituting the values of φ_m from (3.21) into (3.22) yields

$$\operatorname{cap}_p f_m(E) = \omega_{n-1}/I^{p-1},$$

where

$$I = I(r_1, r_2) = \int\limits_{r_1}^{r_2} \frac{dr}{r^{\frac{n-1}{p-1}} q_0^{\frac{1}{p-1}}(r)}.$$

Hence, by Proposition 2.2, all f_m are ring (p, Q_m)-mappings at the origin, and, therefore, they are (p, Q)-homeomorphisms. However,

$$|f_m(x_m)| = \left(1 + \frac{n-p}{p-1} \int\limits_{1/m}^{1} \frac{dt}{t^{\frac{n-1}{p-1}} q_{0,m}^{\frac{1}{p-1}}(t)}\right)^{\frac{p-1}{p-n}} \geq \sigma > 0$$

for all $m \in \mathbb{N}$ and some $\sigma > 0$. Thus, f_m is not equicontinuous at $x_0 = 0$. □

4. Distortion estimates for integrable majorant with additional conditions

In this section we discuss the case when Q is locally integrable in a domain G with a special behavior of the integral (2.3). The following theorem provides a sufficient condition for (p, Q)-mappings to be equicontinuous at a prescribe point.

Theorem 4.1. *Let G and G^* be two domains in $\mathbb{R}^n$, $n \geq 2$, and $\mathcal{F}_Q$ be a family of open discrete ring (p, Q)-mappings $f : G \to G^*$ at a point $x_0 \in G$, $n - 1 < p < n$. Assume that $Q : G \to [0, \infty]$ is a locally integrable function in G such that $\lim\limits_{\varepsilon \to 0} I(x_0, \varepsilon^2, \varepsilon) = \infty$, where $I(x_0, r_1, r_2)$ is defined by (2.3). Then the family $\mathcal{F}_Q$ is equicontinuous at x_0.*

Proof. We start with estimating the n-Lebesgue measure of images of the balls centered at x_0. Consider a condenser $\mathcal{E} = \left(B(x_0, \varepsilon_2), \overline{B(x_0, \varepsilon_1)}\right) \subset G$. By Proposition 2.2,

$$\operatorname{cap}_p f(\mathcal{E}) \leq \frac{\omega_{n-1} C_{x_0}}{I^{p-1}(x_0, r_1, r_2)}. \tag{4.1}$$

Taking $\varepsilon_1 = \sqrt{\varepsilon}$ and $\varepsilon_2 = \sqrt[4]{\varepsilon}$, one gets

$$\operatorname{cap}_p \left(f(B(x_0, \sqrt[4]{\varepsilon})), f(\overline{B(x_0, \sqrt{\varepsilon})})\right) \leq \frac{\omega_{n-1} C_{x_0}}{I^{p-1}(x_0, \sqrt{\varepsilon}, \sqrt[4]{\varepsilon})}. \tag{4.2}$$

On the other hand, the inequality (2.1) yields

$$\operatorname{cap}_p \left(f(B(x_0, \sqrt[4]{\varepsilon})), f(\overline{B(x_0, \sqrt{\varepsilon})})\right) \geq C_{n,p} \left[m(f(B(x_0, \sqrt{\varepsilon})))\right]^{\frac{n-p}{n}}, \tag{4.3}$$

where $C_{n,p}$ is a positive constant depending only on the dimension n and on p.

Combining the inequalities (4.2) and (4.3), one derives the desired bound for the volume of $f(B(x_0,\sqrt{\varepsilon}))$,

$$m(f(B(x_0,\sqrt{\varepsilon}))) \le \frac{\alpha_{n,p} C_{x_0}^{\frac{n}{n-p}}}{I^{\frac{n(p-1)}{n-p}}(x_0,\sqrt{\varepsilon},\sqrt[4]{\varepsilon})}\,. \tag{4.4}$$

Here $\alpha_{n,p}$ also is a positive constant depending only on n and p.

Letting in (4.1), $\varepsilon_1=\varepsilon$ and $\varepsilon_2=\sqrt{\varepsilon}$, one gets

$$\operatorname{cap}_p\left(f(B(x_0,\sqrt{\varepsilon})), f(\overline{B(x_0,\varepsilon)})\right) \le \frac{\omega_{n-1} C_{x_0}}{I^{p-1}(x_0,\varepsilon,\sqrt{\varepsilon})}\,, \tag{4.5}$$

and using the lower estimate for p-capacity given by (2.2),

$$\operatorname{cap}_p\left(f(B(x_0,\sqrt{\varepsilon})), f(\overline{B(x_0,\varepsilon)})\right) \ge \left(\widetilde{C}_{n,p}\frac{d^p(f(\overline{B(x_0,\varepsilon)}))}{m^{1-n+p}(f(B(x_0,\sqrt{\varepsilon})))}\right)^{\frac{1}{n-1}} \tag{4.6}$$

where $\widetilde{C}_{n,p}$ depends on n p.

The estimates (4.5) and (4.6) result in

$$\left(\widetilde{C}_{n,p}\frac{d^p(f(\overline{B(x_0,\varepsilon)}))}{m^{1-n+p}(f(B(x_0,\sqrt{\varepsilon})))}\right)^{\frac{1}{n-1}} \le \frac{\omega_{n-1} C_{x_0}}{I^{p-1}(x_0,\varepsilon,\sqrt{\varepsilon})}.$$

Together with (4.4), this yields

$$d(f(\overline{B(x_0,\varepsilon)})) \le \frac{\alpha_{n,p} C_{x_0}^{\frac{n-1}{p}}}{I^{\frac{(p-1)(n-1)}{p}}(x_0,\varepsilon,\sqrt{\varepsilon})}\left(\frac{\beta_{n,p} C_{x_0}^{\frac{n}{n-p}}}{I^{\frac{n(p-1)}{n-p}}(x_0,\sqrt{\varepsilon},\sqrt[4]{\varepsilon})}\right)^{\frac{1-n+p}{p}} \tag{4.7}$$

with some constants $\alpha_{n,p}$ and $\beta_{n,p}$ depending on n and p. Letting $\varepsilon\to 0$, one derives the assertion of Theorem 4.1 because both integrals in (4.7) tend to ∞. □

Replacing the condition $\lim\limits_{\varepsilon\to 0} I(x_0,\varepsilon^2,\varepsilon)=\infty$ by a stronger one, e.g. $q_{x_0}(r)\le C_{x_0} r^{p-n}$, one can obtain an explicit distortion estimate. This estimate also provides the local logarithmic Hölder continuity of open discrete ring (p,Q)-mappings. We present this as

Corollary 4.1. *Let $f: G\to\mathbb{R}^n$ be a discrete open ring (p,Q)-mapping at $x_0\in G$, $n-1<p<n$, with $Q\in L^1_{\mathrm{loc}}(G)$ satisfying*

$$q_{x_0}(r)\le C_{x_0} r^{p-n} \quad \text{for a.a.} \quad r\in(0,r_0),\ r_0\in(0,\min\{1,d_0\}). \tag{4.8}$$

Then

$$\limsup_{x\to x_0} |f(x)-f(x_0)| \left(\log \frac{1}{|x-x_0|}\right)^{\frac{p-1}{n-p}} \le \lambda_{n,p} C_{x_0}^{\frac{1}{n-p}}, \tag{4.9}$$

with a positive constant $\lambda_{n,p}$ depending only on n and p.

Since by assumption $q_{x_0}(r) \le C_{x_0} r^{p-n}$ for a.a. $r \in (0, r_0)$, $r_0 \in (0, \min\{1, d_0\})$, we obtain from (4.7) that

$$d(f(\overline{B(x_0,\varepsilon)})) \le \frac{\lambda_{n,p} C_{x_0}^{\frac{1}{n-p}}}{\left(\log \frac{1}{\varepsilon}\right)^{\frac{p-1}{n-p}}}.$$

Letting $\varepsilon \to 0$, one derives the estimate (4.9).

The automorphism of the unit ball $\mathbb{B}^n$ defined by (3.18) shows that the estimate (4.9) is sharp with respect to the order.

5. On applications to Orlicz–Sobolev spaces

In this section we apply the results on normality obtained above to homeomorphic mappings which belong to the Orlicz-Sobolev classes.

5.1. p-module of surface families

Let $\mathcal{S}$ be a k-dimensional surface, which means that $\mathcal{S}: D_s \to \mathbb{R}^n$ is a continuous image of an open set $D_s \subset \mathbb{R}^k$. We denote by

$$N(\mathcal{S}, y) = \operatorname{card} \mathcal{S}^{-1}(y) = \operatorname{card}\{x \in D_s : \mathcal{S}(x) = y\}$$

the *multiplicity function* of the surface $\mathcal{S}$ at the point $y \in \mathbb{R}^n$. For a given Borel set $B \subseteq \mathbb{R}^n$ (or, more generally, for a measurable set B with respect to the k-dimensional Hausdorff measure $\mathcal{H}^k$), the k-dimensional Hausdorff area of B in $\mathbb{R}^n$ associated with the surface $\mathcal{S}$ is determined by

$$\mathcal{A}_{\mathcal{S}}(B) = \mathcal{A}^k_{\mathcal{S}}(B) = \int_B N(\mathcal{S}, y)\, d\mathcal{H}^k y,$$

see [7, 3.2.1]. If $\rho : \mathbb{R}^n \to [0, \infty]$ is a Borel function, the integral of ρ over $\mathcal{S}$ is defined by

$$\int_{\mathcal{S}} \rho\, d\mathcal{A} = \int_{\mathbb{R}^n} \rho(y) N(\mathcal{S}, y)\, d\mathcal{H}^k y.$$

Let $\mathcal{S}_k$ be a family of k-dimensional surfaces $\mathcal{S}$ in $\mathbb{R}^n$, $1 \le k \le n-1$ (curves for $k=1$). The p-module of $\mathcal{S}_k$ is defined by (1.3), where the infimum is taken over all Borel measurable functions $\rho \ge 0$ such that

$$\int_{\mathcal{S}} \rho^k\, d\mathcal{A} \ge 1$$

for every $\mathcal{S} \in \mathcal{S}_k$. We call each such ρ an *admissible metric* for $\mathcal{S}_k$ ($\rho \in \operatorname{adm} \mathcal{S}_k$).

Following [19], a metric ρ is said to be *extensively admissible* for $\mathcal{S}_k$ ($\rho \in \operatorname{ext}_p \operatorname{adm} \mathcal{S}_k$) with respect to p-module if $\rho \in \operatorname{adm}(\mathcal{S}_k \backslash \widetilde{\mathcal{S}}_k)$ for any subfamily $\widetilde{\mathcal{S}}_k$ with $\mathcal{M}_p(\widetilde{\mathcal{S}}_k) = 0$ (cf. [11]).

Accordingly, we say that a property $\mathcal{P}$ holds for almost every k-dimensional surface, if $\mathcal{P}$ holds for all surfaces except a family of zero p-module.

The following statement concerns the equivalence of two notions "almost all" related to a family of k-dimensional surfaces depending on a real parameter and to the parameter itself.

Lemma 5.1. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $p \in [n-1, \infty)$ and $x_0 \in D$. The following statements are equivalent:*
(1) *a property $\mathcal{P}$ holds for p-a.e. surfaces $D(x_0, r) := S(x_0, r) \cap D$;*
(2) *$\mathcal{P}$ holds for a.e. $D(x_0, r)$ with respect to the parameter $r \in \mathbb{R}$.*

Proof. The proof follows the lines of Lemma 9.1 in [23].

It suffices to establish the implication (1) $\Rightarrow$ (2) when D is bounded. Assume, in the contrary, that $\mathcal{P}$ holds for p-a.e. surfaces $D(x_0, r) := S(x_0, r) \cap D$, however there is a Borel set $B \subset \mathbb{R}$ of positive one-dimensional Lebesgue measure $m_1(B)$ provided that $\mathcal{P}$ fails for $D(x_0, r)$ at a.e. $r \in B$. If a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is admissible for the family Γ of spheres $S(x_0, r)$, one obtains for its restriction to

$$E = \bigcup_{r \in B} \{x \in D \,:\, |x - x_0| = r\}$$

(by the Hölder inequality),

$$\int\limits_E \rho^{n-1}(x)\; dm(x) \;\leq\; \left(\int\limits_E \rho^p(x)\, dm(x)\right)^{\frac{n-1}{p}} \left(\int\limits_E \; dm(x)\right)^{\frac{p-n+1}{p}}.$$

This yields, together with the Fubini theorem,

$$\int\limits_{\mathbb{R}^n} \rho^p(x)\, dm(x) \;\geq\; \frac{\left(\int\limits_E \rho^{n-1}(x)\, dm(x)\right)^{\frac{p}{n-1}}}{\left(\int\limits_E \; dm(x)\right)^{\frac{p-n+1}{k}}} \;\geq\; \frac{(m_1(B))^{\frac{p}{n-1}}}{c}$$

for some $c > 0$; cf. [27, Theorem 8.1, Ch. III]. Thus, $\mathcal{M}_p(\Gamma) > 0$, in contradiction to (1).

To prove (2) $\Rightarrow$ (1), consider the family Γ_0 of all intersections $D_r := D(x_0, r)$ of the spheres $S(x_0, r)$ with D for which the property $\mathcal{P}$ does not hold. Let R be the set of all $r \in \mathbb{R}$ for which $D_r \in \Gamma_0$. Since $m_1(R) = 0$, one concludes by Fubini's theorem that $m(E) = 0$. Consider the function $\rho_1 : \mathbb{R}^n \to [0, \infty]$ which equals ∞ for all $x \in E$, and 0 otherwise. Then, by the Lusin theorem, there is a Borel function $\rho_2 : \mathbb{R}^n \to [0, \infty]$ such that $\rho_2 = \rho_1$ a.e. in $\mathbb{R}^n$ (see, e.g. [7, Section 2.3.5]), and

$$\mathcal{M}_p(\Gamma_0) \leq \int\limits_E \rho_2^p dm(x) = \int\limits_E \rho_1^p dm(x) = 0.$$

Hence, $\mathcal{M}_p(\Gamma_0) = 0$, which completes the proof of Lemma 5.1. $\square$

5.2. Orlicz-Sobolev classes

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function. The *Orlicz-Sobolev class* $W^{1,\varphi}_{\text{loc}}(D)$ consists of all locally integrable functions f with the first distributional derivatives whose gradient ∇f belongs locally in D to the Orlicz space. Note that by definition $W^{1,\varphi}_{\text{loc}} \subset W^{1,1}_{\text{loc}}$. For $\varphi(t) = t^p$, $p \geq 1$, we shall use the standard notation $f \in W^{1,p}_{\text{loc}}$.

Later on, we also write $f \in W^{1,\varphi}_{\text{loc}}$ for a locally integrable vector-function $f = (f_1, \dots, f_m)$ of n real variables $x_1, \dots, x_n$ if $f_i \in W^{1,1}_{\text{loc}}$ and

$$\int\limits_D \varphi\left(|\nabla f(x)|\right)\, dm(x) < \infty$$

where $|\nabla f(x)| = \sqrt{\sum\limits_{i,j} \left(\frac{\partial f_i}{\partial x_j}\right)^2}$.

Recall that a homeomorphism f between domains D and D' in $\mathbb{R}^n$, $n \geq 2$, is of *finite distortion* if $f \in W^{1,1}_{\text{loc}}$ and

$$\|f'(x)\|^n \leq K_O(x, f) J(x, f)$$

with a.e. finite function K_O where $\|f'(x)\|$ stands for the matrix norm of the Jacobian matrix f' of f at $x \in D$, $\|f'(x)\| = \sup\limits_{h \in \mathbb{R}^n, |h|=1} |f'(x) \cdot h|$, and $J(x, f) = \det f'(x)$ denotes its Jacobian. For the mappings of finite distortion, we refer to [18] and to the reference therein.

In a similar way, we define a counterpart of K_O with respect to the real parameter p, $p \geq 1, p \neq n$, by

$$K_{O,p}(x, f) \;=\; \begin{cases} \frac{\|f'(x)\|^p}{J(x,f)}\,, & J(x, f) \neq 0\,, \\ 0\,, & p > n \text{ and } f'(x) = 0\,, \\ \infty\,, & \text{otherwise.} \end{cases}$$

The following lemma involves a Calderon type condition [4] and shows how relate between the mappings of Orlicz-Sobolev's spaces and (p, Q)-homeomorphisms.

Lemma 5.2. *Let G and G^* be domains in $\mathbb{R}^n$, $n \geq 3$, and let $\varphi : [0,\infty) \to [0,\infty)$ be an increasing function satisfying the following Calderon type condition*

$$\int_1^\infty \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt < \infty. \tag{5.1}$$

Assume that $K_{O,p}(x,f) \in L_{\mathrm{loc}}^{\frac{n-1}{p-n+1}}(G)$, $p > n-1$. Then each homeomorphism $f : G \to G^*$ of finite distortion and such that $f \in W_{\mathrm{loc}}^{1,\varphi}$ is a ring $(\alpha, \widetilde{Q})$-homeomorphism at every point $x_0 \in G$; here $\widetilde{Q}(x) = K_{O,p}^{\frac{n-1}{p-n+1}}(x)$ and $\alpha = \frac{p}{p-n+1}$.

The proof of Lemma 5.2 follows from [29, Theorem 5.1] and [12, Theorem 7.1].

5.3. Equicontinuity of Orlicz-Sobolev classes

Observe that $\alpha = \frac{p}{p-n+1} \in (n-1, n)$, whenever $p \in \left(n, \frac{(n-1)^2}{n-2}\right)$. So one can apply the results obtained in the previous sections to such p and α. For example, Theorem 3.3, Lemma 5.2 and Corollary 3.1 yield:

Theorem 5.1. *Let G and G^* be two domains in $\mathbb{R}^n$, $n \geq 3$, and let $Q : G \to [0,\infty]$ be a Lebesgue measurable function from $L_{\mathrm{loc}}^{\frac{n-1}{p-n+1}\beta}(G)$, with $\beta \geq \frac{n}{n-\alpha}$. Assume that $\varphi : [0,\infty) \to [0,\infty)$ is an increasing function satisfying (5.1). Then the family $\mathcal{F}_{Q,\varphi}$ of all homeomorphisms $W_{\mathrm{loc}}^{1,\varphi}$ of finite distortion such that $K_{O,p}(x) \leq Q(x)$, with $p \in \left(n, \frac{(n-1)^2}{n-2}\right)$, is equicontinuous in G. Moreover, the family $\mathcal{F}_{Q,\varphi}$ is normal.*

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