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On equicontinuity of solutions to the Beltrami equations

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Abstract - It is shown that each homeomorphic $W^{1,1}_{\rm loc}$ solution to the Beltrami equation $\overline{\partial} f = \mu\,\partial f$ is the so-called ring Q–homeomorphism with $Q(z) = K_\mu(z)$ where $K_\mu(z)$ is the dilatation quotient of this equation. On this base, it is stated equicontinuity and normality of families of such solutions under the conditions that $K_\mu(z)$ has a majorant of finite mean oscillation, singularities of logarithmic type or integral constraints of the type $\int \Phi\,(K_\mu(z))\,dx\,dy < \infty$ in a domain $D \subset \mathbb{C}$. The found conditions on the function Φ are not only sufficient but also necessary for equicontinuity and normality of the corresponding families of solutions to the Beltrami equation.

Key words and phrases : Beltrami equations, equicontinuity, normality, lower and ring Q–homeomorphisms.

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1. Introduction

In this paper we present applications of our results on the ring Q–homeomorphisms in the papers [17], [18] and Chapter 7 in the monograph [14] to the study of the problems of equicontinuity and normality for wide classes of solutions for the Beltrami equations with degeneration.

Let D be a domain in the complex plane $\mathbb{C}$, i.e., a connected and open subset of $\mathbb{C}$, and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. (almost everywhere) in D. The **Beltrami equation** is the equation of the form

$$f_{\overline{z}} \;=\; \mu(z) f_z \tag{1.1}$$

where $f_{\overline{z}} = \overline{\partial} f = (f_x + i f_y)/2$, $f_z = \partial f = (f_x - i f_y)/2$, $z = x + iy$, and f_x and f_y are partial derivatives of f in x and y, correspondingly. The function μ is called the **complex coefficient** and

$$K_\mu(z) \;=\; \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{1.2}$$

the **dilatation quotient** for the equation (1.1). The Beltrami equation (1.1) is said to be **degenerate** if $\operatorname{ess\,sup} K_\mu(z) = \infty$. The existence theorem for homeomorphic $W^{1,1}_{\rm loc}$ solutions was established to many degenerate Beltrami equations, see, e.g., related references in the recent monographs [3] and [14] and in the survey [9].

Recall that the **(conformal) modulus** of a family Γ of curves γ in $\mathbb{C}$ is the quantity

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{C}} \rho^2(z) \quad dx\,dy \tag{1.3}$$

where a Borel function $\rho : \mathbb{C} \to [0, \infty]$ is **admissible** for Γ (write $\rho \in \operatorname{adm}\Gamma$), if

$$\int\limits_{\gamma} \rho\ ds \geq 1 \qquad \forall\ \gamma \in \Gamma \tag{1.4}$$

where s is a natural parameter of the length on γ.

Throughout this paper we will use the following notation

$$B(z_0,\, r) = \{z \in \mathbb{C}\,|\, |z_0 - z| < r\}, \quad \mathbb{D} = B(0,\, 1),$$

$$S(z_0,\, r) = \{z \in \mathbb{C}\,|\, |z_0 - z| = r\}, \quad S(r) = S(0,\, r),$$

$$R(r_1, r_2, z_0) = \{z\, \in\, \mathbb{C}\,|\, r_1 < |z - z_0| < r_2\}.$$

Let $E,\ F \subset \overline{\mathbb{C}}$ be arbitrary sets. Denote by $\Gamma(E, F, D)$ the family of all curves $\gamma : [a, b] \to \overline{\mathbb{C}}$ joining E and F in D, i.e., $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ as $t \in (a, b)$.

The following notion is motivated by the ring definition of Gehring for quasiconformal mappings (see, e.g., [7]) introduced first in the plane (see [20]) and extended later on to the space case in [17] (see also Chapters 7 and 11 in [14]).

Given a domain D in $\mathbb{C}$, a (Lebesgue) measurable function $Q : D \to [0, \infty]$, $z_0 \in D$, a homeomorphism $f : D \to \overline{\mathbb{C}}$ is said to be a **ring Q–homeomorphism at the point z_0** if

$$M\left(f\left(\Gamma\left(S_1,\, S_2,\, R(r_1, r_2, z_0)\right)\right)\right) \ \leq \int\limits_{R(r_1,r_2,z_0)} Q(z) \cdot \eta^2(|z - z_0|)\ dx\,dy \tag{1.5}$$

for every ring $R(r_1, r_2, z_0)$ and the circles $S_i = S(z_0, r_i)$, where $0 < r_1 < r_2 < r_0 : \ = \ \operatorname{dist}(z_0, \partial D)$, and every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\ dr \ \geq\ 1\,.$$

f is called a **ring Q–homeomorphism in the domain** D if f is a ring Q–homeomorphism at every point $z_0 \in D$. The notion of ring Q–homeomorphism is closely related to the concept of moduli with weights essentially due to Andreian Cazacu (see, e.g., [1] and references therein).

A continuous mapping γ of an open subset Δ of the real axis $\mathbb{R}$ or a circle into D is called a **dashed line**, see, e.g., 6.3 in [14]. The notion of the modulus of the family Γ of dashed lines γ is defined similarly to (1.3). We say that a property P holds for **a.e.** (almost every) $\gamma \in \Gamma$ if the subfamily of all lines in Γ for which P fails has the modulus zero, cf. [6]. Later on, we also say that a Lebesgue measurable function $\varrho : \mathbb{C} \to [0, \infty]$ is **extensively admissible** for Γ, write $\varrho \in \operatorname{ext}\operatorname{adm}\Gamma$, if (1.4) holds for a.e. $\gamma \in \Gamma$ (see, e.g., 9.2 in [14]).

The following conception was introduced in [12], see also Chapter 9 in [14]. Given domains D and D' in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $z_0 \in \overline{D} \setminus \{\infty\}$, and a measurable function $Q : D \to (0, \infty)$, one says that a homeomorphism $f : D \to D'$ is a **lower Q-homeomorphism at the point** z_0 if

$$M(f\Sigma_\varepsilon) \ \geqslant \inf_{\varrho \in \operatorname{ext}\operatorname{adm}\Sigma_\varepsilon} \int\limits_{D \cap R(\varepsilon,\, \varepsilon_0,\, z_0)} \frac{\varrho^2(z)}{Q(z)}\, dx\, dy \tag{1.6}$$

for every ring $R(\varepsilon,\, \varepsilon_0,\, z_0)$, $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 \in (0, d_0)$, where $d_0 = \sup\limits_{z \in D} |z - z_0|$, and Σ_ε denotes the family of all intersections of the circles $S(z_0, r), r \in (\varepsilon, \varepsilon_0)$, with D.

It was established earlier that a homeomorphism $f : D \to \overline{\mathbb{C}}$ in the class $W^{1,2}_{loc}$ with $K_\mu(z) \in L^1_{loc}(D)$ is a ring Q–homeomorphism with $Q(z) = K_\mu(z)$, (see, e.g., Theorem 4.1 in [14], cf. also [4]) and that a regular homeomorphism of the Sobolev class $W^{1,1}_{loc}$ in the plane with $J_f(z) \neq 0$ a.e. is a ring Q-homeomorphism with $Q(z)$ is equal to the so-called tangential dilatation (see Theorem 3.1. in [21], cf. Lemma 20.9.1 in [3]). Further we show that each homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution of the Beltrami equation (1.1) is a lower Q–homeomorphism as well as a ring Q–homeomorphism with $Q(z) = K_\mu(z)$ and, thus, the whole theory in [17] and [18], see also Chapter 7 in [14], can be applied to such solutions.

2. Preliminaries

First of all, let us give criteria of lower and ring Q-homeomorphisms (see Theorem 2.1 in [12] and Theorem 3.15 in [17], or Theorem 9.2 and 7.2 in [14], correspondingly).

Proposition 2.1. *Let D and D' be domains in $\mathbb{C}$, let $z_0 \in \overline{D} \setminus \{\infty\}$, and let $Q : D \to (0, \infty)$ be a measurable function. A homeomorphism $f : D \to D'$ is a lower Q-homeomorphism at z_0 if and only if*

$$M(f\Sigma_\varepsilon) \ \geq \int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{||\,Q||_{\,1}(r)} \quad \textit{for all } \varepsilon \in (0, \varepsilon_0)\,, \quad \varepsilon_0 \in (0, d_0), \tag{2.1}$$

where

$$d_0 \;=\; \sup_{z\in D} |z-z_0|, \tag{2.2}$$

Σ_ε *denotes the family of all the intersections of the circles* $S(z_0, r)$, $r \in (\varepsilon, \varepsilon_0)$, *with* D, *and*

$$||\,Q||_{\,1}(r) = \int\limits_{D(z_0,r)} Q(z)\; ds \tag{2.3}$$

is the L_1*-norm of* Q *over* $D(z_0, r) = \{z \in D : |\,z - z_0| = r\} = D \cap S(z_0, r)$.

Proposition 2.2. *Let* D *be a domain in* $\mathbb{C}$ *and* $Q : D \to [0,\infty]$ *a measurable function. A homeomorphism* $f : D \to \mathbb{C}$ *is a ring* Q*-homeomorphism at a point* $z_0 \in D$ *if and only if, for every* $0 < r_1 < r_2 < d_0 = \mathrm{dist}\,(z_0, \partial D)$,

$$M(\Gamma(fS_1, fS_2, fD)) \;\le\; \frac{2\pi}{I}, \tag{2.4}$$

where $q_{z_0}(r)$ *is the mean integral value of* $Q(z)$ *over the circle* $|z - z_0| = r$, $S_j = S(z_0, r_j)$, $j = 1, 2$, *and*

$$I \;=\; I(r_1,r_2) \;=\; \int\limits_{r_1}^{r_2} \frac{dr}{rq_{z_0}(r)}\,.$$

Propositions 2.1 and 2.2 now yield the following consequence.

Corollary 2.1. *Let* D *and* D' *be domains in* $\mathbb{C}$, *let* $z_0 \in D$, *and let* $Q : D \to (0,\infty)$ *be a measurable function. If a homeomorphism* $f : D \to D'$ *is a lower* Q*-homeomorphism at* z_0, *then* f *is a ring* Q*–homeomorphism at* z_0.

Indeed, denote by Σ_ε the family of all circles $S(z_0, r)$, $r \in (\varepsilon, \varepsilon_0)$, $\varepsilon_0 \in (0, d_0)$. By Theorem 3.13 in [24], we have

$$M\left(\Gamma\left(fS_\varepsilon,\, fS_{\varepsilon_0},\, f(D)\right)\right) \le \frac{1}{M\left(f\Sigma_\varepsilon\right)} \le \frac{2\pi}{\int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{rq_{z_0}(r)}} \tag{2.5}$$

because $f\Sigma_\varepsilon \subset \Sigma\left(fS_\varepsilon,\, fS_{\varepsilon_0}\right)$, where $\Sigma\left(fS_\varepsilon,\, fS_{\varepsilon_0}\right)$ consists of all closed curves in $f(D)$ that separate fS_ε and fS_{ε_0}.

The following notion was introduced in [10]. Let D be a domain in the complex plane $\mathbb{C}$. A function $\varphi : D \to \mathbb{R}$ has **finite mean oscillation at a point** $z_0 \in D$ if

$$\overline{\lim_{\varepsilon\to 0}} \quad ⨍_{B(z_0,\varepsilon)} |\varphi(z) - \widetilde{\varphi}_\varepsilon(z_0)|\; dx\, dy \;<\; \infty, \tag{2.6}$$

where

$$\widetilde{\varphi}_{\varepsilon}(z_0) = \fint_{B(z_0,\varepsilon)} \varphi(z)\ dx\,dy\ <\ \infty \tag{2.7}$$

is the mean integral value of the function $\varphi(z)$ over the disk $B(z_0, \varepsilon)$. One also says that a function $\varphi : D \to \mathbb{R}$ is of **finite mean oscillation in D** (abbr. $\varphi \in$ FMO(D) or simply $\varphi \in$ **FMO**), if φ has a finite mean oscillation at every point $z_0 \in D$. Note that FMO is not BMO_{loc} (see, e.g., Section 11.2 in [14]).

Recall also that the **spherical (chordal) metric** $h(z', z'')$ in $\overline{\mathbb{C}}$ is equal to $|\pi(z') - \pi(z'')|$ where π is the stereographic projection of $\overline{\mathbb{C}}$ on the sphere $S^2(\frac{1}{2}e_3, \frac{1}{2})$ in $\mathbb{R}^3$, i.e., in the explicit form,

$$h(z', \infty) = \frac{1}{\sqrt{1 + |z'|^2}},\quad h(z', z'') = \frac{|z' - z''|}{\sqrt{1 + |z'|^2}\sqrt{1 + |z''|^2}}\,,\quad z' \neq \infty \neq z''\,.$$

The **spherical diameter of a set** E in $\overline{\mathbb{C}}$ is the quantity

$$h(E) = \sup_{z', z'' \in E} h(z', z'').$$

Given a domain D in $\mathbb{C}$, a family $\mathfrak{F}$ of continuous mappings from D into $\overline{\mathbb{C}}$ is said to be **normal** if every sequence of mappings f_m in $\mathfrak{F}$ has a subsequence f_{m_k} converging to a continuous mapping $f : D \to \overline{\mathbb{C}}$ uniformly on each compact set $C \subset D$. Normality is closely related to the following notion. A family $\mathfrak{F}$ of mappings $f : D \to \overline{\mathbb{C}}$ is said to be **equicontinuous at a point** $z_0 \in D$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $h\left(f(z), f(z_0)\right) < \varepsilon$ for all $f \in \mathfrak{F}$ and $z \in D$ with $|z - z_0| < \delta$. The family $\mathfrak{F}$ is called **equicontinuous** if $\mathfrak{F}$ is equicontinuous at every point $z_0 \in D$. The following version of the Arzela – Ascoli theorem will be useful later on (see, e.g., Section 20.4 in [23]).

Proposition 2.3. *If a family $\mathfrak{F}$ of mappings $f : D \to \overline{\mathbb{C}}$ is equicontinuous, then $\mathfrak{F}$ is normal.*

For every non-decreasing function $\Phi : [0, \infty] \to [0, \infty]$, the **inverse function** $\Phi^{-1} : [0, \infty] \to [0, \infty]$ can be well defined by setting

$$\Phi^{-1}(\tau)\ =\ \inf_{\Phi(t) \geq \tau} t\ . \tag{2.8}$$

As usual, here inf is equal to ∞ if the set of $t \in [0, \infty]$ such that $\Phi(t) \geq \tau$ is empty.

3. The main lemma

The following statement was first proved in [11], Theorem 3.1. We give here its proof for completeness.

Lemma 3.1. *Let f be a homeomorphic $W^{1,1}_{\rm loc}$ solution of the Beltrami equation* (1.1)*. Then f is a lower Q-homeomorphism at each point $z_0 \in \overline{D}$ with $Q(z) = K_\mu(z)$.*

Proof. Let B be the (Borel) set of all points z in D where f has a total differential with $J_f(z) \neq 0$. It is known that B is the union of a countable collection of Borel sets B_l, $l = 1, 2, \dots$, such that $f_l = f|_{B_l}$ is a bi-Lipschitz homeomorphism (see, e.g., Lemma 3.2.2 in [5]). With no loss of generality, we may assume that the B_l are mutually disjoint. Denote also by B_* the set of all points $z \in D$ where f has a total differential with $f'(z) = 0$.

Note that the set $B_0 = D \setminus (B \cup B_*)$ has the Lebesgue measure zero in $\mathbb{C}$ by Gehring-Lehto-Menchoff theorem (see [8] and [16]). Hence by Theorem 2.11 in [13] (see also Lemma 9.1 in [14]), $\text{length}(\gamma \cap B_0) = 0$ for a.e. paths γ in D. Let us show that $\text{length}(f(\gamma) \cap f(B_0)) = 0$ for a.e. circle γ centered at z_0.

The latter follows from absolute continuity of f on closed subarcs of $\gamma \cap D$ for a.e. such circle γ. Indeed, the class $W^{1,1}_{\rm loc}$ is invariant with respect to local quasi-isometries (see, e.g., Theorem 1.1.7 in [15]) and the functions in $W^{1,1}_{\rm loc}$ are absolutely continuous on lines (see, e.g., Theorem 1.1.3 in [15]). Applying say the transformation of coordinates $\log(z - z_0)$, we come to the absolute continuity on a.e. such circle γ.

Thus, $\text{length}(\gamma_* \cap f(B_0)) = 0$ where $\gamma_* = f(\gamma)$ for a.e. circle γ centered at z_0. Now, let $\varrho_* \in \operatorname{adm} f(\Gamma)$ where Γ is the collection of all dashed lines $\gamma \cap D$ for such circles γ and $\varrho_* \equiv 0$ outside $f(D)$. Set $\varrho \equiv 0$ outside D and

$$\varrho(z) \ : \ = \ \varrho_*(f(z))\,(|f_z| + |f_{\bar z}|) \qquad \text{for a.e. } \ z \in D.$$

Arguing piecewise on B_l, we have by Theorem 3.2.5 under $m = 1$ in [5] that

$$\int\limits_{\gamma} \varrho\, ds \ \geqslant \ \int\limits_{\gamma_*} \varrho_*\, ds_* \ \geqslant \ 1 \qquad \text{for a.e. } \ \gamma \in \Gamma$$

because $\text{length}(f(\gamma) \cap f(B_0)) = 0$ and $\text{length}(f(\gamma) \cap f(B_*)) = 0$ for a.e. $\gamma \in \Gamma$, and consequently, $\varrho \in \operatorname{ext} \operatorname{adm} \Gamma$.

On the other hand, again arguing piecewise on B_l, we have the inequality

$$\int\limits_{D} \frac{\varrho^2(z)}{K_\mu(z)}\, dx\, dy \ \leqslant \ \int\limits_{f(D)} \varrho_*^2(w)\ du\, dv$$

because $\varrho(z) = 0$ on B_*. Consequently, we obtain that

$$M(f\Gamma) \ \geqslant \ \inf_{\varrho \in \operatorname{ext} \operatorname{adm} \Gamma} \int\limits_{D} \frac{\varrho^2(z)}{K_\mu(z)}\ dx\, dy\,,$$

i.e., f is really a lower Q-homeomorphism with $Q(z) = K_\mu(z)$. □

Lemma 3.1 and Corollary 2.1 imply the following result.

Theorem 3.1. *Let f be a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution of the Beltrami equation* (1.1)*. Then f is a ring Q-homeomorphism at each point $z_0 \in D$ with $Q(z) = K_\mu(z)$.*

4. Estimates of Distortion

The results of this section are obtained on the base of Theorem 3.1 and the corresponding theorems in the work [17] (see also Chapter 7 in [14]).

Lemma 4.1. *Let D be a domain in $\mathbb{C}$, let D' be a domain in $\overline{\mathbb{C}}$ with $h(\overline{\mathbb{C}} \setminus D') \geq \Delta > 0$, and let $f : D \to D'$ be a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution of the Beltrami equation* (1.1) *at a point $z_0 \in D$. If, for $0 < \varepsilon_0 < \mathrm{dist}(z_0, \partial D)$,*

$$\int\limits_{\varepsilon<|z-z_0|\,<\,\varepsilon_0} K_\mu(z) \cdot \psi_\varepsilon^2(|z-z_0|)\, dx\, dy \;\leq\; c \cdot I^p(\varepsilon)\ , \qquad \varepsilon \in (0, \varepsilon_0), \quad (4.1)$$

where $p \leq 2$ and $\psi_\varepsilon(t)$ is a nonnegative function on $(0,\infty)$ such that

$$0 \;<\; I(\varepsilon) \;=\; \int\limits_{\varepsilon}^{\varepsilon_0} \psi_\varepsilon(t)\ dt < \infty, \qquad \varepsilon \in (0, \varepsilon_0), \quad (4.2)$$

then

$$h(f(z), f(z_0)) \;\leq\; \frac{32}{\Delta}\ \exp\left\{-\left(\frac{2\pi}{c}\right) I^{2-p}(|z-z_0|)\right\} \quad (4.3)$$

for all $z \in B(z_0, \varepsilon_0)$.

Corollary 4.1. *Under the conditions of Lemma* 4.1 *and for $p = 1$,*

$$h(f(z), f(z_0)) \;\leq\; \frac{32}{\Delta}\ \exp\left\{-\left(\frac{2\pi}{c}\right) I(|z-z_0|)\right\}. \quad (4.4)$$

Theorem 4.1. *Let D be a domain in $\mathbb{C}$, let D' be a domain in $\overline{\mathbb{C}}$ with $h(\overline{\mathbb{C}} \setminus D') \geq \Delta > 0$, and let $f : D \to D'$ be a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution of the Beltrami equation* (1.1) *at a point $z_0 \in D$. Then*

$$h(f(z), f(z_0)) \;\leq\; \frac{32}{\Delta}\ \exp\left\{-\int\limits_{|z-z_0|}^{\varepsilon(z_0)} \frac{dr}{r q_{z_0}(r)}\right\} \quad (4.5)$$

for $z \in B(z_0, \varepsilon(z_0))$, where $\varepsilon(z_0) < \mathrm{dist}(z_0, \partial D)$ and $q_{z_0}(r)$ is the mean integral value of $K_\mu(z)$ over the circle $|z - z_0| = r$.

Corollary 4.2. *If*

$$q_{z_0}(r) \le \log \frac{1}{r} \tag{4.6}$$

for $r < \varepsilon(z_0) < \operatorname{dist}(z_0, \partial D)$, *then*

$$h(f(z), f(z_0)) \le \frac{32}{\Delta} \frac{\log \frac{1}{\varepsilon_0}}{\log \frac{1}{|z-z_0|}} \tag{4.7}$$

for all $z \in B(z_0, \varepsilon(z_0))$.

Corollary 4.3. *If*

$$K_\mu(z) \le \log \frac{1}{|z - z_0|}, \qquad z \in B(z_0, \varepsilon(z_0)), \tag{4.8}$$

then (4.7) *holds in the disk* $B(z_0, \varepsilon(z_0))$.

Remark 4.1. If, instead of (4.6) and (4.8), we have the conditions

$$q_{z_0}(r) \le c \cdot \log \frac{1}{r} \tag{4.9}$$

and, correspondingly,

$$K_\mu(z) \le c \cdot \log \frac{1}{|z - z_0|}, \tag{4.10}$$

then

$$h(f(z), f(z_0)) \ \le \ \frac{32}{\Delta} \left[\frac{\log \frac{1}{\varepsilon_0}}{\log \frac{1}{|z-z_0|}} \right]^{1/c}. \tag{4.11}$$

Choosing in Lemma 4.1 $\psi(t) = 1/t$ and $p = 1$, we also have the following conclusion.

Corollary 4.4. *Let* $f : \mathbb{D} \to \mathbb{D}$ *be a homeomorphic* $W^{1,1}_{\mathrm{loc}}$ *solution of the Beltrami equation* (1.1) *such that* $f(0) = 0$ *and*

$$\int\limits_{\varepsilon<|z|<1} K_\mu(z) \ \frac{dx\,dy}{|z|^2} \ \le \ c \ \log \frac{1}{\varepsilon}, \qquad \varepsilon \in (0, 1). \tag{4.12}$$

Then

$$|f(z)| \ \le \ 64 \cdot |z|^{\frac{2\pi}{c}}. \tag{4.13}$$

Theorem 4.2. *Let D be a domain in $\mathbb{C}$, let D' be a domain in $\overline{\mathbb{C}}$ with $h(\overline{\mathbb{C}}\setminus D')\geq\Delta>0$, and let $f:D\to D'$ be a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution of the Beltrami equation (1.1). If $K_\mu(z)\leq Q(z)$ a.e. where Q has finite mean oscillation at a point $z_0\in D$, then*

$$h(f(z),f(z_0))\leq\frac{32}{\Delta}\left\{\frac{\log\frac{1}{\varepsilon_0}}{\log\frac{1}{|z-z_0|}}\right\}^{\beta_0} \tag{4.14}$$

for some $\varepsilon_0<\mathrm{dist}(z_0,\partial D)$ and every $z\in B(z_0,\varepsilon_0)$, where $\beta_0>0$ depends only on the function Q.

5. Criteria of normal families

Now, by Proposition 2.3, on the base of the last section, we obtain the corresponding criteria of normality for solutions to the Beltrami equations (see [2] and [22]).

Given a domain D in $\mathbb{C}$ and a measurable function $Q:D\to[1,\infty]$, let $\mathfrak{B}_{Q,\Delta}(D)$ be the class of all homeomorphic $W^{1,1}_{\mathrm{loc}}$ solutions f of the Beltrami equation (1.1) with $K_\mu(z)\leq Q(z)$ a.e in D with $h(\overline{\mathbb{C}}\setminus f(D))\geq\Delta>0$.

Theorem 5.1. *If $Q\in\mathrm{FMO}$, then $\mathfrak{B}_{Q,\Delta}(D)$ is a normal family.*

Corollary 5.1. *The class $\mathfrak{B}_{Q,\Delta}(D)$ is normal if*

$$\overline{\lim_{\varepsilon\to0}}\;\fint_{B(z_0,\varepsilon)}Q(z)\;\;dxdy\;<\;\infty\qquad\text{for all } z_0\in D. \tag{5.1}$$

Corollary 5.2. *The class $\mathfrak{B}_{Q,\Delta}(D)$ is normal if every $z_0\in D$ is a Lebesgue point of $Q(z)$.*

Theorem 5.2. *Let $\Delta>0$ and let $Q:D\to[0,\infty]$ be a measurable function such that*

$$\int\limits_0^{\varepsilon(z_0)}\frac{dr}{rq_{z_0}(r)}=\infty\qquad\text{for all } z_0\in D \tag{5.2}$$

where $\varepsilon(z_0)<\mathrm{dist}(z_0,\partial D)$ and $q_{z_0}(r)$ denotes the mean integral value of $Q(z)$ over the circle $|z-z_0|=r$. Then $\mathfrak{B}_{Q,\Delta}(D)$ forms a normal family.

Corollary 5.3. *The class $\mathfrak{B}_{Q,\Delta}(D)$ is normal if $Q(z)$ has singularities of the logarithmic type of order not greater than 1 at every point $z\in D$.*

In the theory of mappings called quasiconformal in the mean, conditions of the type

$$\int\limits_{D} \Phi(K(z))\;dx\,dy \;<\; \infty \tag{5.3}$$

are standard for various characteristics K of these mappings (see, e.g., references in Chapter 12 in [14]).

Let D be a fixed domain in $\mathbb{C}$. Given a function $\Phi : [0,\infty] \to [0,\infty]$, $M > 0$, $\Delta > 0$, $\mathfrak{B}^{\Phi}_{M,\Delta}$ denotes the collection of all homeomorphic $W^{1,1}_{\rm loc}$ solutions of the Beltrami equation (1.1) in D such that $h\left(\overline{\mathbb{C}} \setminus f(D)\right) \geq \Delta$ and

$$\int\limits_{D} \Phi\left(K_{\mu}(z)\right) \frac{dx\,dy}{\left(1+|z|^2\right)^2} \;\leq\; M\,. \tag{5.4}$$

On the base of Theorem 3.1, we obtain by Theorem 4.1 in [18] the following result.

Theorem 5.3. *Let $\Phi : [0,\infty] \to [0,\infty]$ be a non-decreasing convex function. If*

$$\int\limits_{\delta_0}^{\infty} \frac{d\tau}{\tau\Phi^{-1}(\tau)} \;=\; \infty \tag{5.5}$$

for some $\delta_0 > \Phi(0)$, then the class $\mathfrak{B}^{\Phi}_{M,\Delta}$ is equicontinuous and, consequently, forms a normal family of mappings for every $M \in (0,\infty)$ and $\Delta \in (0,1)$.

Remark 5.1. Note that the condition

$$\int\limits_{D} \Phi\left(K_{\mu}(z)\right) dx\,dy \leq M \tag{5.6}$$

implies (5.4). Thus, the condition (5.4) is more general than (5.6) and homeomorphic $W^{1,1}_{\rm loc}$ solutions of the Beltrami equation (1.1) with $K_{\mu}(z)$ satisfying (5.6) form a subclass of $\mathfrak{B}^{\Phi}_{M,\Delta}$. Conversely, if the domain D is bounded, then (5.4) implies the condition

$$\int\limits_{D} \Phi\left(K_{\mu}(z)\right) dx\,dy \leq M_* \tag{5.7}$$

where $M_* = M \cdot \left(1+\delta_*^2\right)^2$, $\delta_* = \sup\limits_{z\in D} |z|$.

Theorem 5.1 in [18] shows that the condition (5.5) is not only sufficient but also necessary for equicontinuity (normality) of classes with the integral constraints of the type either (5.4) or (5.7) with a convex non–decreasing Φ. A series of conditions that are equivalent to (5.5) can be found in [19].

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