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ON THE CONVERGENCE OF SPATIAL HOMEOMORPHISMS

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Various theorems on the convergence of general spatial homeomorphisms are proved and, on this basis, convergence theorems for classes of the so-called ring Q-homeomorphisms are obtained. These results will have wide applications to Sobolev's mappings.

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Доказаны различные теоремы о сходимости общих пространственных гомеоморфизмов и, на этой основе, получены теоремы о сходимости для так называемых кольцевых Q–гомеоморфизмов. Эти результаты будут иметь широкие приложения к отображениям классов Соболева.

1. Introduction. We give here foundations of the convergence theory for general homeomorphisms in the space and then develop the convergence theory for the so-called Q-homeomorphisms. The ring Q-homeomorphisms have been introduced first in a plane in connection with the study of the degenerate Beltrami equations, see e.g. the papers [22]–[26] and the monographs [8] and [16]. The theory of ring Q–homeomorphisms is applicable to various classes of mappings with finite distortion intensively investigated in many recent works, see e.g. [13] and [16] and further references therein. The present paper is a natural continuation of our previous works [20] and [21].

Given a family Γ of paths γ in $\mathbb{R}^n$, $n \geq 2$, a Borel function $\rho\colon \mathbb{R}^n \to [0,\infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm}\Gamma$, if $\int_\gamma \rho(x)|dx| \geq 1$ for each $\gamma \in \Gamma$. The *modulus* of Γ is the quantity

$$M(\Gamma) = \inf\Big\{ \int_{\mathbb{R}^n} \rho^n(x)dm(x)\colon \rho \in \operatorname{adm}\Gamma\Big\}.$$

Given a domain D and two subsets E and F of $\overline{\mathbb{R}}^n$, $n \geq 2$, $\Gamma(E,F,D)$ denotes the family of all paths $\gamma\colon [a,b] \to \overline{\mathbb{R}}^n$ which join E and F in D, i.e., $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$. We set $\Gamma(E,F) = \Gamma(E,F,\overline{\mathbb{R}}^n)$ if $D = \overline{\mathbb{R}}^n$. A *ring domain*, or shortly a *ring* in $\overline{\mathbb{R}}^n$, is a domain R in $\overline{\mathbb{R}}^n$ whose complement has two connected components. Let R be a ring in $\overline{\mathbb{R}}^n$. If C_1 and C_2 are the connected components of $\overline{\mathbb{R}}^n \setminus R$, we write $R = R(C_1, C_2)$. The *capacity* of R can be defined by the equality $\operatorname{cap} R(C_1,C_2) = M(\Gamma(C_1,C_2,R))$, see e.g.

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5.49 in [30]. Note also that $M(\Gamma(C_1,C_2,R)) = M(\Gamma(C_1,C_2))$, see e.g. Theorem 11.3 in [29]. A *conformal modulus* of a ring $R(C_1,C_2)$ is defined by

$$\operatorname{mod} R(C_1,C_2) = \left(\frac{\omega_{n-1}}{M(\Gamma(C_1,C_2))}\right)^{1/(n-1)},$$

where ω_{n-1} denotes the area of the unit sphere in $\mathbb{R}^n$, see e.g. (5.50) in [30].

The following notion was motivated by the ring definition of quasiconformality in [7]. Let D be a domain in $\mathbb{R}^n$, $Q\colon D\to(0,\infty)$ be a (Lebesgue) measurable function. Set

$$A(x_0,r_1,r_2) = \{x\in\mathbb{R}^n\colon r_1<|x-x_0|<r_2\}, S(x_0,r_i) = \{x\in\mathbb{R}^n\colon |x-x_0| = r_i\}\ (i\in\{1,2\}).$$

We say (see [20]) for the spatial case, that a homeomorphism f of D into $\overline{\mathbb{R}}^n$ is a *ring Q–homeomorphism at a point* $x_0\in D$ if

$$M\left(\Gamma\left(f(S_1),f(S_2)\right)\right)\le\int_A Q(x)\cdot\eta^n(|x-x_0|)dm(x) \tag{1}$$

for every ring $A=A(x_0,r_1,r_2)$, $0<r_1<r_2<r_0=\operatorname{dist}(x_0,\partial D)$, $S_i=S(x_0,r_i)$, $i\in\{1,2\}$, and for every Lebesgue measurable function $\eta\colon(r_1,r_2)\to[0,\infty]$ such that $\int_{r_1}^{r_2}\eta(r)dr\ge 1$.

If condition (1) holds at every point $x_0\in D$, then we also say that f is a ring Q–homeomorphism in the domain D.

2. On BMO and FMO functions. Recall that a real valued function $\varphi\in \mathrm{L}^1_{\mathrm{loc}}(D)$, given in a domain $D\subset\mathbb{R}^n$, is said to be of *bounded mean oscillation* by John and Nierenberg, abbr. $\varphi\in$ BMO(D) or simply $\varphi\in BMO$, see [10], if

$$\|\varphi\|_* = \sup_{B\subset D}\frac{1}{|B|}\int_B|\varphi(x)-\varphi_B|dm(x)<\infty,$$

where the supremum is taken over all balls B in D and

$$\varphi_B = \frac{1}{|B|}\int_B\varphi(x)dm(x)$$

is the average of the function φ over B. For connections of BMO functions with quasiconformal and quasiregular mappings, see e.g. [1], [2], [11], [17] and [19].

Following [9], we say that a function $\varphi\colon D\to\mathbb{R}$ has *finite mean oscillation* at a point $x_0\in D$ if

$$\overline{\lim_{\varepsilon\to 0}}\frac{1}{|B(x_0,\varepsilon)|}\int_{B(x_0,\varepsilon)}|\varphi(x)-\widetilde{\varphi}_\varepsilon|dm(x)<\infty, \tag{2}$$

where $\widetilde{\varphi}_\varepsilon = \frac{1}{|B(x_0,\varepsilon)|}\int_{B(x_0,\varepsilon)}\varphi(x)dm(x)$ is the average of the function $\varphi(x)$ over the ball $B(x_0,\varepsilon)=\{x\in\mathbb{R}^n\colon |x-x_0|<\varepsilon\}$. Note that under (2) it is possible that $\widetilde{\varphi}_\varepsilon\to\infty$ as $\varepsilon\to 0$.

We also say that a function $\varphi\colon D\to\mathbb{R}$ is of finite mean oscillation in the domain D, abbr. $\varphi\in\mathrm{FMO}(D)$ or simply $\varphi\in FMO$, if φ has finite mean oscillation at every point $x\in D$. Note that FMO is not $\mathrm{BMO}_{\mathrm{loc}}$, see examples in [16], p. 211. It is well–known that $L^\infty(D)\subset\mathrm{BMO}(D)\subset L^p_{\mathrm{loc}}(D)$ for all $1\le p<\infty$, see e.g. [10] and [19], but $\mathrm{FMO}(D)\not\subseteq L^p_{\mathrm{loc}}(D)$ for any $p>1$.

Recall some facts on finite mean oscillation from [9], see also 6.2 in [16].

Proposition 1. *If, for some numbers $\varphi_\varepsilon \in \mathbb{R}, \varepsilon \in (0, \varepsilon_0]$,*

$$\overline{\lim_{\varepsilon \to 0}} \frac{1}{|B(x_0, \varepsilon)|} \int_{B(x_0,\varepsilon)} |\varphi(x) - \varphi_\varepsilon| dm(x) < \infty,$$

then φ has finite mean oscillation at x_0.

Corollary 1. *If, for a point $x_0 \in D$,*

$$\overline{\lim_{\varepsilon \to 0}} \frac{1}{|B(x_0, \varepsilon)|} \int_{B(x_0,\varepsilon)} |\varphi(x)| dm(x) < \infty,$$

then φ has finite mean oscillation at x_0.

Lemma 1. *Let $\varphi\colon D \to \mathbb{R}, n \geq 2$, be a nonnegative function with a finite mean oscillation at $0 \in D$. Then*

$$\int_{\varepsilon<|x|<\varepsilon_0} \frac{\varphi(x) dm(x)}{(|x| \log \frac{1}{|x|})^n} = O\left(\log \log \frac{1}{\varepsilon}\right)$$

as $\varepsilon \to 0$ for a positive $\varepsilon_0 \leq \operatorname{dist}(0, \partial D)$.

This lemma takes an important part in many applications to the mapping theory as well as to the theory of the Beltrami equations, see e.g. the monographs [8] and [16].

3. Convergence of general homeomorphisms. In what follows, we use in $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ the *spherical (chordal) metric* $h(x, y) = |\pi(x) - \pi(y)|$ where π is the stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$, i.e.

$$h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}, \ x \neq \infty, \ y \neq \infty, \quad h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}.$$

It is clear that $\overline{\mathbb{R}}^n$ is homeomorphic to the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$.

The *spherical (chordal) diameter* of a set $E \subset \overline{\mathbb{R}}^n$ is $h(E) = \sup\{h(x, y)\colon x, y \in E\}$. We also define $h(z, E)$ for $z \in \overline{\mathbb{R}}^n$ and $E \subseteq \overline{\mathbb{R}}^n$ as a infimum of $h(z, y)$ over all $y \in E$ and $h(F, E)$ for $F \subseteq \overline{\mathbb{R}}^n$ and $E \subseteq \overline{\mathbb{R}}^n$ as the infimum of $h(z, y)$ over all $z \in F$ and $y \in E$. Later on, we also use the notation $B^*(x_0, \rho)$, $x_0 \in \overline{\mathbb{R}}^n$, $\rho \in (0, 1)$, for the balls $\{x \in \overline{\mathbb{R}}^n\colon h(x, x_0) < \rho\}$ with respect to the spherical metric.

Let us start with a simple consequence of the well–known Brouwer theorem on invariance of domains.

Corollary 2. *Let U be an open set in $\overline{\mathbb{R}}^n$ and let $f\colon U \to \overline{\mathbb{R}}^n$ be continuous and injective. Then f is a homeomorphism of U onto $V = f(U)$.*

Proof. Let $y_0 \in f(D)$ and $x_0{:=}f^{-1}(y_0)$. Set $B = B^*(x_0, \varepsilon_0)$ where $0 < \varepsilon_0 < h(x_0, \partial D)$. Then $\overline{B} \subset D$. Note that the mapping $f_0{:=}f|_{\overline{B}}$ is injective and continuous and maps the compactum $\overline{B}$ into the Hausdorff topological space $\mathbb{R}^n$. Consequently, f_0 is a homeomorphism of $\overline{B}$ onto the topological space $f_0(\overline{B})$ with the topology induced by that of $\mathbb{R}^n$ (see Theorem 41.III.3 in [15]). By the Brouwer theorem on invariance domains (see e.g. Theorem 4.7.16 in [28]), f maps the ball B onto a domain in $\overline{\mathbb{R}}^n$ as a homeomorphism. Hence the mapping $f^{-1}(y)$ is continuous at the point y_0. Thus, $f\colon D \to \overline{\mathbb{R}}^n$ is a homeomorphism. $\square$

The *kernel of a sequence of open sets* $\Omega_l \subset \overline{\mathbb{R}}^n$, $l = 1, 2, \ldots$ is the open set

$$\Omega_0 = \operatorname{Kern} \Omega_l := \bigcup_{m=1}^{\infty} \operatorname{Int} \left(\bigcap_{l=m}^{\infty} \Omega_l \right),$$

where $\operatorname{Int} A$ denotes the set consisting of all inner points of A; in other words, $\operatorname{Int} A$ is the union of all open balls in A with respect to the spherical distance.

The following statement for the plane case can be found in [3], see also Proposition 2.7 in [8].

Proposition 2. *Let $g_l \colon D \to D_l'$, $D_l' := g_l(D)$, be a sequence of homeomorphisms defined on a domain $D \subset \overline{\mathbb{R}}^n$. Suppose that g_l converges as $l \to \infty$ locally uniformly with respect to the spherical (chordal) metric to a mapping $g \colon D \to D' := g(D) \subset \overline{\mathbb{R}}^n$ which is injective. Then g is a homeomorphism and $D' \subset \operatorname{Kern} D_l'$.*

Proof. First of all, the mapping g is continuous as a locally uniform limit of continuous mappings, see e.g. Theorem 13.VI.3 in [14]. Thus, by Corollary 2 g is a homeomorphism.

Now, let y_0 be a point in D'. Consider the spherical ball $B^*(z_0, \rho)$ where $z_0 := g^{-1}(y_0) \in D$ and $\rho < h(z_0, \partial D)$. Then $r_0 := \min_{z \in \partial B^*(z_0, \rho)} h(y_0, g(z)) > 0$. There is an integer N large enough such that $g_l(z_0) \in B^*(y_0, r_0/2)$ for all $l \geq N$ and simultaneously

$$B^*(y_0, r_0/2) \cap g_l(\partial B^*(z_0, \rho)) = B^*(y_0, r_0/2) \cap \partial g_l(B^*(z_0, \rho)) = \varnothing$$

because $g_l \to g$ $(l \to +\infty)$ uniformly on the compact set $\partial B^*(z_0, \rho)$. Hence by the connectedness of balls

$$B^*(y_0, r_0/2) \subset g_l(B^*(z_0, \rho)) \qquad \forall l \geq N,$$

see e.g. Theorem 46.I.1 in [15]. Consequently, $y_0 \in \operatorname{Kern} D_l'$, i.e. $D' \subset \operatorname{Kern} D_l'$ by arbitrariness of y_0. $\square$

Remark 1. In particular, Proposition 2 implies that $D' := g(D) \subset \mathbb{R}^n$ if $D_l' := g_l(D) \subset \mathbb{R}^n$ for all $l = 1, 2, \ldots$.

The following statement for the plane case can be found in the paper [12], see also Lemma 2.16 in the monograph [8].

Lemma 2. *Let D be a domain in $\overline{\mathbb{R}}^n$, $l \in \{1, 2, \ldots\}$, and let f_l be a sequence of homeomorphisms from D into $\overline{\mathbb{R}}^n$ such that f_l converges as $l \to \infty$ locally uniformly with respect to the spherical metric to a homeomorphism f of D into $\overline{\mathbb{R}}^n$. Then $f_l^{-1} \to f^{-1}$ locally uniformly in $f(D)$, too.*

Proof. Given a compactum $C \subset f(D)$, we have by Proposition 2 that $C \subset f_l(D)$ for all $l \geq l_0 = l_0(C)$. Set $g_l = f_l^{-1}$ and $g = f^{-1}$. The locally uniform convergence $g_l \to g$ is equivalent to the so-called continuous convergence, meaning that $g_l(u_l) \to g(u_0)$ for every convergent sequence $u_l \to u_0$ in $f(D)$; see e.g. [5], p. 268 or Theorems 20.VIII.2 and 21.X.4 in [14]. So, let $u_l \in f(D)$, $l \in \{0, 1, 2, \dots\}$ and $u_l \to u_0$ as $l \to \infty$. Let us show that $z_l := g(u_l) \to z_0 := g(u_0)$ as $l \to \infty$.

It is known that every metric space is an $\mathcal{L}^*$-space, i.e. a space with a convergence (see, e.g., Theorem 21.II.1 in [14]), and the Urysohn axiom for compact spaces says that $z_l \to z_0$ as

$l \to \infty$ if and only if, for every convergent subsequence $z_{l_k} \to z_*$, the equality $z_* = z_0$ holds; see e.g. the definition 20.I.3 in [14]. Hence it suffices to prove that the equality $z_* = z_0$ holds for every convergent subsequence $z_{l_k} \to z_*$ as $k \to \infty$. Let D_0 be a subdomain of D such that $z_0 \in D_0$ and $\overline{D_0}$ is a compact subset of D. Then by Proposition 2, $f(D_0) \subset \mathrm{Kern} f_l(D_0)$ and hence u_0 together with its neighborhood belongs to $f_{l_k}(D_0)$ for all $k \geq K$. Thus, with no loss of generality we may assume that $u_{l_k} \in f_{l_k}(D_0)$, i.e. $z_{l_k} \in D_0$ for all $k \in \{1, 2, \dots\}$, and, consequently, $z_* \in D$. Then, by the continuous convergence $f_l \to f$, we have that $f_{l_k}(z_{l_k}) \to f(z_*)$, i.e. $f_{l_k}(g_{l_k}(u_{l_k})) = u_{l_k} \to f(z_*)$. The latter condition implies that $u_0 = f(z_*)$, i.e. $f(z_0) = f(z_*)$ and hence $z_* = z_0$. □

The following statement for the plane case can be found in the paper [26], see also Proposition 2.6 in the monograph [8].

Theorem 1. *Let D be a domain in $\overline{\mathbb{R}}^n$, $n \geq 2$, and let f_m, $m \in \{1, 2, \dots\}$, be a sequence of homeomorphisms of D into $\overline{\mathbb{R}}^n$ converging locally uniformly to a discrete mapping $f\colon D \to \overline{\mathbb{R}}^n$ with respect to the spherical metric. Then f is a homeomorphism of D into $\overline{\mathbb{R}}^n$.*

Proof. First of all, let us show by contradiction that f is injective. Indeed, let us assume that there exist $x_1, x_2 \in D$, $x_1 \neq x_2$, with $f(x_1) = f(x_2)$ and that $x_1 \neq \infty$. Set $B_t = B(x_1, t)$. Let t_0 be such that $\overline{B_t} \subset D$ and $x_2 \notin \overline{B_t}$ for every $t \in (0, t_0]$. By the Jordan–Brower theorem, see e.g. Theorem 4.8.15 in [28], $f_m(\partial B_t) = \partial f_m(B_t)$ splits $\overline{\mathbb{R}}^n$ into two components $C_m{:=}f_m(B_t)$, $C^*_m = \overline{\mathbb{R}}^n \setminus \overline{C_m}$.

By construction $y_m{:=}f_m(x_1) \in C_m$ and $z_m{:=}f_m(x_2) \in C^*_m$. Remark that the ball $B^*(y_m, h(y_m, \partial C_m))$ is contained inside of C_m and, consequently, its closure is inside of $\overline{C_m}$. Hence

$$h(y_m, \partial C_m) < h(y_m, z_m), \quad m \in \{1, 2, \dots\}. \tag{3}$$

By compactness of $\partial C_m = f_m(\partial B_t)$, there is $x_{m,t} \in \partial B_t$ such that

$$h(y_m, \partial C_m) = h(y_m, f_m(x_{m,t})), \quad m \in \{1, 2, \dots\}. \tag{4}$$

By compactness of ∂B_t, for every $t \in (0, t_0]$, there is $x_t \in \partial B_t$ such that $h(x_{m_k,t}, x_t) \to 0$ as $k \to \infty$ for some subsequence m_k. Since the locally uniform convergence of continuous functions in a metric space implies the continuous convergence (see [5], p. 268 or Theorem 21.X.3 in [14]), we have that $h(f_{m_k}(x_{m_k,t}), f(x_t)) \to 0$ as $k \to \infty$. Consequently, from (3) and (4) we obtain that $h(f(x_1), f(x_t)) \leq h(f(x_1), f(x_2))$ $\forall t \in (0, t_0]$. However, by the above assumption $f(x_1) = f(x_2)$ and we have $f(x_t) = f(x_1)$ for every $t \in (0, t_0]$. The latter condition contradicts the discreteness of f. Thus, f is injective.

It remains to show that f and f^{-1} are continuous. The mapping f is continuous as a locally uniform limit of continuous mappings, see e.g. Theorem 13.VI.3 in [14]. Finally, f^{-1} is continuous by Corollary 2. □

4. Convergence of homeomorphisms and moduli. Later on, the following lemma plays a very important role. Its plane analog can be found in the paper [4], see also supplement A1 in the monograph [8].

Lemma 3. *Let f_m, $m \in \{1, 2, \dots\}$, be a sequence of homeomorphisms of a domain $D \subseteq \mathbb{R}^n$ into $\mathbb{R}^n$, $n \geq 2$, converging to a mapping f uniformly on every compact set in D with respect to the spherical metric in $\overline{\mathbb{R}}^n$. Suppose that for every $x_0 \in D$ there exist sequences $R_k > 0$ and $r_k \in (0, R_k)$, $k \in \{1, 2, \dots\}$, such that $R_k \to 0$ as $k \to \infty$ and* $\mathrm{mod}\, f_m\left(A\left(x_0, r_k, R_k\right)\right) \to \infty$

as $k \to \infty$ uniformly with respect to $m \in \{1, 2, \dots\}$. Then the mapping f is either a constant in $\overline{\mathbb{R}}^n$ or a homeomorphism of D into $\mathbb{R}^n$.

Proof. Assume that f is not constant. Let us consider the open set V consisting of all points in D which have neighborhoods where f is a constant and show that $f(x) \neq f(x_0)$ for every $x_0 \in D \setminus V$ and $x \neq x_0$. Without loss of generality, we may assume that $f(x_0) \neq \infty$. Now, let us fix a point $x_* \neq x_0$ in $D \setminus V$ and choose $k \in \{1, 2, \dots\}$ such that $R{:=}R_k < |x_* - x_0|$ and

$$\operatorname{mod} f_m\left(A(x_0, r, R)\right) > \left(\omega_{n-1}/\tau_n(1)\right)^{1/(n-1)}, \ \forall m \in \{1, 2, \dots\} \tag{5}$$

for $r = r_k$ where $\tau_n(s)$ denotes the capacity of the Teichmüller ring $R_{T,n}(s){:=} \left[\mathbb{R}^n \setminus \{te_1 \colon t \geq s\}, \ [-e_1, 0]\right], \ s \in (0, \infty)$.

Let $c_m \in f_m(S(x_0, R))$ and $b_m \in f_m(S(x_0, r))$ be such that

$$\min_{w \in f_m(S(x_0,R))} |w - f_m(x_0)| = |c_m - f_m(x_0)|, \quad \max_{w \in f_m(S(x_0,r))} |w - f_m(x_0)| = |b_m - f_m(x_0)|.$$

Since f_m is a homeomorphism, the set $f_m(A(x_0, r, R))$ is a ring domain $\mathfrak{R}_m = (C_m^1, C_m^2)$, where $a_m{:=}f_m(x_0)$ and $b_m \in C_m^1$, c_m and $\infty \in C_m^2$. Applying Lemma 7.34 in [30] with $a = a_m$, $b = b_m$ and $c = c_m$, we obtain that

$$\operatorname{cap} \mathfrak{R}_m = M(\Gamma(C_m^1, C_m^2)) \geq \tau_n\left(\frac{|a_m - c_m|}{|a_m - b_m|}\right). \tag{6}$$

Note that the function $\tau_n(s)$ is strictly decreasing (see Lemma 7.20 in [30]). Thus, it follows from (5) and (6) that

$$\frac{|a_m - c_m|}{|a_m - b_m|} \geq \tau_n^{-1}\left(\operatorname{cap} \mathfrak{R}_m\right) > \tau_n^{-1}(\tau_n(1)) = 1.$$

Hence there is a spherical ring $A_m = \{y \in \mathbb{R}^n \colon \rho_m < |y - f_m(x_0)| < \rho_m^*\}$ in the ring domain $\mathfrak{R}_m$ for every $m \in \{1, 2, \dots\}$. Since f is not locally constant at x_0, we can find a point x' in the ball $|x - x_0| < r$ with $f(x_0) \neq f(x')$. The ring A_m separates $f_m(x_0)$ and $f_m(x')$ from $f_m(x_*)$ and, thus, $|f_m(x') - f_m(x_0)| \leq \rho_m$ and $|f_m(x_*) - f_m(x_0)| \geq \rho_m^*$. Consequently, $|f_m(x') - f_m(x_0)| \leq |f_m(x_*) - f_m(x_0)|$ for all $m \in \{1, 2, \dots\}$. Under $m \to \infty$ we have then $0 < |f(x') - f(x_0)| \leq |f(x_*) - f(x_0)|$ and hence $f(x_*) \neq f(x_0)$.

It remains to show that the set V is empty. Let us assume that V has a nonempty component V_0. Then $f(x) \equiv z$ for every $x \in V_0$ and some $z \in \overline{\mathbb{R}}^n$. Note that $\partial V_0 \cap D \neq \varnothing$ by connectedness of D, because $f \not\equiv \text{const}$ in D and the set $D \setminus \overline{V_0}$ is also open. If $x_0 \in \partial V_0 \cap D$, then by continuity, $f(x_0) = z$ contradicting the assertion established in the first part of the proof because $x_0 \in D \setminus V$.

Thus, we have proved that the mapping f is injective if f is not constant. But f is continuous as a locally uniform limit of continuous mappings f_m, see Theorem 13.VI.3 in [14], and then by Corollary 2 f is a homeomorphism. Finally, by Remark 1 $f(D) \subset \mathbb{R}^n$ and the proof is complete. $\square$

Lemma 4. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $Q_m \colon D \to (0, \infty)$ be measurable functions, f_m, $m \in \{1, 2, \dots\}$, be a sequence of ring Q_m-homeomorphisms of D into $\mathbb{R}^n$ converging locally uniformly to a mapping f. Suppose*

$$\int_{\varepsilon < |x - x_0| < \varepsilon_0} Q_m(x) \cdot \psi^n(|x - x_0|) dm(x) = o(I^n(\varepsilon, \varepsilon_0)) \quad \forall x_0 \in D, \tag{7}$$

where $o(I^n(\varepsilon,\varepsilon_0))/I^n(\varepsilon,\varepsilon_0)\to 0$ *as* $\varepsilon\to 0$ *uniformly with respect to* m *for* $\varepsilon_0<\mathrm{dist}(x_0,\partial D)$ *and a measurable function* $\psi(t)\colon(0,\varepsilon_0)\to[0,\infty]$ *such that*

$$0<I(\varepsilon,\varepsilon_0):=\int_{\varepsilon}^{\varepsilon_0}\psi(t)dt<\infty\quad\forall\varepsilon\in(0,\varepsilon_0). \tag{8}$$

Then the mapping f *is either a constant in* $\overline{\mathbb{R}}^n$ *or a homeomorphism into* $\mathbb{R}^n$.

Remark 2. In particular, the conclusion of Lemma 4 holds for Q-homeomorphisms f_m with a measurable function $Q\colon D\to(0,\infty)$ such that

$$\int_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x)\cdot\psi^n(|x-x_0|)dm(x)=o(I^n(\varepsilon,\varepsilon_0))\qquad\forall x_0\in D. \tag{9}$$

Proof. By Luzin's theorem, there exists a Borel function $\psi_*(t)$ such that $\psi(t)=\psi_*(t)$ for a.e. $t\in(0,\varepsilon_0)$, see e.g. 2.3.6 in [6]. Since $Q_m(x)>0$ for all $x\in D$ we have from (7) that $I(\varepsilon,a)\to\infty$ for every fixed $a\in(0,\varepsilon_0)$ and, in particular, $I(\varepsilon,a)>0$ for every $\varepsilon\in(0,b)$ and some $b=b(a)\in(0,a)$. Given $x_0\in D$ and a sequence of such numbers $b=\varepsilon_k\to 0$ as $k\to\infty$, $k\in\{1,2,\ldots\}$, consider the sequence of the Borel measurable functions $\rho_{\varepsilon,k}$ defined by

$$\rho_{\varepsilon,k}(x)=\begin{cases}\psi_*(|x-x_0|)/I(\varepsilon,\varepsilon_k), & \varepsilon<|x-x_0|<\varepsilon_k,\\ 0, & \text{otherwise.}\end{cases}$$

Note that the function $\rho_{\varepsilon,k}(x)$ is admissible for $\Gamma_{\varepsilon,k}:=\Gamma(S(x_0,\varepsilon),S(x_0,\varepsilon_k),A(x_0,\varepsilon,\varepsilon_k))$ because

$$\int_{\gamma}\rho_{\varepsilon,k}(x)|dx|\geq\frac{1}{I(\varepsilon,\varepsilon_k)}\int_{\varepsilon}^{\varepsilon_k}\psi(t)dt=1$$

for all (locally rectifiable) curves $\gamma\in\Gamma_{\varepsilon,k}$ (see Theorem 5.7 in [29]). Then by the definition of ring Q–homeomorphisms

$$M(f_m(\Gamma_{\varepsilon,k}))\leq\frac{1}{I^n(\varepsilon,\varepsilon_k)}\int_{\varepsilon<|x-x_0|<\varepsilon_0}Q(x)\cdot\psi^n(|x-x_0|)dm(x) \tag{10}$$

for all $m\in\mathbb{N}$. Note that $\frac{1}{I^n(\varepsilon,\varepsilon_k)}=\alpha_{\varepsilon,k}\cdot\frac{1}{I^n(\varepsilon,\varepsilon_0)}$, where $\alpha_{\varepsilon,k}:=\left(1+\frac{I(\varepsilon_k,\varepsilon_0)}{I(\varepsilon,\varepsilon_k)}\right)^n$ is independent on m and bounded as $\varepsilon\to 0$. Then it follows from (7) and (10) that there exists $\varepsilon_k^*\in(0,\varepsilon_k)$ such that for all $M(f_m(\Gamma_{\varepsilon_k^*,k}))\leq 2^{-k}$ $\forall m\in\mathbb{N}$. Applying Lemma 3 we obtain the desired conclusion. □

The next important statements follows from Lemma 4.

Theorem 2. *Let* D *be a domain in* $\mathbb{R}^n$, $n\geq 2$, $Q\colon D\to(0,\infty)$ *a Lebesgue measurable function and let* f_m, $m\in\{1,2,\ldots\}$, *be a sequence of ring* Q*-homeomorphisms of* D *into* $\mathbb{R}^n$ *converging locally uniformly to a mapping* f. *Suppose that* $Q\in FMO$. *Then the mapping* f *is either a constant in* $\overline{\mathbb{R}}^n$ *or a homeomorphism into* $\mathbb{R}^n$.

Proof. Let $x_0\in D$. We may consider further that $x_0=0\in D$. Choosing a positive $\varepsilon_0<\min\{\mathrm{dist}\,(0,\partial D)\,,e^{-1}\}$, we obtain by Lemma 1 for the function $\psi(t)=\frac{1}{t\log\frac{1}{t}}$ that

$$\int_{\varepsilon<|x|<\varepsilon_0}Q(x)\cdot\psi^n(|x|)dm(x)=O\Big(\log\log\frac{1}{\varepsilon}\Big).$$

Note that $I(\varepsilon,\varepsilon_0):=\int_{\varepsilon}^{\varepsilon_0}\psi(t)dt=\log\frac{\log\frac{1}{\varepsilon}}{\log\frac{1}{\varepsilon_0}}$. Now the desired conclusion follows from Lemma 4. □

The following conclusions can be obtained on the basis of Theorem 2, Proposition 1 and Corollary 1.

Corollary 3. *In particular, the limit mapping f is either a constant in $\overline{\mathbb{R}}^n$ or a homeomorphism of D into $\mathbb{R}^n$ whenever*

$$\overline{\lim_{\varepsilon\to 0}}\frac{1}{|B(x_0,\varepsilon)|}\int_{B(x_0,\varepsilon)} Q(x)dm(x)<\infty \quad \forall x_0\in D$$

or whenever every $x_0\in D$ is a Lebesgue point of Q.

Theorem 3. *Let D be a domain in $\mathbb{R}^n$, $n\geq 2$, and let $Q\colon D\to(0,\infty)$ be a measurable function such that*

$$\int_0^{\varepsilon(x_0)}\frac{dr}{rq_{x_0}^{\frac{1}{n-1}}(r)}=\infty \qquad \forall x_0\in D \tag{11}$$

for a positive $\varepsilon(x_0)<\operatorname{dist}(x_0,\partial D)$ where $q_{x_0}(r)$ denotes the average of $Q(x)$ over the sphere $|x-x_0|=r$. Suppose that f_m, $m\in\{1,2,\ldots\}$, is a sequence of ring Q-homeomorphisms from D into $\mathbb{R}^n$ converging locally uniformly to a mapping f. Then the mapping f is either a constant in $\overline{\mathbb{R}}^n$ or a homeomorphism into $\mathbb{R}^n$.

Proof. Fix $x_0\in D$ and set $I=I(\varepsilon,\varepsilon_0)=\int_\varepsilon^{\varepsilon_0}\psi(t)dt$, $\varepsilon\in(0,\varepsilon_0)$, where

$$\psi(t)=\begin{cases}1/[tq_{x_0}^{\frac{1}{n-1}}(t)], & t\in(\varepsilon,\varepsilon_0),\\ 0\ , & t\notin(\varepsilon,\varepsilon_0).\end{cases}$$

Note that $I(\varepsilon,\varepsilon_0)<\infty$ for every $\varepsilon\in(0,\varepsilon_0)$. Indeed, by Theorem 3.15 in [20] on the criterion of ring $Q-$homeomorphisms, we have that

$$M\Big(f\Big(\Gamma(S(x_0,\varepsilon),S(x_0,\varepsilon_0),A(x_0,\varepsilon,\varepsilon_0))\Big)\Big)\leq\frac{\omega_{n-1}}{I^{n-1}}. \tag{12}$$

On the other hand, by Lemma 1.15 in [18], we see that

$$M\Big(\Gamma(f(S(x_0,\varepsilon)),f(S(x_0,\varepsilon_0)),f(A(x_0,\varepsilon,\varepsilon_0)))\Big)>0.$$

Then it follows from (12) that $I<\infty$ for every $\varepsilon\in(0,\varepsilon_0)$. In view of (11), we obtain that $I(\varepsilon,\varepsilon_*)>0$ for all $\varepsilon\in(0,\varepsilon_*)$ with some $\varepsilon_*\in(0,\varepsilon_0)$. Finally, simple calculations show that (9) holds, in fact,

$$\int_{\varepsilon<|x-x_0|<\varepsilon_*}Q(x)\cdot\psi^n(|x-x_0|)dm(x)=\omega_{n-1}\cdot I(\varepsilon,\varepsilon_*)$$

and $I(\varepsilon,\varepsilon_*)=o\left(I^n(\varepsilon,\varepsilon_*)\right)$ by (11). The rest follows by Lemma 4. □

Corollary 4. *In particular, the conclusion of Theorem 3 holds if $q_{x_0}(r)=O\left(\log^{n-1}\frac{1}{r}\right)$ for all $x_0\in D$.*

Corollary 5. *Under assumptions of Theorem 3, the mapping f is either a constant in $\overline{\mathbb{R}}^n$ or a homeomorphism into $\mathbb{R}^n$ provided $Q(x)$ has singularities only of the logarithmic type of the order which is not more than $n-1$ at every point $x_0\in D$.*

Theorem 4. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and $Q\colon D \to (0,\infty)$ be a measurable function such that*

$$\int_{\varepsilon<|x-x_0|<\varepsilon_0} \frac{Q(x)}{|x-x_0|^n} dm(x) = o\left(\log^n \frac{1}{\varepsilon}\right) \qquad \forall x_0 \in D \tag{13}$$

as $\varepsilon \to 0$ for some positive number $\varepsilon_0 = \varepsilon(x_0) < \operatorname{dist}(x_0, \partial D)$. Suppose that f_m, $m \in \{1,2,\ldots\}$, is a sequence of ring Q-homeomorphisms from D into $\overline{\mathbb{R}}^n$ converging locally uniformly to a mapping f. Then the limit mapping f is either a constant in $\overline{\mathbb{R}}^n$ or a homeomorphism into $\mathbb{R}^n$.

Proof. The conclusion follows from Lemma 4 by the choice $\psi(t) = \frac{1}{t}$. □

For every nondecreasing function $\Phi\colon [0,\infty] \to [0,\infty]$, the *inverse function* $\Phi^{-1}\colon [0,\infty] \to [0,\infty]$ can be well defined by setting $\Phi^{-1}(\tau) = \inf_{\Phi(t)\geq\tau} t$. As usual, here inf is equal to ∞ if the set of all $t \in [0,\infty]$ such that $\Phi(t) \geq \tau$ is empty. Note that the function Φ^{-1} is nondecreasing, too. Note also that if $h\colon [0,\infty] \to [0,\infty]$ is a sense–preserving homeomorphism and $\varphi\colon [0,\infty] \to [0,\infty]$ is a nondecreasing function, then

$$(\varphi \circ h)^{-1} = h^{-1} \circ \varphi^{-1}. \tag{14}$$

Theorem 5. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, let $Q\colon D \to (0,\infty)$ be a measurable function and $\Phi\colon [0,\infty] \to [0,\infty]$ be a nondecreasing convex function. Suppose that*

$$\int_D \Phi\left(Q(x)\right) \frac{dm(x)}{\left(1+|x|^2\right)^n} \leq M < \infty \tag{15}$$

and

$$\int_\delta^\infty \frac{d\tau}{\tau\left[\Phi^{-1}(\tau)\right]^{\frac{1}{n-1}}} = \infty \tag{16}$$

for some $\delta > \Phi(0)$. Suppose that f_m, $m \in \{1,2,\ldots\}$, is a sequence of ring Q-homeomorphisms of D into $\mathbb{R}^n$ converging locally uniformly to a mapping f. Then the mapping f is either a constant in $\overline{\mathbb{R}}^n$ or a homeomorphism into $\mathbb{R}^n$.

Proof. It follows from (15)–(16) and Theorem 3.1 in [21] that the integral in (11) is divergent for some positive $\varepsilon(x_0) < \operatorname{dist}(x_0, \partial D)$. The rest follows by Theorem 3. □

Remark 3. We may assume in Theorem 5 that the function $\Phi(t)$ is convex not on the whole segment $[0,\infty]$ but only on the segment $[t_*,\infty]$ where $t_* = \Phi^{-1}(\delta)$. Indeed, every non-decreasing function $\Phi\colon [0,\infty] \to [0,\infty]$ which is convex on the segment $[t_*,\infty]$ can be replaced with a non-decreasing convex function $\Phi_*\colon [0,\infty] \to [0,\infty]$ in the following way. Set $\Phi_*(t) \equiv 0$ for $t \in [0,t_*]$, $\Phi(t) = \varphi(t)$ for $t \in [t_*,T_*]$ and $\Phi_* \equiv \Phi(t)$ for $t \in [T_*,\infty]$, where $\tau = \varphi(t)$ is the line passing through the point $(0,t_*)$ and touching the graph of the function $\tau = \Phi(t)$ at a point $(T_*,\Phi(T_*))$, $T_* \in (t_*,\infty)$. By the construction, we have that $\Phi_*(t) \leq \Phi(t)$ for all $t \in [0,\infty]$ and $\Phi_*(t) = \Phi(t)$ for all $t \geq T_*$ and, consequently, conditions (15) and (16) hold for Φ_* under the same M and every $\delta > 0$.

Furthermore, by the same reasons it is sufficient to assume that the function Φ is only minorized by a nondecreasing convex function Ψ on a segment $[T,\infty]$ such that

$$\int_\delta^\infty \frac{d\tau}{\tau\left[\Psi^{-1}(\tau)\right]^{\frac{1}{n-1}}} = \infty \tag{17}$$

for some $T \in [0,\infty)$ and $\delta > \Psi(T)$. Note that condition (17) can be written in terms of the function $\psi(t) = \log \Psi(t)$

$$\int_{\Delta}^{\infty} \psi(t) \frac{dt}{t^{n'}} = \infty \tag{18}$$

for some $\Delta > t_0 \in [T,\infty]$, where $t_0{:=}\sup_{\psi(t)=-\infty} t$, $t_0 = T$ if $\psi(T) > -\infty$, and where $\frac{1}{n'} + \frac{1}{n} = 1$, i.e., $n' = 2$ for $n = 2$, n' is decreasing in n and $n' = n/(n-1) \to 1$ as $n \to \infty$, see Proposition 2.3 in [21]. It is clear that if the function ψ is nondecreasing and convex, then the function $\Phi = e^{\psi}$ is so but the inverse conclusion generally speaking is not true. However, the conclusion of Theorem 5 is valid if $\psi^m(t)$, $t \in [T,\infty]$, is convex and (18) holds for ψ^m under some $m \in \mathbb{N}$ because $e^{\tau} \geq \tau^m/m!$ for all $m \in \mathbb{N}$.

Corollary 6. *In particular, the conclusion of Theorem 5 is valid if, for some $\alpha > 0$,*

$$\int_D e^{\alpha Q^{\frac{1}{n-1}}(x)} \frac{dm(x)}{(1+|x|^2)^n} \leq M < \infty.$$

The same is true for any function $\Phi = e^{\psi}$, where $\psi(t)$ is a finite product of the function αt^{β}, $\alpha > 0$, $\beta \geq 1/(n-1)$, and some of the functions $[\log(A_1+t)]^{\alpha_1}$, $[\log\log(A_2+t)]^{\alpha_2}, \ldots$, $\alpha_m \geq -1$, $A_m \in \mathbb{R}$, $m \in \mathbb{N}$, $t \in [T,\infty]$, $\psi(t) \equiv \psi(T)$, $t \in [0,T]$.

Remark 4. For further applications, integral conditions (15) and (16) for Q and Φ can be written in other forms that are more convenient for some cases. Namely, by (14) with $h(t) = t^{\frac{1}{n-1}}$ and $\varphi(t) = \Phi(t^{n-1})$, $\Phi = \varphi \circ h$, the couple of conditions (15) and (16) is equivalent to the following couple

$$\int_D \varphi\left(Q^{\frac{1}{n-1}}(x)\right) \frac{dm(x)}{(1+|x|^2)^n} \leq M < \infty \tag{19}$$

and

$$\int_{\delta}^{\infty} \frac{d\tau}{\tau \varphi^{-1}(\tau)} = \infty \tag{20}$$

for some $\delta > \varphi(0)$. Moreover, by Theorem 2.1 in [27] the couple of the conditions (19) and (20) is in turn equivalent to the next couple

$$\int_D e^{\psi\left(Q^{\frac{1}{n-1}}(x)\right)} \frac{dm(x)}{(1+|x|^2)^n} \leq M < \infty \quad \text{and} \quad \int_{\Delta}^{\infty} \psi(t) \frac{dt}{t^2} = \infty$$

for some $\Delta > t_0$, where $t_0{:=}\sup_{\psi(t)=-\infty} t$, $t_0 = 0$ if $\psi(0) > -\infty$.

Finally, as it follows from Lemma 4 all the results of this section are valid if f_m are Q_m-homeomorphisms and the above conditions on Q hold for Q_m uniformly with respect to the parameter $m \in \{1,2,\ldots\}$.

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